Stochastic functional differential equations with infinite delay

Shaobo Zhou\textsuperscript{a}, Zhiyong Wang\textsuperscript{b,\ast}, Dan Feng\textsuperscript{c}
\textsuperscript{a} School of Mathematics and Statistics, Huazhong University of Science and Technology, Wuhan 430074, China
\textsuperscript{b} School of Applied Mathematics, University of Electronic Science and Technology of China, Chengdu 610054, China
\textsuperscript{c} College of Computer Science, Huazhong University of Science and Technology, Wuhan 430074, China

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\textbf{ABSTRACT}

The stability and boundedness of the solution for stochastic functional differential equation with finite delay have been studied by several authors, but there is almost no work on the stability of the solutions for stochastic functional differential equations with infinite delay. The main aim of this paper is to close this gap. We establish criteria of \(p\)th moment \(\psi_{\gamma}(t)\)-bounded for neutral stochastic functional differential equations with infinite delay and exponentially stable criteria for stochastic functional differential equations with infinite delay, and we also illustrate the result with an example.

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1. Introduction and preliminary results

Stochastic differential equations (SDE in short) are well known to model from among areas of science and engineering, wherein quite often the future state of such systems depends not only on the present state but also on its past history (delay) leading to stochastic functional differential equations (SFDE in short) with delay rather than SDEs. In the recent years, there is an increasing interest in stochastic drift equation with finite delay, we here mention Li et al. \cite{1}, Mao \cite{2–6}, Shen and Liao \cite{7}, Shen et al. \cite{8,9}, Liu and Xia \cite{10}, Randjelovic and Jankovic \cite{11}. For example, Mao \cite{5} studied the stability of stochastic functional differential equations with finite delay

\[ d\left[ x(t) - u(x_t) \right] = f(x_t, t) + g(x_t, t) \, dw(t). \]

Motivated by the above works, Young et al. \cite{12}, Wei and Wang \cite{13} and Zhou and Xue \cite{14} have generalized the SDE from finite delay to infinite delay, and they have proved the existence and uniqueness of solutions to SFDE with infinite delay. However, there is no work on the boundedness and stability of the solutions for stochastic functional differential equation with infinite delay (ISFDE for short).

It is well known that the classical and powerful technique applied in the study of stability is based on a stochastic version of the Lyapunov direct method. However, as finding Lyapunov functional can be difficult when applying the above method, it is important to find some other more applicable criteria to verify the required type of stability. In the paper, we shall extend the results from papers \cite{11} and \cite{12} by Liu and Randjelovic referring to exponential stability in mean square and \(p\)th moment for NSFDE with finite delay. We establish criteria of \(p\)th moment \(\psi_{\gamma}(t)\)-bounded for neutral stochastic functional differential equations with infinite delay and exponentially stable criteria for stochastic functional differential equations with infinite delay, and we also illustrate the result with an example.

\textsuperscript{\ast} Corresponding author.
\textsuperscript{\ast\ast} Corresponding author.
Throughout this paper, unless otherwise specified, we let \((\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)\) be a complete probability space with a filtration \(\{\mathcal{F}_t\}_{t \geq 0}\), satisfying the usual conditions (i.e. it is right continuous and \(\mathcal{F}_0\) contains all \(P\)-null sets). Let \(w(t) = (w_1(t), \ldots, w_m(t))^T\) be an \(m\)-dimensional Brownian motion defined on the probability space. If \(A\) is a vector or matrix, its transpose is denoted by \(A^T\). If \(A\) is a matrix, its trace norm is denoted by \(\|A\| = \sqrt{\text{trace}(A^T A)}\), while its operator norm is denoted by \(\|A\| = \sup\{|Ax| : |x| = 1\}\) (without any confusion with \(\|\psi\|\)). Let \(L^p((-\infty, 0]; R^d)\) be the family of Borel measurable \(R^d\)-valued functions \(\psi(s) (-\infty < s \leq 0)\) with the norm \(\|\psi\|_{L^p} = (\int_{-\infty}^0 \|\psi(s)\|^p ds)^{\frac{1}{p}} < \infty\).

Let \(\mathcal{W}((\infty, 0]; R^d)\) be the family of Borel measurable bounded non-negative functions \(\eta(s) (-\infty < s \leq 0)\) such that \(\int_{-\infty}^0 \eta(s) ds = 1\) (the weighting functions). Furthermore, denote by \(\mathcal{GB}_{\mathcal{F}_0}((\infty, 0]; R^d)\) the family of continuous bounded \(R^d\)-valued stochastic process \(\xi = (\xi(s), -\infty < s \leq 0)\) such that \(\xi(s)\) is \(\mathcal{F}_0\)-measurable for all \(-\infty < s \leq 0\). Consider a \(d\)-dimensional neutral stochastic functional differential equations with infinite delay

\[
d[x(t) - u(x_t)] = [h(t, x(t)) + f(t, x_t)] dt + g(t, x_t) dw(t), \tag{1}
\]

with the initial data \(x_0 = \xi = (\xi(0)) = (\xi(\theta) : -\infty < \theta \leq 0)\) in \(\mathcal{GB}_{\mathcal{F}_0}((\infty, 0]; R^d)\), where \(x_t = \{x(t + \theta) : -\infty < \theta \leq 0\}\) which is regarded as an \(L^p((-\infty, 0]; R^d)\)-valued stochastic process, moreover

\[
u : L^p((-\infty, 0]; R^d) \rightarrow R^d, \quad h : R_+ \times R^d \rightarrow R^d, \quad f : R_+ \times L^p((-\infty, 0]; R^d) \rightarrow R^d, \quad g : R_+ \times L^p((-\infty, 0]; R^d) \rightarrow R^{d \times m}
\]

are Borel measurable. An \(\mathcal{F}_t\)-adapted process \(x(t) (x_0) : -\infty < t < +\infty\) is said to be the solution of Eq. (1) if it satisfies the initial condition and corresponding integral equation holds a.s., i.e.

