Singular *M*-Matrices and Inverse Positivity

C. D. Meyer, Jr., and M. W. Stadelmaier
Department of Mathematics
North Carolina State University
Raleigh, North Carolina 27607

Submitted by Robert Plemmons

ABSTRACT

For a matrix decomposable as $A=sI-B$, where $B\geqslant 0$, it is well known that $A^{-1}\geqslant 0$ if and only if the spectral radius $\rho(B)>s$. An extension of this result to the singular case $\rho(B)=s$ is made by replacing A^{-1} by $[A+t(I-AA^{D})]^{-1}$, where A^{D} is the Drazin generalized inverse.

1. INTRODUCTION

Let $\mathbf{R}^{n\times n}$ denote the set of real $n\times n$ matrices, and consider the subset of M-matrices which are defined as follows.

DEFINITION 1. If $A\in\mathbf{R}^{n\times n}$ can be written in the form $A=sI-B$ where $B\geqslant 0$ and $s\geqslant\rho(B)$, then A is called an M-matrix. [$\rho(\cdot)$ denotes spectral radius.]

If A is a *singular* M-matrix, then A must have a decomposition of the form $A=\rho(B)I-B$ for some $B\geqslant 0$. M-matrices form a subset of the so-called Z-matrices. They are denoted by $Z^{n\times n}$ and are those matrices for which $a_{ij}\leqslant 0$, $i\neq j$.

The set of of M-matrices, first studied by Minkowski [6] and later by Ostrowski [8], occur naturally in many different branches of applied mathematics. M-matrices are found in finite difference or finite element methods for partial differential equations, Markov chains, production and growth models in economics, and linear complementarity problems in operations research.

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Because the concept of an M-matrix has been applied to so many diverse areas, much of the terminology and notation has not been standardized. Furthermore, in spite of the fact that the study of M-matrices is relatively old, there is not yet a text which one can refer to in order to obtain the complete theory. The forthcoming text [3] by Berman and Plemmons and material contained in [9] and [11] can provide the needed background.

For nonsingular M-matrices, a central role is played by the concept of inverse positivity, which is defined below.

DEFINITION 2. A nonsingular matrix $A \in \mathbf{R}^{n \times n}$ is said to be *inverse positive* if $A^{-1} \geqslant 0$. If $A^{-1} > 0$, then A is said to be *strictly inverse positive.*

The result which links the concept of inverse positivity to *nonsingular* M-matrices is the following well-known theorem.

THEOREM A.

(i) *If A is a nonsingular M-matrix, then A is inverse positive.*

(i′) *If $A \in Z^{n \times n}$, then the converse of* (i) *is also true.*

(ii) *If A is an* irreducible *nonsingular M-matrix, then A is* strictly *inverse positive.*

(ii′) *If $A \in Z^{n \times n}$, then the converse of* (ii) *is also true.*

Although there is nothing very deep in this theorem, it has proven itself to be extremely useful in many different settings.

For *singular* matrices, the problem of linking the concept of inverse positivity to M-matrices is much more complex. The Drazin inverse [2] seems to be of value in the singular case.

DEFINITION 2. For $A \in \mathbf{R}^{n \times n}$, the *index* of A, denoted by $\operatorname{Ind}(A)$, is the smallest nonnegative integer k such that $\operatorname{Rank}(A^{k+1}) = \operatorname{Rank}(A^k)$. The *Drazin inverse* of A, denoted by A^D, is the unique solution of the equations $A^{k+1}X = A^k$, $XAX = X$, and $AX = XA$ where $k = \operatorname{Ind}(A)$.

The best extension of Theorem A which, to this time, has appeared was given by Rothblum [12] and Plemmons [9]. They made use of the Drazin generalized inverse and formulated their results in terms of matrices considered as operators restricted to certain subspaces. These results are stated as follows.

For $A \in Z^{n \times n}$ with $\operatorname{Ind}(A) = k$, *the following two statements are equivalent to saying that A is an M-matrix*: (i) *A^D restricted to $R(A^k)$ is a nonnegative operator.* (ii) *A restricted to $R(A^k)$ is monotone.* (*$R(\cdot)$ denotes the range.*)

The fact that, for singular matrices, it is necessary to consider operators obtained by the restriction of A or A^D to $R(A^k)$ can limit the usefulness of this result.

We will give an extension of Theorem A to the singular case, which deals with inverse positivity directly (i.e., in the sense of Definition 2) without having to consider the matrices involved as operators restricted to $R(A^k)$, or any other subspace.

2. STATEMENT OF MAJOR RESULTS AND A SEQUENCE OF LEMMAS

Our first two major results are stated directly below. They are extensions of Theorem A to include the case where A is singular. The extensions are accomplished by considering a special kind of perturbation.

