Available online at www.sciencedirect.com
ScienceDirect
Applied Mathematics Letters 21 (2008) 669–674
Applied Mathematics Letters
www.elsevier.com/locate/aml

The variational iteration method for studying the Klein–Gordon equation

Elcin Yusufoğlu

Dumlupinar University, Art-Science Faculty, Department of Mathematics, Kütahya, Turkey

Received 7 September 2006; received in revised form 2 May 2007; accepted 2 July 2007

Abstract

In this work, we use He's variational iteration method for solving linear and nonlinear Klein–Gordon equations. Also, the results are compared with those obtained by Adomian's decomposition method (ADM). The results reveal that the method is very effective and simple.

Keywords: Variational iteration method; Adomian's decomposition method; Klein–Gordon equations; Correction functional; Lagrange multiplier

1. Introduction

Nonlinear phenomena occur in a wide variety of scientific applications such as plasma physics, solid state physics, fluid dynamics and chemical kinetics [1]. Because of the increased interest in the theory of solitary waves, a broad range of analytical and numerical methods have been used in the analysis of these scientific models.

First, the variational iteration method was proposed by He in 1998 [2–5] and was successfully applied to autonomous ordinary differential equation [6], to nonlinear partial differential equations with variable coefficients [7], to Schrödinger–KdV, generalized KdV and shallow water equations [8], to Burgers' and coupled Burgers' equations [9], to the linear Helmholtz partial differential equation [10] and recently to nonlinear fractional differential equations with Caputo differential derivative [11], and other fields [12–14].

The numerical treatment of the Klein–Gordon equation

$$u_{tt} - u_{xx} = -F(u), \tag{1}$$

subject to initial conditions

$$u(x,0) = f(x), \qquad u_t(x,0) = g(x), \tag{2}$$

has been under consideration, where $F(u)$ is a linear or nonlinear function. The equation has attracted much attention in studying solitons and condensed matter physics, in investigating the interaction of solitons in collisionless plasma,

E-mail address: eyusufoglu@dumlupinar.edu.tr.

0893-9659/$ - see front matter
doi:10.1016/j.aml.2007.07.023

the recurrence of initial states, and in examining the nonlinear wave equations [15]. Some projection methods for numerical treatment of (1) are given in [16–18].

In this work, a new application of He's variational iteration method is applied to solve linear and nonlinear Klein–Gordon equations. This application does not have secular terms and in a special case, ADM is obtained [5]. Kaya [19,20], El-Sayed [21], and Wazwaz [22] have implemented ADM [23–25] to solve the nonlinear Klein–Gordon equation. Comparisons are made between standard ADM and He's variational iteration method and between the exact solution and the proposed method. The results reveal that the proposed method is very effective and simple and can be applied to other nonlinear problems.

2. He's variational iteration method

To illustrate the basic concepts of the variational iteration method, we consider the following general nonlinear system:

$$Lu(x) + Nu(x) = g(x), \tag{3}$$

where L is a linear operator part while N is the nonlinear operator part, and $g(x)$ is a known analytic function.

According to the variational iteration method, a correction functional can be constructed as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda \{Lu_n(\xi) + N\tilde{u}_n(\xi) - g(\xi)\}\, \mathrm{d}\xi, \tag{4}$$

where λ is a general multiplier [7], which can be identified optimally via the variational theory [7,12], the subscript n denotes the nth approximation, and $\tilde{u}_n$ is considered as a restricted variation [7], i.e., $\delta\tilde{u}_n = 0$.

The initial guess can be freely chosen with possible unknown constants; it can also be solved from its corresponding linear homogeneous equation

$$Lu_0(x) = 0. \tag{5}$$

The variational iteration method can solve effectively, easily, and accurately a large class of nonlinear problems with approximations converging rapidly. For linear problems, exact solutions can be obtained in only one iteration step due to the fact that the Lagrange multiplier can be obtained by just one iteration, because λ can be exactly identified.

3. Applications

To achieve the goal of this work, we first start with the linear Klein–Gordon equation.