\[
x(t) - u(x_t) = \xi(0) - u(x_0) + \int_0^t [h(s, x(s)) + f(s, x_s)] \, ds + \int_0^t g(s, x_s) \, dw(s) \quad \text{a.s.} \quad t \geq 0.
\]

Zhou and Xue [14] have been proved the basic existence-and-uniqueness to the solutions of Eq. (1). And we need to introduce two useful inequalities and two useful notations.

**Lemma.** (See [15].)

(i) \((x + y)^p \leq (1 + \varepsilon)^{p-1}(x^p + \varepsilon^{1-p} y^p), \varepsilon > 0\).

(ii) \((\sum x_i)^p \leq C_p \sum x_i^p, C_p = 1 \text{ when } 0 < p \leq 1, C_p = n^{p-1} \text{ when } p \geq 1\).

**Definition 1.** (See [15].) The solution of Eq. (1) is said to be \(\psi^p(t)\)-bounded in \(p\)th moment if there exists an increasing function \(\psi(t) \in C^1(-\infty, +\infty)\) (i.e. \(\psi'(t) > 0\)) such that

\[
\limsup_{t \to \infty} \frac{\ln E|x(t; \xi)|^p}{\ln \psi(t)} \leq \gamma
\]

for all \(\xi \in GB_{\mathcal{F}_0}((\infty, 0]; R^d)\) and \(p > 0, \gamma > 0\). The solution of Eq. (1) is said to be almost surely \(\psi^p(t)\)-bounded if there exists increasing function \(\psi(t)\) (i.e. \(\psi'(t) > 0\)) such that

\[
\limsup_{t \to \infty} \frac{\ln |x(t; \xi)|}{\ln \psi(t)} \leq \gamma \quad \text{a.s.}
\]

for all \(\xi \in GB_{\mathcal{F}_0}((\infty, 0]; R^d)\) and \(\gamma > 0\).

**Definition 2.** (See [5].) Eq. (1) is said to be exponentially stable in \(p\)th moment if

\[
\limsup_{t \to \infty} \frac{1}{t} \log(E|x(t; \xi)|^p) < 0
\]

for all \(\xi \in GB_{\mathcal{F}_0}((\infty, 0]; R^d)\). The equation is said to be almost surely exponentially stable if

\[
\limsup_{t \to \infty} \frac{1}{t} \log(|x(t; \xi)|) < 0 \quad \text{a.s.}
\]

for all \(\xi \in GB_{\mathcal{F}_0}((\infty, 0]; R^d)\).
2. Neutral stochastic functional differential equations with infinite delay

In this section, we shall give the criteria of $p$th moment $\psi(t)^p$-bounded for neutral stochastic functional differential equations with infinite delay (1).

**Theorem 1.** Assume that $p \geq 2$ and there exist a constant $k \in (0, 1)$, $0 < \lambda_2 < \lambda_1$ and functions $\eta_1(.)$, $\eta \in \mathcal{W}((\infty, 0]; \mathbb{R}_+)$ and a functional $u : L^p((\infty, 0]; \mathbb{R}^d) \to \mathbb{R}^d$ such that

\[
|u(\varphi)|^p \leq k \int_{-\infty}^{0} \eta(s)|\varphi(s)|^p ds, \\

p|\varphi(0) - u(\varphi)|^{p-2} \left[ (\varphi(0) - u(\varphi))^T (h(t, \varphi(0)) + f(t, \varphi)) + \frac{1}{2} |g(s, \varphi)|^2 \right] \\
+ \frac{p(p-2)}{2} |\varphi(0) - u(\varphi)|^{p-4} \left[ (\varphi(0) - u(\varphi))^T g(t, \varphi) \right] \leq -\lambda_1 |\varphi(0)|^p + \lambda_2 \int_{-\infty}^{0} \eta_1(v)|\varphi(v)|^p dv,
\]

for all $\varphi \in L^p((\infty, 0]; \mathbb{R}^d)$ and $t \geq 0$. Then the trivial solution of Eq. (1) is the $p$th moment $\psi^p(t)$ bounded, in the sense that

\[
\limsup_{t \to \infty} \frac{\ln E|x(t; \xi)|^p}{\ln \psi(t)} \leq \gamma,
\]

where $\gamma \leq \frac{\lambda_1 - \lambda_2}{k \psi_1(0) - 2^{p-1} \psi(t)}$, $0 < \psi(t) \leq \psi(t)$, $\psi''(t) > 0$, $\psi''(t) \psi(t) > \psi(t)$.

