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NONLINEAR TRANSPORT IN MOVING FLUIDS

G. ADOMIAN
General Analytics Corp.
155 Clyde Road, Athens, GA 30605, U.S.A.
R. MEYERS
U.S. Army Research Laboratory, White Sands Missile Range 88002

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Abstract—The time-dependent spread of contaminants in moving fluids is normally studied by computer-intensive discretized procedures which have some disadvantages. Application of the decomposition method allows a continuous, convenient, accurate procedure which works and extends to nonlinear and stochastic partial differential equations as well.

ONE-DIMENSIONAL ADVECTION EQUATION

The simplest case is modelled by

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{\alpha \partial u}{\partial x} &= 0, && 0 < t \le T, \\ u(x,0) &= f(x), && 0 \le x \le 1, \\ u(x,t) &= g(t), && \alpha > 0. \end{aligned}$$

By decomposition, we have

$$u = u(x,0) - L_t^{-1} \alpha \left(\frac{\partial}{\partial x}\right) \sum_{n=0}^{\infty} u_n,$$

where $L_t^{-1} = \frac{\partial}{\partial t}$, $L_t^{-1} = \int_0^t (\cdot)\, dt$, $u = \sum_{n=0}^{\infty} u_n$, $u(x,0)$ is identified as u_0, and $f(x)$ is assumed differentiable as necessary. Then,

$$\begin{aligned} u_0 &= f(x), \\ u_1 &= -\alpha\, L_t^{-1} \left(\frac{\partial}{\partial x}\right) f(x) = -\alpha\, t\, f'(x), \\ u_2 &= \left(\frac{\alpha^2 t^2}{2!}\right) f''(x), \\ &\vdots \\ u_n &= (-1)^n \left(\frac{\alpha^n t^n}{n!}\right) f^{(n)}(x), \end{aligned}$$

so that

$$u = \sum_{n=0}^{\infty} (-1)^n \left(\frac{\alpha^n t^n}{n!}\right) f^{(n)}(x).$$

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Then,

$$\phi_{m+1} = \sum_{n=0}^{m} (-1)^n \left(\frac{\alpha^n t^n}{n!}\right) f^{(n)}(x) \tag{1}$$

is an $(m+1)$-term approximation to u satisfying the equation and condition at $t = 0$ using the t-dimension "partial solution" [1]. The x-dimension partial solution is derived by

$$\frac{\alpha \partial u}{\partial x} = -\frac{\partial u}{\partial t},$$

$$L_x u = -\alpha^{-1} \left(\frac{\partial}{\partial t}\right) \sum_{n=0}^{\infty} u_n,$$

$$u = u(0,t) - \alpha^{-1} L_x^{-1} \left(\frac{\partial}{\partial t}\right) \sum_{n=0}^{\infty} u_n.$$

Consequently,

$$\begin{aligned} u_0 &= g(t), \\ u_1 &= -\alpha^{-1} L_x^{-1} \left(\frac{\partial}{\partial t}\right) u_0 = -\alpha^{-1} x\, g'(t), \\ u_2 &= \alpha^{-2} \left(\frac{x^2}{2!}\right) g^{(2)}(t), \\ &\vdots \\ \phi_{m+1} &= \sum_{n=0}^{m-1} (-1)^n \alpha^{-n} \left(\frac{x^n}{n!}\right) g^{(n)}(t). \end{aligned} \tag{2}$$

Either (1) or (2) represents the solution.