THEOREM 1. *If A is an M-matrix, then there is a number $c>0$ such that $A+t(I-AA^D)$ is inverse positive when $t\in(0,c)$. If $A\in Z^{n\times n}$, then the converse is also true.*

THEOREM 2. *If A is an* irreducible *M-matrix, then there is a number $c>0$ such that $A+t(I-AA^D)$ is* strictly *inverse positive when $t\in(0,c)$. If $A\in Z^{n\times n}$, then the converse is also true.*

Clearly, when A is nonsingular, these reduce to the statements given in Theorem A, so that they are indeed extensions of that theorem to the singular case.

The proofs of Theorems 1 and 2 are somewhat involved. In order to facilitate reading the proof, we will develop it through a sequence of lemmas. (Some are of interest in their own right.)

LEMMA 1. *If A is an irreducible singular M-matrix, then*

$$\operatorname{Ind}(A)=1 \tag{1.1}$$

and

$$I-AA^D>0. \tag{1.2}$$

Proof. The hypothesis guarantees that A can be written as $A=\rho(B)I-B$, where B is nonnegative and irreducible. By applying the Perron-Frobenius theorem to B, one obtains that 0 is a simple eigenvalue for A. Thus,

$$\mathrm{Ind}(A)=1=\mathrm{Dim}[N(A)].$$

Furthermore, the Perron-Frobenius theorem guarantees that there exist positive right and left eigenvectors for B corresponding to $\rho(B)$. That is, there exist $x\in\mathbf{R}^{n\times 1}$ and $y^T\in\mathbf{R}^{1\times n}$ such that

$$\begin{aligned} Ax=0 \quad &\text{and} \quad x>0,\\ y^TA=0 \quad &\text{and} \quad y^T>0. \end{aligned} \tag{1.3}$$

Moreover, x and y^T in (1.3) can be chosen so that

$$y^Tx=1.$$

Let P denote the matrix $P=xy^T$. Clearly,

$$P^2=P>0 \quad \text{and} \quad \mathrm{Rank}(P)=1.$$

Furthermore,

$$R(P)=N(A) \quad \text{and} \quad N(P)=R(A).$$

The first equality follows from the fact that $\{x\}$ is a basis for both $R(P)$ and $N(A)$. The second equality follows because $R(A)\subseteq N(P)$ and

$$\mathrm{Dim}[N(P)]=n-1=n-\mathrm{Dim}[N(A)]=\mathrm{Dim}[R(A)].$$

We have now demonstrated that $P>0$ and P is the projector onto $N(A)$ along $R(A)$. Since $\mathrm{Ind}(A)=1$, it must be the case that

$$I-AA^D=P>0. \qquad \blacksquare$$

Equation (1.1) also appears in the work of Plemmons [10] but was originally due to Schneider [13].

In what follows, we fill find it convenient to use some special notation, which we will now define.

NOTATION. For a matrix valued function $F(t)=[f_{ij}(t)]$ the notation

$$\lim F(t)>0$$

will mean that there exists numbers $c>0$ and $\delta>0$ such that $f_{ij}(t)>\delta$, for all i and j, whenever $t\in(0,c)$.

Notice that the expression $\lim F(t)>0$ can be meaningful regardless of whether or not the actual limit $\lim_{t\to 0^+} F(t)$ exists. In the case when $\lim_{t\to 0^+} F(t)$ actually exists, then the statements $\lim F(t)>0$ and $\lim_{t\to 0^+} F(t)>0$ are equivalent. [Basically, $\lim F(t)>0$ means that the entries of $F(t)$ are strictly positive for small $t>0$ and no entry of $F(t)$ approaches 0 as $t\to 0^+$.] Clearly, if $F_1(t)\geqslant F_2(t)$ on some interval $(0,p)$, then $\lim F_2(t)>0$ implies $\lim F_1(t)>0$.

LEMMA 2. *If A is an irreducible M-matrix, then*

$$(A+tI)^{-1}>0 \qquad \text{for} \quad t>0, \tag{2.1}$$

$$\lim t^k(A+tI)^{-1}>0 \qquad \text{when} \quad 0\leqslant k\leqslant \operatorname{Ind}(A), \tag{2.2}$$

and

$$\lim_{t\to 0^+} t^k(A+tI)^{-1}=0 \qquad \text{when} \quad k>\operatorname{Ind}(A). \tag{2.3}$$

Proof. If A is an M-matrix, then $A+tI$ is a nonsingular M-matrix for each $t>0$. For this lemma, A is also irreducible. From Theorem A, we know that $A+tI$ must be strictly inverse positive. Thus, (2.1) is established.