**Proof.** For the given initial data $\xi \in \mathcal{GBF}_2((\infty, 0]; \mathbb{R}^d)$, let $x(t) = x(t, \xi)$ be the solution of Eq. (1). According to Lemma, we have

\[
E|x(t)|^p \leq \frac{1}{(1 - \theta)^p - 1} E|x(t) - u(x_t)|^p + \frac{1}{\theta^p - 1} E|u(x_t)|^p,
\]

where $\theta = \theta_{1}(\xi)$. Applying the Ito formula, we may obtain

\[
E\psi^{-\gamma} |x(t) - u(x_t)|^p = E\psi^{-\gamma} |x(0) - u(x_0)|^p - E \int_{0}^{t} \gamma \psi^{-\gamma} \psi_1(s)|x(s) - u(x_s)|^p ds \\
+ pE \int_{0}^{t} \psi^{-\gamma} |x(s) - u(x_s)|^{p-2} \left[ (x(s) - u(x_s))^T (h(s, x_s) + f(s, x_s)) + \frac{1}{2} |g(s, x_s)|^2 \right] ds \\
+ \frac{p(p-2)}{2} E \int_{0}^{t} \psi^{-\gamma} |x(s) - u(x_s)|^{p-4} \left[ (x(s) - u(x_s))^T g(s, x_s) \right] ds \\
+ pE \int_{0}^{t} \psi^{-\gamma} |x(s) - u(x_s)|^{p-2} \left[ (x(s) - u(x_s))^T g(s, x_s) \right] ds \\
+ pE \int_{0}^{t} \psi^{-\gamma} |x(s) - u(x_s)|^{p-4} \left[ (x(s) - u(x_s))^T g(s, x_s) \right] ds,
\]

where $\psi_1(t) = \psi^{-1}(t)$, the last integral is equal to zero, by using of condition (3), we have

\[
E\psi^{-\gamma} |x(t) - u(x_t)|^p \leq E\psi^{-\gamma} |x(0) - u(x_0)|^p - E \int_{0}^{t} \gamma \psi^{-\gamma} \psi_1(s)|x(s) - u(x_s)|^p ds \\
- \lambda_1 E \int_{0}^{t} \psi^{-\gamma} |x(s)|^p ds + \lambda_2 E \int_{0}^{t} \psi^{-\gamma} \eta_1(v)|x_s(v)|^p dv ds \\
\leq E\psi^{-\gamma} |x(0) - u(x_0)|^p - E \int_{0}^{t} \gamma \psi^{-\gamma} \psi_1(s)|x(s) - u(x_s)|^p ds \\
- \lambda_1 E \int_{0}^{t} \psi^{-\gamma} |x(s)|^p ds + \lambda_2 E \int_{0}^{t} \psi^{-\gamma} \eta_1(v)|x_s(v)|^p dv ds.
\]
By the inequality \(|a| + |b|^p \leq 2^{p-1}|a|^p + 2^{p-1}|b|^p\), we have
\[
|x(s)|^p = |x(s) - u(x_0) + u(x_0)|^p
\leq \left( |x(s) - u(x_0)| + |u(x_0)| \right)^p
\leq 2^{p-1}|x(s) - u(x_0)|^p + 2^{p-1}|u(x_0)|^p.
\]
according to condition (2) and the above inequality, we have
\[
|x(s) - u(x_0)|^p \geq 2^{1-p}|x(s)|^p - |u(x_0)|^p
\geq 2^{1-p}|x(s)|^p - k \int_{-\infty}^{0} \eta(v)|x_1(v)|^p dv.
\]
Substitute this into (6), then
\[
E \psi(t)^{-\gamma} |x(t) - u(x_0)|^p
\leq E \psi(0)^{-\gamma} |x(0) - u(x_0)|^p - E \int_{0}^{t} \gamma \psi(s)^{-\gamma} \psi_1(s) \left[ 2^{1-p}|x(s)|^p - k \int_{-\infty}^{0} \eta(v)|x(s+v)|^p dv \right] ds
- \lambda_1 E \int_{0}^{t} \psi(s)^{-\gamma} |x(s)|^p ds + \lambda_2 E \int_{0}^{t} \psi(s)^{-\gamma} \int_{-\infty}^{0} \eta_1(v)|x(s+v)|^p dv ds
= E \psi(0)^{-\gamma} |x(0) - u(x_0)|^p + E \int_{0}^{t} \left[ -2^{1-p} \gamma \psi_1(s) - \lambda_1 \right] \psi(s)^{-\gamma} |x(s)|^p ds
+ \int_{0}^{t} k \gamma \psi(s)^{-\gamma} \int_{-\infty}^{0} \eta(v)|x(s+v)|^p dv ds + \lambda_2 E \int_{0}^{t} \psi(s)^{-\gamma} \int_{-\infty}^{0} \eta_1(v)|x(s+v)|^p dv ds.
\]
Note that \(\psi_1(s) < 1\) and \(\psi(s)^{-\gamma}\) is a decreasing function (since \(\psi'(s) > 0\)), we estimate the previous integral
\[
E \int_{0}^{t} \psi(s)^{-\gamma} \int_{-\infty}^{0} \eta(v)|x(s+v)|^p dv ds
= \int_{0}^{t} \psi(s)^{-\gamma} \int_{-\infty}^{s} \eta(v-s)|x(v)|^p dv ds
= E \int_{-\infty}^{t} \int_{0}^{s} \psi(s)^{-\gamma} \eta(v-s) ds |x(v)|^p dv
\leq E \int_{-\infty}^{t} \int_{0}^{s} \psi(s)^{-\gamma} \eta(v-s) ds |x(v)|^p dv + E \int_{0}^{t} \int_{-\infty}^{0} \psi(s)^{-\gamma} \eta(v-s) ds |x(v)|^p dv
\leq \psi(0)^{-\gamma} E \int_{-\infty}^{t} \int_{0}^{s} \eta(v-s) ds |x(v)|^p dv + E \int_{0}^{t} \psi(s)^{-\gamma} |x(v)|^p dv
\leq \psi(0)^{-\gamma} E \int_{-\infty}^{t} |x(v)|^p dv + E \int_{0}^{t} \psi(v)^{-\gamma} |x(v)|^p dv.
\]
Similarly
\[
E \int_{0}^{t} \psi(s)^{-\gamma} \int_{-\infty}^{0} \eta_1(v)|x(s+v)|^p dv ds \leq \psi(0)^{-\gamma} E \int_{-\infty}^{t} |x(v)|^p dv + E \int_{0}^{t} \psi(v)^{-\gamma} |x(v)|^p dv.
\]
Note that $\psi'(s) = \frac{\psi'(s(\psi(s)^{-\gamma}))}{\psi'(\gamma)} > 0$, that is, $\psi_1(s)$ is an increasing function and $\psi_1(s) > \psi_1(0)$, $s \in (0, t)$. Substitute (8) and (9) into (7), and by $\psi_1(t) < 1$, the result is