ADVECTION-DIFFUSION

Let $\xi(x,y,z,t)$ represent concentration. Let the fluid velocity be $\overline{u}$ with components u, v, w in R^3 and assume an incompressible fluid [2,3]

$$\frac{\partial \xi}{\partial t} + \overline{u} \cdot \nabla \xi = D \nabla^2 \xi,$$

where D is the diffusion constant (which is a constant for a particular fluid or contaminant, temperature and pressure), $\xi(x,y,z,0)$ is a given initial condition and various boundary conditions are possible, e.g., $\xi \to 0$ as $x,y,z \to \infty$, or $\xi(t)$ is specified on a boundary Γ, or, we have a preassigned flux at Γ. We have

$$\frac{\partial \xi}{\partial t} = D \left\{ \frac{\partial^2 \xi}{\partial x^2} + \frac{\partial^2 \xi}{\partial y^2} + \frac{\partial^2 \xi}{\partial z^2} \right\} - \frac{u \partial \xi}{\partial x} - \frac{v \partial \xi}{\partial y} - \frac{w \partial \xi}{\partial z}. \tag{3}$$

By decomposition [4,5], using $L = \frac{\partial}{\partial t}$ and $L^{-1} = \int_0^t (\cdot)\, dt$,

$$\begin{aligned} \xi = \xi(t=0) &+ DL^{-1} \left(\frac{\partial^2}{\partial x^2}\right) \sum_{n=0}^{\infty} \xi_n + DL^{-1} \left(\frac{\partial^2}{\partial y^2}\right) \sum_{n=0}^{\infty} \xi_n \\ &+ DL^{-1} \left(\frac{\partial^2}{\partial z^2}\right) \sum_{n=0}^{\infty} \xi_n - L^{-1} u \left(\frac{\partial}{\partial x}\right) \sum_{n=0}^{\infty} \xi_n \\ &- L^{-1} v \left(\frac{\partial}{\partial y}\right) \sum_{n=0}^{\infty} \xi_n - L^{-1} w \left(\frac{\partial}{\partial z}\right) \sum_{n=0}^{\infty} \xi_n. \end{aligned}$$

Thus,

$$\begin{aligned} \xi_0 &= \xi(t=0) = f(x,y,z), \\ \xi_{m+1} &= DL^{-1} \nabla^2 \xi_m - L^{-1} \overline{u} \cdot \nabla \xi_m, \end{aligned}$$

for $m \geq 0$. Now all components are determined and we can write $\phi_N(\xi) = \sum_{m=0}^{N-1} \xi_m$ as an approximation to ξ improving as N increases.

STOCHASTIC CASE

If we have turbulent motion of the fluid, we can have random fluctuations of the concentration and hydrodynamical variables, hence statistical characteristics, would become necessary. $\phi_N(\xi)$ becomes a series of stochastic terms and we form $\langle\phi_n(\xi)\rangle$ to get the expectation as functions of average velocity components. The customary treatments of turbulent motion lead to a lack of closure and untenable assumptions which are avoided by using decomposition. Thus, in the above method, u, v, w, and ξ are replaced by their corresponding steady states plus quantities representing fluctuations from the steady states. Thus,

$$\xi = \overline{\xi} + \xi', \quad u = \overline{u} + u', \quad v = \overline{v} + v', \quad w = \overline{w} + w'.$$

Statistical averaging causes terms such as

$$D\nabla^2 \langle \xi' \rangle, \quad \left(\frac{\partial}{\partial t}\right)\langle \xi' \rangle, \quad u\left(\frac{\partial}{\partial x}\right)\langle \xi' \rangle, \quad \langle u' \rangle \left(\frac{\partial}{\partial x}\right)\overline{\xi}$$

etc. to vanish. We then have

$$\left(\frac{\partial}{\partial t}\right)\overline{\xi} + \overline{u}\left(\frac{\partial}{\partial x}\right)\overline{\xi} + \overline{v}\left(\frac{\partial}{\partial y}\right)\overline{\xi} + \overline{w}\left(\frac{\partial}{\partial z}\right)\overline{\xi}$$

$$= D\,\nabla^2 \overline{\xi} - \overline{u'\left(\frac{\partial}{\partial x}\right)\xi'} - \overline{v'\left(\frac{\partial}{\partial y}\right)\xi'} - \overline{w'\left(\frac{\partial}{\partial z}\right)\xi'}.$$