Example 1 (*El-Sayed [21]*). Consider the linear form $F(u) = -u$ in Eq. (1); therefore we set

$$u_{tt} - u_{xx} = u, \tag{6}$$

subject to initial conditions

$$u(x, 0) = 1 + \sin(x), \qquad u_t(x, 0) = 0. \tag{7}$$

According to Eq. (4), we can construct a correction functional as follows:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda \{u_{n\tau\tau}(x,\tau) + \tilde{u}_{nxx}(x,\tau) - u_n(x,\tau)\}\, \mathrm{d}\tau. \tag{8}$$

Making the above correction functional stationary, and noting that $\delta\tilde{u}_n = 0$, we get

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) + \delta \int_0^t \lambda \{u_{n\tau\tau}(x,\tau) + \tilde{u}_{nxx}(x,\tau) - u_n(x,\tau)\}\, \mathrm{d}\tau, \tag{9}$$

or

$$\delta u_{n+1}(x,t) = \delta u_n(x,t) - \lambda' \delta u_n(x,\tau)\big|_{\tau=t} + \lambda \delta u_{n\tau}(x,\tau)|_{\tau=t} + \int_0^t \left(\lambda'' - \lambda\right) \delta u_n(x,\tau)\mathrm{d}\tau| , \tag{10}$$

which yields the following stationary conditions:

$$\lambda'' - \lambda = 0, \tag{11}$$

$$1 - \lambda'\big|_{\tau=t} = 0, \tag{12}$$

$$\lambda|_{\tau=t} = 0. \tag{13}$$

The general Lagrange multiplier, therefore, can be identified as

$$\lambda(\tau) = \sinh(\tau - t). \tag{14}$$

As a result, we obtain the following iteration formula:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \sinh(\tau - t)\{u_{n\tau\tau}(x,\tau) + u_{nxx}(x,\tau) - u_n(x,\tau)\}\,\mathrm{d}\tau. \tag{15}$$

In the first step, by iteration formula (15) with initial approximation

$$u_0(x,t) = u(x,0) + tu_t(x,0) = 1 + \sin x \tag{16}$$

we have

$$u_1(x,t) = \sin x + \cosh t, \tag{17}$$

which is the general solution of initial value problem (6) and (7). The solution for the variational iteration method in question was solved via ADM in [21] and the solution was obtained as

$$u(x,t) = \sin x + 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \frac{t^6}{6!} + \frac{t^8}{8!} + \cdots. \tag{18}$$

This shows that the application of the variational iteration method reduces the computational work considerably.

Now consider the following nonlinear Klein–Gordon equation with nonhomogeneous initial conditions, which was recently solved by ADM [21].

Example 2 (*El-Sayed [21]*). We consider now a nonlinear example, i.e., $F(u) = u^2$, with nonhomogeneous initial conditions [22], namely

$$u_{tt} - u_{xx} = -u^2, \tag{19}$$

$$u(x,0) = 1 + \sin x, \qquad u_t(x,0) = 0. \tag{20}$$

To solve Eq. (19) by means of the variational iteration method, we construct a correction functional which reads

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda\{u_{n\tau\tau}(x,\tau) - \tilde{u}_{nxx}(x,\tau) + \tilde{u}_n^2(x,\tau)\}\mathrm{d}\tau, \tag{21}$$

where $\delta\tilde{u}_n$ is considered as a restricted variation. Its stationary conditions can be obtained as follows:

$$\lambda'' = 0, \tag{22}$$

$$1 - \lambda'\big|_{\tau=t} = 0, \tag{23}$$

$$\lambda|_{\tau=t} = 0. \tag{24}$$

This in turn gives

$$\lambda(\tau) = \tau - t. \tag{25}$$

Substituting this value of the Lagrangian multiplier into functional (21) gives the iteration formula

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\tau - t)\{u_{n\tau\tau}(x,\tau) - u_{nxx}(x,\tau) + u_n^2(x,\tau)\}\mathrm{d}\tau. \tag{26}$$

As stated before, we can use any selective function for $u_0(x,t)$; preferentially we use the initial condition (20), i.e.

Table 1
Comparison between the value u for the solution of the Klein–Gordon equation for He's variational iteration method (VIM) and ADM in Ref. [21] at various values of (x, t)

x	$t = 0.1$		$t = 0.2$		$t = 0.3$	
	ADM	VIM	ADM	VIM	ADM	VIM
0.0	0.9949999861	0.9950000249	0.9799991162	0.9800015775	0.9549900052	0.9550176534
0.1	1.093291132	1.093291179	1.073723730	1.073726319	1.073723730	1.073726319
0.2	1.190502988	1.190503087	1.166134875	1.166138050	1.125945576	1.125974851
0.3	1.285668610	1.285668848	1.256326130	1.256331032	1.208114007	1.208147932
0.4	1.377844211	1.377844710	1.343423788	1.343432104	1.287043874	1.287088824
0.5	1.466118315	1.466119219	1.426594492	1.426608263	1.362025218	1.362089477
0.6	1.549620480	1.549621939	1.505052082	1.505073495	1.432404521	1.432497282
0.7	1.627529538	1.627531694	1.578063673	1.578094808	1.497587424	1.497717706
0.8	1.699081273	1.699084244	1.644954933	1.644997540	1.557040327	1.557215916
0.9	1.763575490	1.763579356	1.705114628	1.705169916	1.610291023	1.610517519
1.0	1.820382425	1.820387216	1.757998450	1.758066925	1.656928567	1.657208637