$$
\begin{align*}
E \psi(t)^{-\gamma} |x(t) - u(x_t)|^p & \leq E \psi(0)^{-\gamma} |x(0) - u(x_0)|^p + E \int_0^t [-21^{-p} \psi_1(s) - \lambda_1] \psi(s)^{-\gamma} |x(s)|^p \, ds \\
& \quad + k\gamma \left[ \psi(0)^{-\gamma} E \int_0^t |x(v)|^p \, dv + E \int_0^t \psi(v)^{-\gamma} |x(v)|^p \, dv \right] \\
& \quad + \lambda_2 \left[ \psi(0)^{-\gamma} E \int_0^t |x(v)|^p \, dv + E \int_0^t \psi(v)^{-\gamma} |x(v)|^p \, dv \right] \\
\leq E \psi(0)^{-\gamma} |x(0) - u(x_0)|^p + [k\gamma + \lambda_2] \psi(0)^{-\gamma} E \int_0^t |x(v)|^p \, dv \\
& \quad + E \int_0^t [-21^{-p} \psi_1(s) \gamma - \lambda_1 + k\gamma + \lambda_2] \psi(s)^{-\gamma} |x(s)|^p \, ds \\
\end{align*}
$$

$$
= c_1 + (-21^{-p} \psi_1(0) \gamma - \lambda_1 + k\gamma + \lambda_2) E \int_0^t \psi(s)^{-\gamma} |x(s)|^p \, ds.
$$

(10)

where $c_1 = E \psi(0)^{-\gamma} |x(0) - u(x_0)|^p + [k\gamma + \lambda_2] \psi(0)^{-\gamma} E \int_0^t |x(v)|^p \, dv$. Since $\lambda_1 \geq -21^{-p} \psi_1(0) \gamma + k\gamma + \lambda_2$, then $E|x(t) - u(x_t)|^p \leq c_1 \psi(t)^\gamma$, by (4), for $t \geq 0$.

$$
E \psi(t)^{-\gamma} |x(t)|^p \leq \frac{1}{(1 - \theta)^{p-1}} E \psi(t)^{-\gamma} |x(t) - u(x_t)|^p + \frac{1}{\theta^{p-1}} E \psi(t)^{-\gamma} |u(x_t)|^p
$$

$$
\leq \frac{c_1}{(1 - \theta)^{p-1}} + \frac{k\psi(t)^{-\gamma}}{\theta^{p-1}} \int_0^t \eta(v) |x_t(v)|^p \, dv
$$

$$
\leq \frac{c_1}{(1 - \theta)^{p-1}} + \frac{k\psi(t)^{-\gamma}}{\theta^{p-1}} \int_0^t \eta(t) |x(t + v)|^p \, dv
$$

$$
\leq \frac{c_1}{(1 - \theta)^{p-1}} + \frac{k\psi(t)^{-\gamma}}{\theta^{p-1}} \int_{-\infty}^t \eta(t) |x(s)|^p \, ds
$$

$$
\leq \frac{c_1}{(1 - \theta)^{p-1}} + \frac{k\psi(t)^{-\gamma}}{\theta^{p-1}} E \left[ \sup_{-\infty < s \leq t} |x(s)|^p \right]
$$

$$
\leq \frac{c_1}{(1 - \theta)^{p-1}} + \frac{k\psi(t)^{-\gamma}}{\theta^{p-1}} E \|\xi\|^p + \frac{k\psi(t)^{-\gamma}}{\theta^{p-1}} E \left[ \sup_{0 < s \leq t} |x(s)|^p \right]
$$

$$
\leq \frac{c_1}{(1 - \theta)^{p-1}} + \frac{k\psi(t)^{-\gamma}}{\theta^{p-1}} E \|\xi\|^p + \frac{k\psi(t)^{-\gamma}}{\theta^{p-1}} E \left[ \sup_{0 < s \leq t} |x(s)|^p \right].
$$

i.e.

$$
\left(1 - \frac{k}{\theta^{p-1}}\right) E \left[ \psi^{-\gamma} \sup_{0 < s \leq t} |x(s)|^p \right] \leq \frac{c_1}{(1 - \theta)^{p-1}} + \frac{k\psi(t)^{-\gamma}}{\theta^{p-1}} E \|\xi\|^p.
$$