When A is nonsingular, $\operatorname{Ind}(A)=0$, and the statements (2.2) and (2.3) are evident. When A is singular, we know from Lemma 1 that $\operatorname{Ind}(A)=1$. Therefore, (2.2) corresponds to the cases when $k=0$ or 1, and (2.3) corresponds to the case when $k\geqslant 2$. We will make use of a result of Meyer [5, Corollary 3.1] which states that

$$\lim_{t\to 0} t^k(A+tI)^{-1}=\begin{cases} I-AA^D & \text{when} \quad k=\operatorname{Ind}(A)=1, \\ 0 & \text{when} \quad k>\operatorname{Ind}(A). \end{cases} \tag{2.4}$$

Now,

For $k=1$,

$$\lim_{t\to 0^+} t^k(A+tI)^{-1} = \lim_{t\to 0^+} t(A+tI)^{-1} = I - AA^D > 0$$

by (2.4) and Lemma 1.

For $k=0$, this follows from the previous case (by dividing by t). Thus, (2.2) is established.

For $k \geqslant 2$,

$$\lim_{t\to 0^+} t^k(A+tI)^{-1} = 0$$

by (2.4) and (1.1). Thus, (2.3) is established. ■

Let A be an M-matrix. A is either irreducible or reducible. If A is reducible, one can permute rows and corresponding columns to write A as

$$A = P^T \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ 0 & A_{22} & \cdots & A_{2m} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & A_{mm} \end{bmatrix} P,$$

where each A_{ii} is either irreducible or zero, and $A_{ij} \leqslant 0$, $i \neq j$. The block triangular matrix given above is often called the *standard form for* A. Whenever we are dealing with reducible matrices, we will assume A is in standard form and disregard the permutation matrix P, since all of the following results are independent of the permutation P. If A is irreducible, then A is already in standard form (i.e., no permutations are necessary and there is only a single block, namely $A_{11} = A$).

LEMMA 3. *Let A be any real matrix in standard form, and let $c > 0$ be a number such that $(A+tI)^{-1}$ exists when $t \in (0,c)$. If $B(t)$ is the block diagonal matrix*

$$B(t) = \begin{bmatrix} (A_{11}+tI)^{-1} & 0 & \cdots & 0 \\ 0 & (A_{22}+tI)^{-1} & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & (A_{mm}+tI)^{-1} \end{bmatrix}$$

and C is the nilpotent matrix

$$C=\begin{bmatrix} 0 & -A_{12} & \cdots & -A_{1m} \\ 0 & 0 & \ddots & \vdots \\ \vdots & & \ddots & -A_{m-1,m} \\ 0 & \cdots & 0 & 0 \end{bmatrix},$$

then

$$(A+tI)^{-1}=\sum_{k=0}^{m-1}\left[B(t)C\right]^k B(t) \qquad \text{on } (0,c). \tag{3.1}$$

If A is an M-matrix, then

$$C\geqslant 0 \quad \text{and} \quad B(t)\geqslant 0 \text{ on } (0,c), \tag{3.2}$$

$$(A+tI)^{-1}B^{-1}(t)\geqslant 0 \quad \text{and} \quad (A+tI)^{-1}\geqslant 0 \qquad \text{on } (0,c), \tag{3.3}$$

$$(A+tI)^{-1}\geqslant \frac{1}{m}(A+tI)^{-1}B^{-1}(t)(A+tI)^{-1} \qquad \text{on } (0,c). \tag{3.4}$$

Proof. Let us write B in place of $B(t)$ on $(0,c)$. To prove (3.1), use the fact that BC is nilpotent to write

$$\sum_{k=0}^{m-1}(BC)^k B=(I-BC)^{-1}B=(B^{-1}-C)^{-1}=(A+tI)^{-1}.$$

To prove (3.2), simply notice $C\geqslant 0$ because A is an M-matrix. Next, observe that A_{ii} is an irreducible M-matrix and apply (2.1). To prove (3.3), write

$$(A+tI)^{-1}B^{-1}=\sum_{k=0}^{m-1}(BC)^k\geqslant 0, \qquad \text{since} \quad B\geqslant 0 \text{ and } C\geqslant 0.$$

Furthermore $(A+tI)^{-1}\geqslant 0$, since $(A+tI)^{-1}=[(A+tI)^{-1}B^{-1}]B$. Finally, (3.4) is obtained by writing

$$\begin{aligned}(A+tI)^{-1}B^{-1}(A+tI)^{-1}&=\left[I+BC+\cdots+(BC)^{m-1}\right]^2B\\ &=\left[I+2(BC)+\cdots+m(BC)^{m-1}\right]B \qquad \text{since} \quad (BC)^m=0\\ &\leqslant m\left[I+BC+\cdots+(BC)^{m-1}\right]B \qquad \text{since} \quad B,C\geqslant 0\\ &=m(A+tI)^{-1}. \qquad \blacksquare\end{aligned}$$

When A is in standard form A^D, $A+tI$, and A^k are block triangular matrices. We use the notation A_{ij}, $(A^D)_{ij}$, $(A+tI)_{ij}$, and $(A^k)_{ij}$ to represent the matrices which appears as the (i,j)-block in A, A^D, $(A+tI)$, and A^k, respectively.