$$u_0(x,t) = u(x,0) + t u_t(x,0) = 1 + \sin x. \tag{27}$$

Consequently, on using (26), the following successive approximations are obtained:

$$u_1(x,t) = 1 + \sin x - \frac{t^2}{2!}\left(1 + 3\sin x + \sin^2 x\right), \tag{28}$$

$$u_2(x,t) = 1 + \sin x - \frac{t^2}{2!}\left(1 + 3\sin x + \sin^2 x\right) + \frac{t^4}{4!}\left(11 + 12\sin x + 2\sin^2 x\right)\sin x + \cdots, \tag{29}$$

$$\begin{aligned} u_3(x,t) \;=\; & 1 + \sin x - \frac{t^2}{2!}\left(1 + 3\sin x + \sin^2 x\right) + \frac{t^4}{4!}\left(11 + 12\sin x + 2\sin^2 x\right)\sin x \\ & + \frac{t^6}{6!}\left(18 - 57\sin x - 160\sin^2 x - 82\sin^3 x - 10\sin 4x\right) + \cdots, \end{aligned} \tag{30}$$

$$\begin{aligned} u_4(x,t) \;=\; & 1 + \sin x - \frac{t^2}{2!}\left(1 + 3\sin x + \sin^2 x\right) + \frac{t^4}{4!}\left(11 + 12\sin x + 2\sin^2 x\right)\sin x \\ & + \frac{t^6}{6!}\left(18 - 57\sin x - 160\sin^2 x - 82\sin^3 x - 10\sin 4x\right) \\ & + \frac{t^8}{8!}\left(-356 - 27\sin x + 2304\sin^2 x + 2692\sin^3 x + 884\sin^4 x + 80\sin^5 x\right) + \cdots. \end{aligned} \tag{31}$$

It is clear that this approximation can be used for numerical purposes only because a closed form solution is not obtainable. To illustrate the above results, we present a numerical experiment to compare our approximate solution and the results obtained by using ADM in [21]. The comparison between the fourth-iteration solution of the variational iteration method and five terms of ADM are given in Table 1.

Example 3. We finally close our analysis by studying the Klein–Gordon equation

$$u_{tt} - u_{xx} + \frac{3}{4}u - \frac{3}{2}u^3 = 0, \tag{32}$$

with initial conditions

$$u(x,0) = -\operatorname{sech} x, \qquad u_t(x,0) = \frac{1}{2}\operatorname{sech}(x)\tanh(x). \tag{33}$$

The exact solution of Eq. (32) is (see Ref. [26])

$$u(x,t) = -\operatorname{sech}\left(x + \frac{1}{2}t\right). \tag{34}$$

Table 2
The absolute error for different values of (x, t)

x	$t = 0.1$		$t = 0.3$		$t = 0.5$	
	VIM	ADM	VIM	ADM	VIM	ADM
1.0	4.809×10^{-12}	5.201×10^{-11}	3.177×10^{-8}	4.427×10^{-8}	1.904×10^{-6}	2.325×10^{-5}
2.0	2.607×10^{-13}	3.362×10^{-11}	1.651×10^{-9}	6.142×10^{-9}	9.522×10^{-8}	7.570×10^{-6}
3.0	4.985×10^{-14}	2.379×10^{-11}	3.211×10^{-10}	3.528×10^{-10}	1.877×10^{-8}	4.958×10^{-7}
4.0	2.774×10^{-15}	1.509×10^{-11}	1.788×10^{-11}	2.774×10^{-11}	1.046×10^{-9}	1.545×10^{-8}
5.0	1.292×10^{-16}	1.496×10^{-11}	8.318×10^{-13}	8.682×10^{-12}	4.862×10^{-11}	3.039×10^{-9}
6.0	2.315×10^{-18}	2.471×10^{-12}	1.741×10^{-14}	1.430×10^{-13}	8.485×10^{-13}	1.554×10^{-9}
7.0	1.403×10^{-18}	2.250×10^{-12}	9.126×10^{-15}	5.498×10^{-13}	5.383×10^{-13}	5.924×10^{-10}
8.0	6.288×10^{-19}	1.613×10^{-13}	4.081×10^{-15}	1.514×10^{-13}	2.404×10^{-13}	2.194×10^{-10}
9.0	2.369×10^{-19}	1.541×10^{-13}	1.537×10^{-15}	4.975×10^{-14}	9.055×10^{-14}	8.073×10^{-11}
10.0	8.743×10^{-20}	1.108×10^{-14}	5.674×10^{-16}	4.353×10^{-14}	3.341×10^{-14}	2.968×10^{-11}