Let $\theta^p = k < 1$, then

$$
E \left[ \sup_{0 < s \leq t} |x(s)|^p \right] \leq \left( \frac{c_1}{1 - k^p} + \frac{k^p}{1 - k^p} E \|\xi\|^p \psi(t)^{-\gamma} \right) \psi(t)^\gamma = C \psi(t)^\gamma,
$$

where $C = \frac{c_1}{1 - k^p} + \frac{k^p}{1 - k^p} E \|\xi\|^p \psi(t)^{-\gamma}$. The proof is complete. □
Corollary 1. Let condition (2) hold with \( k \in (0, 1) \) and \( \eta \in \mathcal{W}((\infty, 0]; R^+), \) let also there exist functions \( \psi(t) \) satisfying the conditions of Theorem 1, let also there exist positive constants \( l_1, l_2, l_3, l_4 \) and functions \( \eta_2(s), \eta_3(s) \in \mathcal{W}((\infty, 0]; R^+) \) such that

\[
x^T h(t, x(t)) \leq -l_1 |x(t)|^2, \quad |h(t, x(t))|^p \leq l_2 |x(t)|^2, \quad |f(t, \varphi)|^p \leq l_3 \int_{-\infty}^{0} \eta_2(s) |\varphi(s)|^p \, ds, \quad |g(t, \varphi)|^p \leq l_4 \int_{-\infty}^0 \eta_3(s) |\varphi(s)|^p \, ds,
\]

for all \( t \geq 0, x(t) \in R^d \) and \( \psi \in L^p((\infty, 0]; R^d). \) If the following condition is valid:

\[
0 \leq (p + 2)(\sqrt{k_{12}^2} + \sqrt{k_{35}^2}) + 4 \sqrt{k_{35}^2} + \sqrt{k_{12}^2} + (p - 1)k_4 + (p - 2)k_6 \left( \frac{\sqrt{k_{12}^2} + \sqrt{k_{35}^2}}{\sqrt{k_{12}}} + \frac{1 + \sqrt{k_{12}}}{\sqrt{k_{35}}} + p - 1 \right)
\]

\[
< 2pl_1 - (p + 2)(\sqrt{k_{12}^2} + \sqrt{k_{35}^2}) - (p - 2) \left( \frac{\sqrt{k_{12}^2} + \sqrt{k_{35}^2}}{\sqrt{k_{12}}} + \frac{1 + \sqrt{k_{12}}}{\sqrt{k_{35}}} + p - 1 \right)
\]

then the trivial solution of Eq. (1) is the \( p \)th moment \( \psi^T(t) \)-bounded.

Proof. We mainly check condition (3) of Theorem 1, that is

\[
W(t) = p|\psi(0) - u(\varphi)|^{p-2} \left[ |\psi(0) - u(\varphi)|^2 t^T [h(t, \psi(0)) + f(t, \varphi)] + \frac{1}{2} |g(s, \varphi)|^2 \right]
\]

\[
+ \frac{p(p-2)}{2} |\psi(0) - u(\varphi)|^{p-4} |\psi(0) - u(\varphi)|^2 g(t, \varphi)^2
\]

\[
\leq p|\psi(0) - u(\varphi)|^{p-2} \left[ |\psi(t, \psi(0)) - u(\varphi)| t^T h(t, \psi(t)) + \psi(t, \psi(0)) t^T f(t, \varphi) - u(\varphi) t^T f(t, \varphi) + \frac{1}{2} |g(t, \varphi)|^2 \right]
\]

\[
+ \frac{p(p-2)}{2} |\psi(0) - u(\varphi)|^{p-2} |g(t, \varphi)|^2
\]

\[
\leq p|\psi(0) - u(\varphi)|^{p-2} \left[ -l_1 |\psi(t)|^2 + |u(\varphi)||h(t, \psi(0))| + |\psi(0)||f(t, \varphi)| + |u(\varphi)||f(t, \varphi)| + \frac{p-1}{2} |g(t, \varphi)|^2 \right].
\]

By using the inequalities \((|a|+|b|)^p \leq 2^{p-1}(|a|^{p-2}+|b|^{p-2})\) and \(|a|+|b| \leq \sqrt{a^2+b^2}\), we have

\[
W(t) \leq 2^{p-4} \left[ |\psi(0)|^{p-2} + |u(\varphi)|^{p-2} \right] \left[ -2l_1 |\psi(0)|^2 + \sqrt{\frac{k_{12}}{k}} |\psi(0)|^2 + \sqrt{\frac{k}{k_{12}}} |f(t, \varphi)|^2 + \sqrt{\frac{1}{k_{12}}} |f(t, \varphi)|^2 + (p - 1)|g(t, \varphi)|^2 \right]
\]

\[
+ \sqrt{\frac{k_{12}}{k}} |\psi(0)|^2 + \sqrt{\frac{k}{k_{12}}} |f(t, \varphi)|^2 + \sqrt{\frac{1}{k_{12}}} |f(t, \varphi)|^2 + (p - 1)|g(t, \varphi)|^2
\]

\[
\leq 2^{p-4} \left[ -2l_1 |\psi(0)|^2 + \sqrt{\frac{k_{12}}{k}} |\psi(0)|^2 + \sqrt{\frac{k}{k_{12}}} |f(t, \varphi)|^2 + \sqrt{\frac{1}{k_{12}}} |f(t, \varphi)|^2 + (p - 1)|g(t, \varphi)|^2 \right]
\]

\[
+ \sqrt{\frac{k_{12}}{k}} |\psi(0)|^2 + \sqrt{\frac{k}{k_{12}}} |f(t, \varphi)|^2 + (p - 1)|\psi(0)|^{p-2} |g(t, \varphi)|^2
\]

\[
- 2l_1 |\psi(0)|^2 |u(\varphi)|^{p-2} + \sqrt{\frac{k_{12}}{k}} |u(\varphi)|^p + \sqrt{\frac{k}{k_{12}}} |h(t, \psi(0))|^2 |u(\varphi)|^{p-2}
\]

\[
+ \sqrt{\frac{k_{12}}{k}} |u(\varphi)|^p + \sqrt{\frac{k}{k_{12}}} |f(t, \varphi)|^2 + \sqrt{\frac{1}{k_{12}}} |f(t, \varphi)|^2 |u(\varphi)|^{p-2}
\]

\[
+ \frac{1}{k_{12}} |u(\varphi)|^p + (p - 1)|u(\varphi)|^{p-2} |g(t, \varphi)|^2.
\]