LEMMA 4. *Let A be an M-matrix in standard form, and let $r(i)=$ $\operatorname{Ind}(A_{ii})$. If $(A^D)_{ij}\neq 0$, then*

$$\lim t^{r(i)}\left[(A+tI)^{-1}\right]_{ij}>0.$$

Proof. It is well known that A^D is a polynomial in A (see [2]). By using this fact one can see that $(A^D)_{ij}\neq 0$ implies that $(A^q)_{ij}\neq 0$ for some $q\leqslant n$. Therefore, by direct expansion of A^q, one can find a set of indices $S=\{i=h_0<h_1<\cdots<h_p=j\}$ such that $\prod_{i=1}^{q}A_{u_{i-1}u_i}\neq 0$, where $u_{i-1}\leqslant u_i$ and $\{u_i\}_{i=1}^{q}=S$. This implies that no factor of this product is a zero matrix, so that if C is as defined in Lemma 3, then

$$C_{h_{l-1}h_l}\geqslant 0 \text{ and } C_{h_{l-1}h_l}\neq 0 \qquad \text{for} \quad 1\leqslant l\leqslant p. \tag{4.1}$$

From (2.2) we know that

$$\lim\left(A_{ff}+tI\right)^{-1}>0 \qquad \text{for each } f,$$

$$\lim t^{r(i)}\left(A_{ii}+tI\right)^{-1}>0. \tag{4.2}$$

Let $B=B(t)$ and C be as in Lemma 3, and write

$$F_1(t)=t^{r(i)}\left[(A+tI)^{-1}\right]_{ij}=t^{r(i)}\left[\sum_{k=0}^{m-1}(BC)^kB\right]_{ij} \qquad \text{by (3.1)}$$

$$=t^{r(i)}\left[\sum_{k=0}^{n}(BC)^kB\right]_{ij} \qquad \text{since } (BC)^m=(BC)^{m+1}=\cdots=0$$

$$\geqslant t^{r(i)}\left[(BC)^pB\right]_{ij} \qquad \text{by (3.2)}$$

$$\geqslant t^{r(i)}\left(B_{ii}C_{ih_1}B_{h_1h_1}C_{h_1h_2}\cdots B_{h_{p-1}h_{p-1}}C_{h_{p-1}j}\right)B_{jj}$$

by just expanding $(BC)^pB$

$$=\left[t^{r(i)}\left(A_{ii}+tI\right)^{-1}C_{ih_1}\right]\left[\left(A_{h_1h_1}+tI\right)^{-1}C_{h_1h_2}\right]$$

$$\cdots\left[\left(A_{h_{p-1}h_{p-1}}+tI\right)^{-1}C_{h_{p-1}j}\right]\left[\left(A_{jj}+tI\right)^{-1}\right].$$

Call this product $F_2(t)$. By using (4.1) together with (4.2) one obtains the fact that $\lim F_2(t) > 0$. Since $F_1(t) \geqslant F_2(t)$ for $t > 0$, it follows that $\lim F_1(t) > 0$, and the lemma is proven. ∎

LEMMA 5. *For every M-matrix A, there exists a number $c > 0$ such that*

$$(A+tI)^{-1}(I+tA^D) \geqslant 0 \qquad \text{when} \quad t \in (0,c).$$

Proof. Assume A is in standard form, and let $B = B(t)$ be as defined in Lemma 3. Write

$$\begin{aligned}(A+tI)^{-1}(I+tA^D) &= (A+tI)^{-1} + t(A+tI)^{-1}A^D \\ &\geqslant \frac{1}{m}(A+tI)^{-1}B^{-1}(A+tI)^{-1} \\ &\quad + t(A+tI)^{-1}A^D \qquad \text{by (3.4)} \\ &= (A+tI)^{-1}B^{-1}\left[\frac{1}{m}(A+tI)^{-1} + tBA^D\right]. \qquad (5.1)\end{aligned}$$

From (3.3), we know that $(A+tI)^{-1}B^{-1} \geqslant 0$, so that we need only to prove that $[(1/m)(A+tI)^{-1} + tBA^D] \geqslant 0$. In order to do this, we will show each block of this matrix is nonnegative (i.e., we show $[(1/m)(A+tI)^{-1} + tBA^D]_{ij} \geqslant 0$). Since B is block diagonal, one can write

$$[tBA^D]_{ij} = t(A_{ii}+tI)^{-1}[A^D]_{ij},$$

so that

$$\begin{aligned}&\left[\frac{1}{m}(A+tI)^{-1} + tBA^D\right]_{ij} \\ &= \frac{1}{m}\left[(A+tI)^{-1}\right]_{ij} + t(A_{ii}+tI)^{-1}[A^D]_{ij}. \qquad (5.2)\end{aligned}$$