The last three correlation terms involve correlations of velocities and concentration which are unknown. Then the procedure is to let u_i' for $i = 1, 2, 3$ denote u, v, w, and x_j for $j = 1, 2, 3$ represent x, y, z and write terms as being proportional to a mean gradient of the concentration in terms of a "turbulent diffusion tensor" $-K_{ij}(x_j, t)\frac{\partial \xi}{\partial x_j}$. To clarify the difficulty, consider the operator format $Lu + Ru = g$ or $u = L^{-1}g - L^{-1}Ru$. If we average, we have $\langle u \rangle = L^{-1}\langle g \rangle - L^{-1}\langle Ru \rangle$. We can think of g as an input to a system containing R. The output u can be statistically independent of g, but not of R. By decomposition, one writes $u = L^{-1}g - L^{-1}R\sum u_n = L^{-1}g - L^{-1}R\,L^{-1}g + L^{-1}R\,L^{-1}R\,L^{-1}g - \cdots$. Averaging is no problem since g is statistically independent of R.

NONLINEAR TRANSPORT

Let's consider the equation $L\xi + R\xi + N\xi = g$ where

$$L = \frac{\partial}{\partial t}, \quad N\xi = f(\xi), \quad R = \overline{u}\cdot\nabla - D\nabla^2.$$

Let $\xi = \sum_{n=0}^{\infty} \xi_n$ and $N\xi = \sum_{n=0}^{\infty} A_n$. Then,

$$\xi = L_t^{-1}\,g - L_t^{-1}\,R\sum_{n=0}^{\infty}\xi_n - L_t^{-1}\sum_{n=0}^{\infty}A_n,$$

where

$$\xi_0 = L_t^{-1}\,g$$
$$\xi_{m+1} = -L_t^{-1}\,R\,\xi_m - L_t^{-1}\,A_m,$$

for $m \geq 0$. Then $\phi_{m+1} = \sum_{n=0}^{m}\xi_n$ which converges to $\xi = \sum_{n=0}^{\infty}\xi_m$. Further generalizations are straightforward. We can, for example, consider $Fu = g$ where

$$Fu = L_t u + L_x u + L_y u + L_z u + Ru + Nu = g,$$

and solve for $L_t u$, $L_x u$, $L_y u$ or $L_z u$ which would simply treat the other operator terms as the remainder operator R and would require the appropriate given boundary conditions [6]. The case of stochastic ξ or stochastic processes in the R term leads to a stochastic ϕ_N which can be averaged or from which expectations and covariances can be found. Convergence has been proven by Cherruault [7,8]. The solutions are verifiable by checking that the original equation and the given conditions are satisfied.

Since the concern here is solution of physical systems, unbounded or pathological inputs and conditions are of no interest. If the model equation and the conditions are physically correct and consistent, a solution is obtained which is unique and accurate. If numerical results are calculated, one sees the approach to a stable solution for the desired number of decimal places. If conditions on one variable are better known than the others, we consider the appropriate operator equation which can yield the solution.

REFERENCES

1. G. Adomian and R. Rach, Equality of partial solutions in the decomposition method for linear or nonlinear partial differential equations, *Computers Math. Applic.* **10** (12), 9–12 (1990).
2. R.V. Ozmidov, *Diffusion of Contaminants in the Ocean*, Kluwer, Dordrecht, (1991).
3. J. Noye, Some explicit three-level finite-difference simulations of advection, *Math. and Comput. in Simulation* **32**, 359–372 (1990).
4. G. Adomian, *Nonlinear Stochastic Operator Equations*, Academic Press, New York, (1986).
5. G. Adomian, R. Rach and R. Meyers, An efficient methodology for the physical sciences, *Kybernetes* **20** (6), 104–113 (1991).
6. G. Adomian, A review of the decomposition method and some recent results for nonlinear equations, *Computers Math. Applic.* **21** (5), 101–127 (1991); (Errata in Mathl. Comput. Modelling, in press).
7. Y. Cherruault, Convergence of Adomian's method, *Kybernetes* **18** (2), 31–38 (1989).
8. Y. Cherruault, G. Saccamondi and B. Some, New results for convergence of Adomian's method applied to integral equations, *Mathl. Comput. Modelling* **16** (2), 85–93 (1992).