Apply Young inequality \(|a|^2|b|^{p-2} = (|a|^p)^{\frac{p}{p-2}} (|b|^{p-2})^{\frac{2}{p-2}} \leq \frac{1}{p} |a|^p + \frac{p-2}{p} |b|^{p-2}\) to estimate the terms of the form \(|a|^2|b|^{p-2}\). Then on the basis of (11)–(12) we obtain
Theorem 2. stable criteria for the equation. Wei and Wang [13] have checked its existence-and-uniqueness of the equation. In this section we will give exponentially conditions of Theorem 1 are valid, the trivial solution of Eq. (1) is the

3. Stochastic functional differential equations with infinite delay

Let \( u(x_t) = 0 \), Eq. (1) deduces to stochastic functional differential equations with infinite delay:

\[
dx(t) = (b(t, x_t) + f(x_t, t)) \, dt + g(x_t, t) \, dw(t).
\]

Wei and Wang [13] have checked its existence-and-uniqueness of the equation. In this section we will give exponentially stable criteria for the equation.

**Theorem 2.** Let there exist a strictly increasing differentiable function \( \lambda(t) \uparrow \infty \) a.s. \( t \to \infty \), satisfying \( 0 < \lambda'(t) \leq \lambda(t), \lambda''(t) < 0 \) for \( t \geq 0 \). Assume that there exist a function \( \eta(t) \in \mathcal{W}((0, \infty); R^+) \) and constants \( 0 \leq \lambda_2 < \lambda_1 \) such that

\[
W(t) \leq 2^{p-4} \left[ -2l \, t \, p + \frac{p}{p} |\varphi(0)|^p + \frac{2}{p} |u(\varphi)|^p + \sqrt{k} |\varphi(0)|^p \\
+ \frac{1}{\sqrt{k}} \left| p - 2 \right| |\varphi(0)|^p + \frac{2}{p} |f(t, \varphi)|^p \right] \\
+ \sqrt{k} |\varphi(0)|^p + \frac{1}{\sqrt{k}} \left| p - 2 \right| |\varphi(0)|^p + \frac{2}{p} |f(t, \varphi)|^p + (p - 1) \left( \frac{p - 2}{p} |\varphi(0)|^p + \frac{2}{p} |g(t, \varphi)|^p \right) \\
+ \int_{-\infty}^{0} \eta_2(s) |\varphi(s)|^p \, ds + 2^{p-2} (p - 1) l_4 \int_{-\infty}^{0} \eta_3(s) |\varphi(s)|^p \, ds \\
= a_0 |\varphi(0)|^p + \int_{-\infty}^{0} (a_1(s) + a_2 \eta_2(s) + a_3 \eta_3(s)) |\varphi(s)|^p \, ds,
\]

where

\[
a_0 = 2^{p-4} \left[ -2l \, t \, p + (p - 2) \left( \frac{\sqrt{\lambda_2} + \sqrt{\lambda_1}}{\sqrt{k}} + \frac{\sqrt{k} + 1}{\sqrt{\lambda_3}} + p - 1 \right) \right],
\]

\[
a = 2^{p-4} \left[ (p + 2) \sqrt{k} (\sqrt{\lambda_2} + \sqrt{\lambda_3}) + (p - 2) k \left( \sqrt{k} d + \sqrt{\lambda_3} + \frac{\sqrt{k} + 1}{\sqrt{\lambda_3}} + p - 1 \right) \right],
\]

\[
a_2 = 2^{p-2} \sqrt{\lambda_3} (\sqrt{k} + 1), \quad a_3 = 2^{p-2} (p - 1) l_4,
\]

\[
\eta_1 = \frac{a_1(s) + a_2 \eta_2(s) + a_3 \eta_3(s)}{a_1 + a_2 + a_3}.
\]

Condition (13) implies that \( a_0 < 0 \) and \( a + a_2 + a_3 > 0 \). Put \( \lambda_1 = -a_0, \lambda_2 = a + a_2 + a_3, \) then \( 0 \leq \lambda_2 < \lambda_1. \) Thus all the conditions of Theorem 1 are valid, the trivial solution of Eq. (1) is the \( p \)th moment \( \psi^p(t) \)-bounded. \( \square \)