By using this together with (3.3), it is clear that

$$\left[\frac{1}{m}(A+tI)^{-1} + tBA^D\right]_{ij} = \frac{1}{m}\left[(A+tI)^{-1}\right]_{ij} \geqslant 0 \qquad \text{when} \quad [A^D]_{ij} = 0. \tag{5.3}$$

If $[A^D]_{ij} \neq 0$, use (5.2) and consider the following expression:

$$t^{r(i)}\left[\frac{1}{m}(A+tI)^{-1}+tBA^D\right]_{ij}$$

$$=\frac{1}{m}t^{r(i)}\left[(A+tI)^{-1}\right]_{ij}+t^{r(i)+1}(A_{ii}+tI)^{-1}\left[A^D\right]. \tag{5.4}$$

As $t \to 0^+$, (2.3) guarantees that

$$t^{r(i)+1}(A_{ii}+tI)^{-1}\left[A^D\right]_{ij} \to 0,$$

and Lemma 4 guarantees that

$$\lim \frac{1}{m}t^{r(i)}\left[(A+tI)^{-1}\right]_{ij} > 0.$$

Therefore, for sufficiently small $t>0$, the expression in (5.4) has strictly positive entries. Since $t>0$, it therefore must also be the case that, for sufficiently small t,

$$\left[\frac{1}{m}(A+tI)^{-1}+tBA^D\right]_{ij} > 0 \qquad \text{when} \quad \left[A^D\right]_{ij} \neq 0. \tag{5.5}$$

By combining (5.3) with (5.5), we have demonstrated that

$$\left[\frac{1}{m}(A+tI)^{-1}+BA^D\right] \geqslant 0,$$

so that the desired result now follows from (5.1). If A is not in standard form, then there exists a permutation P such that $A=P^TA_sP$, where A_s is in standard form.

It is a very simple computation to show that

$$(A_s+tI)^{-1}(I+tA_s^D) \geqslant 0 \quad \text{implies} \quad (A+tI)^{-1}(I+tA^D) \geqslant 0. \qquad \blacksquare$$

LEMMA 6. *If A is an M-matrix, then for every $t>0$, the expression*

$$G(t)=(A+tI)^{-1}(I+tA^D)=A^D+(A+tI)^{-1}(I-AA^D)$$

is nonsingular. Furthermore, there exists a number $c>0$ *such that* $G(t)\geqslant 0$ *when* $t\in(0,c)$.

Proof. The proof is by Lemma 5 and some direct calculation. The details are omitted.

3. PROOF OF MAJOR RESULTS

We can now very easily tie Lemmas 1–6 together and construct a proof of Theorems 1 and 2, along with some other interesting results concerning M-matrices.

Proof of Theorem 1. Assume A is an M-matrix. By direct computation, it is easy to verify that

$$[A+t(I-AA^D)]^{-1}=A^D+(A+tI)^{-1}(I-AA^D).$$

By Lemma 6, it follows that there is a number $c>0$ such that $A+t(I-AA^D)$ is inverse positive when $t\in(0,c)$. Now assume that $A\in Z^{n\times n}$ and that $A+t(I-AA^D)$ is inverse positive on some interval $(0,c)$. Let $s\geqslant \max a_{ii}$, and let $B=sI-A$. Since $A\in Z^{n\times n}$, it is clear that $B\geqslant 0$, so that the Perron-Frobenius theorem can be used to guarantee that there exists a vector $x\geqslant 0$ such that $Bx=\rho x$, where $\rho=\rho(B)$. Thus, $Ax=(s-\rho)x$, which in turn implies that $x\in R(A^k)$ where $k=\operatorname{Ind}(A)$. It now follows that

$$[A+t(I-AA^D)]x=Ax=(s-\rho)x \qquad \text{for all } t,$$

so that

$$(s-\rho)[A+t(I-AA^D)]^{-1}x=x\geqslant 0 \qquad \text{for} \quad t\in(0,c).$$

Since $[A+t(I-AA^D)]^{-1}\geqslant 0$ on $(0,c)$ and $x\geqslant 0$, it must be the case that $s-\rho\geqslant 0$ or $s\geqslant\rho$. We have shown that A can be written as $A=sI-B$ where $s\geqslant\rho(B)$, and therefore, by Definition 1, A must be an M-matrix. ∎

Proof of Theorem 2. Suppose first that A is an irreducible M-matrix. If A is nonsingular, then the desired result reduces to the well-known fact that $A^{-1}>0$. If A is a singular M-matrix, then Lemma 1 guarantees that