According to He's method the following variational iteration formula in the t-direction can be obtained:

$$u_0(x,t) = u(x,0) + tu_t(x,0) = -\operatorname{sech} x + \frac{1}{2}t\operatorname{sech}(x)\tanh(x), \tag{35}$$

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\tau - t)\left\{u_{n\tau\tau}(x,\tau) - u_{nxx}(x,\tau) + \frac{3}{4}u_n(x,\tau) - \frac{3}{2}u_n^3(x,\tau)\right\}\mathrm{d}\tau. \tag{36}$$

Using the above iteration formula (36), we can directly obtain the other components as follows:

$$\begin{aligned} u_1(x,t) &= -\operatorname{sech} x + \frac{1}{2}t\operatorname{sech}(x)\tanh(x) - \frac{t^2}{8}\left(\operatorname{sech}(x) - 2\operatorname{sech}^3(x)\right) \\ &\quad + \frac{t^3}{48}\operatorname{sech}(x)\tanh(x)\left(1 - 6\operatorname{sech}^2(x)\right) + \cdots, \end{aligned} \tag{37}$$

$$\begin{aligned} u_2(x,t) &= -\operatorname{sech} x + \frac{1}{2}t\operatorname{sech}(x)\tanh(x) - \frac{t^2}{8}\left(\operatorname{sech}(x) - 2\operatorname{sech}^3(x)\right) \\ &\quad + \frac{t^3}{48}\operatorname{sech}(x)\tanh(x)\left(1 - 6\operatorname{sech}^2(x)\right) - \frac{t^4}{384}\operatorname{sech} x\left(1 - 20\operatorname{sech}^2(x) + 24\operatorname{sech}^4(x)\right) \\ &\quad + \frac{t^5}{3840}\operatorname{sech}(x)\tanh(x)\left(1 - 60\operatorname{sech}^2(x) + 120\operatorname{sech}^4(x)\right) + \cdots, \end{aligned} \tag{38}$$

$$\begin{aligned} u_3(x,t) &= -\operatorname{sech} x + \frac{1}{2}t\operatorname{sech}(x)\tanh(x) - \frac{t^2}{8}\left(\operatorname{sech}(x) - 2\operatorname{sech}^3(x)\right) \\ &\quad + \frac{t^3}{48}\operatorname{sech}(x)\tanh(x)\left(1 - 6\operatorname{sech}^2(x)\right) - \frac{t^4}{384}\operatorname{sech} x\left(1 - 20\operatorname{sech}^2(x) + 24\operatorname{sech}^4(x)\right) \\ &\quad + \frac{t^5}{3840}\operatorname{sech}(x)\tanh(x)\left(1 - 60\operatorname{sech}^2(x) + 120\operatorname{sech}^4(x)\right) \\ &\quad - \frac{t^6}{46\,080}\operatorname{sech}(x)\left(1 - 182\operatorname{sech}^2(x) + 840\operatorname{sech}^4(x) - 720\operatorname{sech}^6(x)\right) \\ &\quad + \frac{t^7}{645\,120}\operatorname{sech}(x)\tanh(x)\left(1 - 546\operatorname{sech}^2(x) + 4200\operatorname{sech}^4(x) - 5040\operatorname{sech}^6(x)\right) + \cdots, \end{aligned} \tag{39}$$

and so on. The rest of the components of the iteration formula (36) can be obtained in a similar way. The nth approximation converges to the exact solution, which has been obtained by using the hyperbolic function method by Zhao et al. [26]. In order to verify the efficiency of the proposed method in comparison with the exact solution and Adomian decomposition method [23], we report the absolute errors for different values of x and t.

The differences between the third-iteration solution of the variational iteration method and four terms of the Adomian decomposition method with the exact solution are shown in Table 2.

A very good agreement between the results from the variational iteration method and the exact solution was observed, which confirms the validity of He's variational iteration method. In comparison with the results of ADM, one can see that third iteration of the variational iteration method is more effective than four terms of ADM. The present method overcomes the difficulty arising in calculating Adomian polynomials and further the computation time is effectively reduced.

4. Conclusion

The Klein–Gordon equations have been analyzed using the variational iteration method. All the examples show that the variational iteration method is a powerful mathematical tool for solving Klein–Gordon equation. It is also a promising method for solving other nonlinear equations. This method solves the problem without any need for discretization of the variables; therefore, it is not affected by computation round off errors and one does not face the need for large computer memory and time. In our work, we made use of the Maple package to calculate the series obtained from the variational iteration method.