3. Stochastic functional differential equations with infinite delay
\[
\begin{align*}
& p |\varphi(0)|^p - 2 \left[ \varphi(0)^T [h(t, \varphi(0)) + f(t, \varphi)] + \frac{1}{2} g(t, \varphi) \right]^2 + \frac{p(p - 2)}{2} |\varphi(0)|^{p-4} |\varphi(0)^T g(t, \varphi)|^2 \\
& \leq -\lambda_1 |\varphi(0)|^p + \lambda_2 \frac{\lambda'(t)}{\lambda(t)} \int_{-\infty}^{0} \eta_1(s)|\varphi(s)|^p \, ds \\
& \quad \text{for all } t > 0. \text{ Then the trivial solution of Eq. (14) is the } p \text{th moment exponentially stable for } \gamma = 1 \land (\lambda_1 - \lambda_2), \text{ in the sense that} \\
& \lim_{t \to \infty} \frac{\ln E|x(t; \xi)|^p}{\ln \lambda(t)} \leq -\gamma.
\end{align*}
\]
Substitute (18) into (17), the result is
\[
E e^{\gamma t} |x(t)|^p \leq E e^{\gamma t} |x(0)|^p + E \int_0^t (\gamma t - \lambda_1) e^{\gamma s} |x(s)|^p \, ds 
+ \lambda_2 \left( \mu'(0) e^{\gamma t} E \|x\|_p^p + E \int_0^t \mu'(s) e^{\gamma s} |x(s)|^p \, ds \right) 
\leq E e^{\gamma t} |x(0)|^p + \lambda_2 \mu'(0) e^{\gamma t} E \|x\|_p^p + E \int_0^t \left[ \gamma t - \lambda_1 + \lambda_2 \mu'(s) \right] e^{\gamma s} |x(s)|^p \, ds 
= c_1 + (\gamma - \lambda_1 + \lambda_2) E \int_0^t e^{\gamma s} |x(s)|^p \, ds, 
\]
(19)

note that \( \mu'(s) < 1 \) and \( \gamma < \lambda_1 - \lambda_2 \), \( E|x(t)|^p \leq c_1 e^{-\gamma t} \). \( \square \)

From the above process of the proof, we should note that, if we choose \( \lambda(t) = e^t \) as usual, the conditions of Theorem 2 cannot claim that the assertion is valid.

**Corollary 2.** Assume that there exist functions \( \lambda(t) \) satisfying the conditions of Theorem 2, let also there exist positive constants \( l_1, l_2, l_3, l_4 \) and functions \( \eta_2(s), \eta_3(s) \in \mathcal{W}((\infty, 0); R_+) \) such that
\[
x^T h(t, x(t)) \leq -l_1 |x(t)|^2, \quad |h(t, x(t))|^2 \leq l_2 |x(t)|^2, 
\]
(20)
\[
|f(t, \varphi)|^p \leq l_3 \int_{-\infty}^0 \eta_2(s) |\varphi(s)|^p \, ds, \quad |g(t, \varphi)|^p \leq l_4 \int_{-\infty}^0 \eta_3(s) |\varphi(s)|^p \, ds. 
\]
(21)

for all \( t \geq 0, x(t) \in R^d \) and \( \varphi \in L^p((\infty, 0); R^d) \). If the following condition is valid:
\[
l_1 p - \frac{p-2}{2} \sqrt{\frac{1}{l_3}} - \frac{p}{3} \sqrt{\frac{1}{l_3}} - \frac{(p-2)(p-1)}{2} > \sqrt{l_3} + (p-1)l_4 > 0, 
\]
(22)
then the trivial solution of Eq. (1) is the \( p \)th moment exponentially stable.

**Proof.** We mainly check condition (15) of Theorem 2, that is
\[
A(t) = p|\varphi(0)|^{p-2} \left[ \varphi(0)^T \left[ h(t, \varphi(0)) + f(t, \varphi) \right] + \frac{1}{2} |g(t, \varphi)|^2 \right] + \frac{p(p-2)}{2} |\varphi(0)|^{p-2} |g(t, \varphi)|^2 
\leq p|\varphi(0)|^{p-2} \left[ -l_1 |\varphi(0)|^2 + |\varphi(0)||f(t, \varphi)| + \frac{p-1}{2} |g(t, \varphi)|^2 \right]. 
\]

By using the inequalities \( |a| + |b|^p \leq 2^{p-2}(|a|^{p-2} + |b|^{p-2}) \) and \( |a||b| \leq |a|^2 + |b|^2 \), we have
\[
A(t) \leq p|\varphi(0)|^{p-2} \left[ -l_1 |\varphi(0)|^2 + \frac{1}{l_3} |f(t, \varphi)|^2 + \frac{1}{2} \sqrt{l_3} |\varphi(0)|^2 + \frac{p-1}{2} |g(t, \varphi)|^2 \right] 
\leq \frac{p}{2} \left[ -2l_1 |\varphi(0)|^p + \frac{1}{l_3} |\varphi(0)|^{p-2} |f(t, \varphi)|^2 + \frac{1}{\sqrt{l_3}} |\varphi(0)|^p + (p-1) |\varphi(0)|^{p-2} |g(t, \varphi)|^2 \right] 
\leq \frac{p}{2} \left[ -2l_1 |\varphi(0)|^p + \frac{2}{l_3} |\varphi(0)|^{p-2} |f(t, \varphi)|^p + \frac{2}{\sqrt{l_3}} |\varphi(0)|^p + (p-1) |\varphi(0)|^{p-2} |g(t, \varphi)|^p \right] 
\leq \frac{p}{2} \left[ -2l_1 |\varphi(0)|^p + \frac{p-2}{p} \frac{1}{l_3} |\varphi(0)|^p + \frac{2}{p} \sqrt{l_3} |f(t, \varphi)|^p + \frac{2}{p} |g(t, \varphi)|^p \right] 
\leq \frac{p}{2} \left[ -2l_1 |\varphi(0)|^p + \frac{2(p-2)(p-1)}{p} |\varphi(0)|^p + \frac{2(p-1)}{p} |g(t, \varphi)|^p \right] 
\]}
Similarly, by applying the Holder inequality, then

\[ k \text{ of Eqs. (1) and (14) in which } \gamma = \eta(\lambda_1 \wedge (\lambda - \lambda_2)). \]

Thus all the conditions of Theorem 2 are valid, the trivial solution of Eq. (14) is the \( p \)th moment exponentially stable for \( \gamma = 1 \wedge (\lambda_1 - \lambda_2). \]

4. Example

To examine the validity of the preceding concept and results by applying Corollary 4, we will give the following example of Eqs. (1) and (14) in which \( \omega(t) \) is a one-dimensional Brownian motion.