$\mathrm{Ind}(A)=1$ and $I-AA^D>0$. For this case, it is easy to verify that

$$[A+t(I-AA^D)]^{-1}=A^D+\frac{1}{t}(I-AA^D) \qquad \text{for all} \quad t\neq 0.$$

Clearly, since $I-AA^D>0$, one can find a sufficiently small $c>0$ so that $A^D+(1/t)(I-AA^D)>0$ when $t\in(0,c)$. Conversely, if $A\in Z^{n\times n}$ and $A+t(I-AA^D)$ is *strictly* inverse positive, then Theorem 1 guarantees that A is an m-matrix. To prove that A is also irreducible, assume otherwise. That is, assume there is a permutation P such that A can be written as

$$A=P^T\left[\begin{array}{c|c} A_1 & A_2 \\ \hline 0 & A_3 \end{array}\right]P.$$

It is well known (and easily seen) that A^D, $I-AA^D$, and $(A+tI)^{-1}$ can be written in a similar form. Since (as in the proof of Theorem 1)

$$[A+t(I-AA^D)]^{-1}=A^D+(A+tI)^{-1}(I-AA^D),$$

it is evident that the reducibility of A will contradict the *strict* inverse positivity of $A+t(I-AA^D)$. ∎

COROLLARY. *If $A\in Z^{n\times n}$, then there exists a number $c>0$ such that*

$$[A+t(I-AA^D)]^{-1}\geqslant 0 \quad\Leftrightarrow\quad (A+tI)^{-1}\geqslant 0$$

for $t\in(0,c)$. If $A\in Z^{n\times n}$, then there exists a $c>0$ such that

$$[A+t(I-AA^D)]^{-1}>0 \quad\Leftrightarrow\quad (A+tI)^{-1}>0$$

for $t\in(0,c)$.

Proof. Suppose first there is a $c>0$ such that $[A+t(I-AA^D)]^{-1}\geqslant 0$ when $t\in(0,c)$. By Theorem 1, this is equivalent to saying that A is an M-matrix, which in turn is equivalent to saying $(A+tI)^{-1}\geqslant 0$ for all $t>0$. (See [9] or [3].) Conversely, suppose $(A+tI)^{-1}\geqslant 0$ on some interval $(0,c_1)$. Since $A+tI\in Z^{n\times n}$ and is inverse positive, it follows that $A+tI$ is an M-matrix for all $t\in(0,c_1)$. It is easy to show that this implies that A is an M-matrix, so that Theorem 1 can be used to arrive at the desired result. The proof of the second part is similar. ∎

4. GENERALIZED INVERSE POSITIVITY

If A is a nonsingular M-matrix, then $A^{-1} \geqslant 0$. However, if A is a singular matrix, then analogous statements concerning the nonnegativity of the common generalized inverses do not hold. That is, one can give examples of singular M-matrices where the Moore-Penrose inverse, the inverses defined by a subset of the Penrose equations, and even the Drazin inverse are not nonnegative matrices. However, if one examines the matrix

$$G(t) = [A + t(I - AA^D)]^{-1} = A^D + (A + tI)^{-1}(I - AA^D)$$
$$= (A + tI)^{-1}(I + tA^D)$$

of Theorem 1 and Lemma 6, it is evident that for a singular M-matrix A, $G(t)$ is a "weak" Drazin inverse for A which is nonnegative. It is a "weak" Drazin inverse in the sense that it satisfies two of the three defining conditions for A^D. Namely, $G(t)$ satisfies

$$A^{k+1}G(t) = A^k, \qquad k \geqslant \operatorname{Ind}(A)$$

and

$$AG(t) = G(t)A.$$

However, it is easy to give examples to show $G(t)AG(t) \neq G(t)$. "Weak" Drazin inverses have been studied previously in [4], where their elementary properties and some of their applications have been given.

We will exploit this concept further, but before we proceed we make the following definition.

Definition 4. Let $A \in \mathbf{R}^{n \times n}$. If $X \in \mathbf{R}^{n \times n}$ is a matrix such that

$$XA^{k+1} = A^k \qquad \text{for every} \quad k \geqslant \operatorname{Ind}(A),$$

then X is called a *left weak inverse for* A. Similarly, if $X \in \mathbf{R}^{n \times n}$ is a matrix such that

$$A^{k+1}X = A^k \qquad \text{for every} \quad k \geqslant \operatorname{Ind}(A),$$

then X is called a *right weak inverse for* A. If X is both a left and right weak inverse, then X is simply said to be a *weak inverse for* A.