References

[1] M.J. Ablowitz, P.A. Clarkson, Solitons, Nonlinear Evolution Equations and Inverse Scattering Transform, Cambridge University Press, Cambridge, 1990.
[2] J.H. He, A new approach to nonlinear partial differential equations, Comm. Nonlinear Sci. Numer. Simul. 2 (1997) 230–235.
[3] J.H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods Appl. Mech. Engrg. 167 (1998) 57–68.
[4] J.H. He, Approximate solution of nonlinear differential equations with convolution product nonlinearities, Comput. Methods Appl. Mech. Engrg. 167 (1998) 69–73.
[5] J.H. He, Variational iteration method—A kind of non-linear analytical technique: Some examples, Internat. J. Non-Linear Mech. 34 (1999) 699–708.
[6] J.H. He, Variational iteration method for autonomous ordinary differential systems, Appl. Math. Comput. 114 (2000) 115–123.
[7] J.H. He, Variational principle for some nonlinear partial differential equations with variable coefficients, Chaos Solitons Fractals 19 (2004) 847–851.
[8] M.A. Abdou, A.A. Soliman, New applications of variational iteration method, Physica D 211 (2005) 1–8.
[9] M.A. Abdou, A.A. Soliman, Variational iteration method for solving Burgers' and coupled Burgers' equations, J. Comput. Appl. Math. 181 (2005) 245–251.
[10] S. Momani, S. Abuasad, Application of He's variational iteration method to Helmholtz equation, Chaos Solitons Fractals 27 (2006) 1119–1123.
[11] Z.M. Odibat, S. Momani, Application of variational iteration method to nonlinear differential equations of fractional order, Internat. J. Nonlinear Sci. Numer. Simul 7 (2006) 27–34.
[12] J.H. He, Some asymptotic methods for strongly nonlinear equations, Internat. J. Modern Phys. B 20 (2006) 1141–1199.
[13] E.M. Abulwafa, M.A. Abdou, A.A. Mahmoud, The solution of nonlinear coagulation problem with mass loss, Chaos Solitons Fractals 26 (2006) 313–330.
[14] N. Bildik, A. Konuralp, The use of variational iteration method, differential transform method and Adomian decomposition method for solving different types of nonlinear partial differential equations, Internat. J. Nonlinear Sci. Numer. Simul. 7 (2006) 65–70.
[15] R.K. Dodd, I.C. Eilbeck, J.D. Gibbon, H.C. Morris, Solitons and Nonlinear Wave Equations, Academic, London, 1982.
[16] A.Q. Khaliq, B. Abukhodair, Q. Sheng, M.S. Ismail, A predictor–corrector scheme for sine–Gordon equation, Numer. Methods Partial Differential Equations 16 (2000) 133–146.
[17] M.A.M. Lynch, Large amplitude instability in finite difference approximations to the Klein–Gordon equation, Appl. Numer. Math. 31 (1999) 173–182.
[18] X. Lu, R. Schmid, Symplectic integration of sine–Gordon type systems, Math. Comput. Simulation 50 (1999) 255–263.
[19] D. Kaya, An implementation of the ADM for generalized one-dimensional Klein–Gordon equation, Appl. Math. Comput. 166 (2005) 426–433.
[20] D. Kaya, S.M. El-Sayed, A numerical solution of the Klein—Gordon equation and convergence of the decomposition method, Appl. Math. Comput. 156 (2004) 341–353.
[21] S.M. El-Sayed, The decomposition method for studying the Klein–Gordon equation, Chaos Solitons Fractals 18 (2003) 1025–1030.
[22] A.M. Wazwaz, The modified decomposition method for analytic treatment of differential equations, Appl. Math. Comput. 173 (2006) 165–176.
[23] G. Adomian, Solving Frontier Problems of Physics: The Decomposition Methods, Kluwer Academic Publishers, Boston, 1994.
[24] A.M. Wazwaz, A reliable modification of Adomian decomposition method, Appl. Math. Comput. 102 (1997) 77–86.
[25] A.M. Wazwaz, A new algorithm for calculating Adomian polynomials for nonlinear operators, Appl. Math. Comput. 111 (2000) 53–69.
[26] Xueqin Zhao, Hongyan Zhi, Yaxuan Yu, Hongqing Zhang, A new Riccati equation expansion method with symbolic computation to construct new travelling wave solution of nonlinear differential equations, Appl. Math. Comput. 172 (2006) 24–39.