Example 1. Assume \( \omega(t) \) is a one-dimensional Brownian motion, and let \( x(t) = (x_1(t) \ x_2(t))^T, \ x_t(s) = (x_1^1(s) \ x_2^2(s))^T, \)

\[ u(x_t) = \int_{-\infty}^{0} e^{(\kappa + 1) s} (x_1^1(s) \ x_2^2(s))^T ds, \]

\[ h(t, x(t)) = (-a x_1^1(t) - a x_2^2(t) + e^{(\kappa + 1) t}|x_1^1(t)|)^T, \]

\[ f(t, x_t) = \int_{-\infty}^{0} e^{(k_2 + 1) s} (x_1^1(s) \ x_2^2(s))^T ds, \]

\[ g(t, x_t) = \int_{-\infty}^{0} e^{(k_3 + 1) s} (x_1^1(s) \ x_2^2(s))^T ds, \]

where \( k_2, k_3, \kappa, a = \text{const} > 0. \) It is easy to check that

\[ x(t)^T h(t, x(t)) \leq \left( a - \frac{1}{2} \right) |x(t)|^2, \quad |h(t, x(t))|^2 \leq (a + 1)^2 |x(t)|^2. \]

By applying Holder inequality, then

\[ |u(x_t)|^p \leq \left( \int_{-\infty}^{0} e^{\xi x_1^1} ds \right)^{\frac{p}{q}} \left( \int_{-\infty}^{0} |e^{\xi x_2^2}| ds \right)^{\frac{p}{q}} \]

\[ = p \left( \frac{1}{qk} \right)^{\frac{1}{q}} \int_{-\infty}^{0} |e^{\xi x_1^1}| ds \]

\[ = p \left( \frac{1}{qk} \right)^{\frac{1}{q}} \int_{-\infty}^{0} |e^{\xi x_1^2}| ds. \]

Similarly
\[ |f(t, x_t)|^p \leq \left( \int_{-\infty}^{t} e^{\frac{1}{2}qk_3} ds \right)^{\frac{p}{q}} \left( \int_{-\infty}^{t} e^{\frac{1}{2}qk_2} ds \right)^{\frac{p}{q}} \leq p \left( \frac{1}{qk_2} \right)^{\frac{p}{q}} \int_{-\infty}^{t} e^{p s} |x_t(s)|^p ds,
\]
\[ |g(t, x_t)|^p \leq \left( \int_{-\infty}^{t} e^{\frac{1}{2}qk_3} ds \right)^{\frac{p}{q}} \left( \int_{-\infty}^{t} e^{\frac{1}{2}qk_2} ds \right)^{\frac{p}{q}} \leq p \left( \frac{1}{qk_3} \right)^{\frac{p}{q}} \int_{-\infty}^{t} e^{p s} |x_t(s)|^p ds.
\]

Then \( l_1 = a - \frac{1}{2}, l_2 = (a + 1)^2, l_3 = p \left( \frac{p-1}{2qk_3} \right)^{\frac{p}{q}} = p \left( \frac{p-1}{2qk_3} \right)^{p-1}, l_4 = p \left( \frac{p-1}{p k_3} \right)^{p-1}, k = p \left( \frac{p-1}{p k_3} \right)^{p-1}, \) condition (13) becomes
\[ 2l_1 p - (p - 2) \left( \frac{\sqrt{\sqrt{k_2} + \sqrt{k_3}}}{\sqrt{k}} + \frac{\sqrt{k_1} + 1}{\sqrt{k_3}} + p - 1 \right) - (p + 2) \left( \sqrt{\sqrt{k_2} + \sqrt{k_3}} \right) \]
\[ \geq (p + 2) \sqrt{k_1} \left( \sqrt{\sqrt{k_2} + \sqrt{k_3}} \right) + (p - 2) k \left( \sqrt{\sqrt{k_2} + \sqrt{k_3}} + \frac{\sqrt{k_1} + 1}{\sqrt{k_3}} + p - 1 \right) + 4 \left( \sqrt{\sqrt{k_2} + \sqrt{k_3}} \right) + 4(p - 1) l_4.
\]

Let us specify \( p = 2, k_2 = k_3 = \kappa = 16, \) then \( k = l_3 = l_4 = \frac{1}{16}. \) If substituting these into (13), one gets
\[ 4 \left( a - \frac{1}{2} \right) + 4 \left[ \frac{1}{4} (a + 1) + \frac{1}{4} \right] \geq 4 \left[ \frac{1}{4} (a + 1) + \frac{1}{4} \times \frac{1}{4} \right] + 4 \times \frac{1}{4} \left( \frac{1}{4} + 1 \right)
\]
\[ + 4 \times \frac{1}{16}.
\]
i.e., \( a \geq 25/8. \) If substituting these into (22), one gets \( a > 5/8. \)

References