Notice that the commutivity condition in the definition of the Drazin inverse was not required in the definition of weak inverses. Indeed, it is possible for a matrix X to be a weak inverse for A but $AX \neq XA$. (It is easy to construct such examples.) Imposing the extra condition $AX = XA$ on any of the matrices of Definition 4 would, in general, constitute a rather severe restriction.

In the above terminology, the class of matrices defined by $G(t) = [A + t(I - AA^D)]^{-1}$ represents a class of nonnegative, nonsingular, commuting weak inverses for A, provided that A is an M-matrix.

THEOREM 3. *Let $A \in Z^{n \times n}$ with* $\operatorname{Ind}(A) = k$. *The following statements are equivalent.*

(1) *A is an M-matrix.*
(2) *There exists a nonnegative, nonsingular, commuting weak inverse for A. (Namely, $G = [A + t(I - AA^D)]^{-1} = A^D + (A + tI)^{-1}(I - AA^D)$.)*
(3) *There exists a nonnegative weak inverse for A.*
(4) *There exists a nonnegative* left *weak inverse for A.*
(4′) *A^D is nonnegative on $R(A^k)$. ($x \in R(A^k)$ and $x \geqslant 0 \Rightarrow A^D x \geqslant 0$.)*
(5) *There exists a nonnegative* right *weak inverse for A.*

Proof. We first show the chain of implications

$$(1) \Rightarrow (2) \Rightarrow (3) \begin{cases} \Rightarrow (4) \Rightarrow (4') \\ \Rightarrow (5) \end{cases}$$

holds for arbitrary $A \in R^{n \times n}$. The implication (1)⇒(2) holds because, by Lemma 6, there is a number $c > 0$ such that if $t \in (0, c)$, then

$$\left[A + t(I - AA^D)\right]^{-1} = A^D + (A + tI)^{-1}(I - AA^D)$$

is a nonnegative, nonsingular, commuting weak inverse for A. To prove that (4)⇒(4′), assume W is a matrix such that $W \geqslant 0$ and $WA^{k+1} = A^k$. Then

$$WA^k = WA^{k+1}A^D = A^k A^D = A^D A^k.$$

If $x \geqslant 0$ and $x \in R(A^k)$, write $x = A^k y$, $y \in R^n$, so that

$$A^D x = A^D A^k y = WA^k y = Wx \geqslant 0.$$

The remaining implications are obvious.

Now let $A \in Z^{n \times n}$. (4′)⇒(1) was proven by Rothblum [12]. To complete the chain we observe that (5)⇒(1) can be proven similarly. ■

The fact that (4′)⇒(4) for arbitrary matrices has recently been proven by Neumann and Plemmons [7], using a well-known consistency theorem of Farkas [1]. With appropriate substitutions, this theorem states that the following are equivalent:

(a) $\exists X \geqslant 0$ such that $XA = B = CA$.
(b) $Ay \geqslant 0 \Rightarrow (CA)y = By \geqslant 0$.

Condition (b) states that C is nonnegative on $R(A)$. By replacing A with A^k and C with A^D, it can be seen that (4) and (4′) are equivalent.

The question arises to what extent the assumption that $A \in Z^{n \times n}$ is necessary. Since the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 1 & 1 \end{bmatrix}$$

satisfies condition (4) but not (5), it is clear that, in general, (4)$\not\Rightarrow$(3). We do not know whether (3)⇒(2). Finally, (2)$\not\Rightarrow$(1), in general, since the matrix

$$N = \begin{bmatrix} 0 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

is not an M-matrix.

5. COMPLEMENTARY PERTURBATIONS

By examing the expression $A + t(I - AA^D)$, one readily sees that what has been done is to perturb the matrix A by a "complementary" matrix in order to produce a nonsingular matrix. If A is an M-matrix, then this complementary perturbation is inverse positive.

Below, we show that one can perform slightly more general perturbations than the one described above and still retain the property of inverse positivity in the case of M-matrices. First, we formulate a definition.

DEFINITION 5. Let $A \in \mathbf{R}^{n \times n}$ have $\mathrm{Ind}(A) = k$. For a number $c > 0$ and for $E \in \mathbf{R}^{n \times n}$, the matrix valued function

$$S(t) = A + tE, \qquad t \in (0, c),$$

is called a *complementary perturbation of* A when

(i) $S(t)$ is nonsingular on $(0, c)$ and
(ii) $A^D E = E A^D \geqslant 0$.

When (i) holds but E only satisfies the condition that $A^D E \geqslant 0$, then $S(t)$ is called a *right* complementary perturbation of A. Similarly, when (i) holds and $EA^D \geqslant 0$, then $S(t)$ is called a *left* complementary perturbation of A.

When A is singular, this is a special case of what is commonly known as a *singular perturbation* [since $\lim_{t \to 0} S(t)$ is singular], which is used in the study of singular systems of algebraic as well as differential equations. In these applications, it is sometimes desirable to perturb a singular coefficient matrix by a "complementary" matrix so that the perturbed system is nonsingular on some interval $(0, c)$. Questions concerning the behavior of the perturbed system as $t \to 0$ are of fundamental importance. We will show there is a definite relationship between singular M-matrices and complementary singular perturbations.

THEOREM 4. *For* $A \in Z^{n \times n}$ *with* $k = \mathrm{Ind}(A)$, *the following statements are equivalent.*

(1) *A is an M-matrix.*

(2) *There exists a complementary perturbation of A which is inverse positive.*

(3) *There exists a* left *complementary perturbation of A which is inverse positive.*

(4) *There exists a* right *complementary perturbation of A which is inverse positive.*

Proof. We show that

$$(1) \Rightarrow (2) \left\{ \begin{array}{c} \Rightarrow (3) \Rightarrow \\ \Rightarrow (4) \Rightarrow \end{array} \right\} (1).$$

First, (1) implies $[A + t(I - AA^D)]^{-1} \geqslant 0$, by Lemma 6, and hence (2) is established. (2)$\Rightarrow$(3) and (2)$\Rightarrow$(4) are trivial. To prove (3)$\Rightarrow$(1), one must first observe that if E is a matrix such that $(A+tE)^{-1}$ exists when $t\in(0,c)$, then

$$A^D+(A+tE)^{-1}(I-AA^D)=(A+tE)^{-1}(I+tEA^D) \qquad \text{when} \quad t\in(0,c).$$

This can be verified by multiplying both sides by $A+tE$. If $A+tE$ is a left complementary perturbation which is inverse positive, then the above equality implies that

$$A^D+(A+tE)^{-1}(I-AA^D)\geqslant 0 \qquad \text{when} \quad t\in(0,c).$$

Since $A^D+(A+tE)^{-1}(I-AA^D)$ is also a left weak inverse for A, Theorem 3 guarantees that A is an M-matrix. The proof of (4)⇒(1) is similar. ■

6. AN OPEN QUESTION AND CONCLUDING REMARKS

For every M-matrix A, the matrix $A+t(I-AA^D)$ is inverse positive on some interval $(0,c)$. However, for a general M-matrix A, an expression for the maximal value for c is not known. For the special case when $\mathrm{Ind}(A)=1$ the problem of finding the maximal c is easy to solve, because $[A+t(I-AA^D)]^{-1}=A^D+(1/t)(I-AA^D)$ for all t. Therefore the maximal c is the largest t for which $(I-AA^D)\geqslant -tA^D$. When $\mathrm{Ind}(A)>1$, the problem seems more difficult.

REFERENCES

1 A. Ben-Israel, Linear equations and inequalities on finite dimensional, real or complex, vector spaces: a unified theory, *J. Math. Anal. Appl.* **27** (1969), 367–389.
2 A. Ben-Israel and T. N. E. Greville, *Generalized Inverses—Theory and Applications*, Wiley-Interscience, New York, 1973.
3 A. Berman and R. J. Plemmons, *Nonnegative Matrices in the Mathematical Sciences*, Academic, New York, to be published.
4 S. L. Campbell and C. D. Meyer, Jr., Weak Drazin inverses, *Linear Algebra and Appl.*, to be published.
5 C. D. Meyer Jr., Limits and the index of a square matrix, *SIAM J. Appl. Math.* **26** (1974), 469–478.
6 H. Minkowski, Zur Theorie der Einkerten in der Algebraischen Zahlkörpern, *Nachr. Königlichen Gesellschaft Wiss. Göttingen, Math.-Phys. Kl.* 1900, 90–93.

7 M. Neumann and R. J. Plemmons, Generalized inverse-positivity and splittings of M-matrices, to be published.
8 A. Ostrowski, Über die Determinanten mit Überwiegender Hauptdiagonale, *Comment. Math. Helv.* **10** (1937), 69–96.
9 R. J. Plemmons, A survey of the theory and applications of M-matrices, Tech. Rep. CS-76-15, Computer Science Department, Univ. of Tennessee, Knoxville, Tenn.
10 R. J. Plemmons, M-matrices leading to semi-convergent splittings, *Linear Algebra and Appl.* **15** (1976), 243–252.
11 R. J. Plemmons, M-matrix characterizations—I. Nonsingular M-matrices, *Linear Algebra and Appl.* **18** (1977), 185–188.
12 U. G. Rothblum, An Index classification of M-matrices, Tech. Rep. No. 82, School of Organization and Management, Yale Univ., New Haven, Conn.; *Linear Algebra and Appl.*, to be published.
13 H. Schneider, The elementary divisors associated with 0 and a singular M-matrix, *Proc. Edinburgh Math. Soc.* **10** (1956), 108–122.

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