A Dirichlet Boundary Value Problem for a Generalized Ginzburg-Landau Equation

HONGJUN GAO*
Department of Mathematics, Nanjing Normal University
Nanjing 210097, P.R. China
gaoj@pine.njnu.edu.cn

C. BUTF
Department of Mathematics, Wellesley College
Wellesley, MA 02482, U.S.A.
cbu@wellesley.edu

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Abstract—We study the following generalized 1D Ginzburg-Landau equation on \( \Omega = (0, \infty) \times (0, \infty) \):
\[
  u_t = (1 + i\mu)u_{xx} + (a_1 + ib_1)|u|^2u_x + (a_2 + ib_2)u^2u_x - (1 + iv)|u|^4u
\]
with initial and Dirichlet boundary conditions \( u(x, 0) = h(x), \ u(0, t) = Q(t) \). Under suitable conditions, we prove that there is a unique \( H^1 \) weak solution that exists for all time. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

The cubic Ginzburg-Landau equation (GL)
\[
  u_t = (1 + i\mu)u_{xx} - (1 + iv)|u|^2u + \gamma u
\]
(1.1)
is often used to describe the amplitude evolution of instability waves in a large variety of dissipative systems, for example, in fluid dynamics. It frequently occurs as the leading term in an asymptotic expansion of envelope solutions for exact models such as Navier-Stokes equation [1]. It is of physical interest to carry the expansion to second order [2,3]. The leads to the following generalized GL:
\[
  u_t = (1 + i\mu)u_{xx} + (a_1 + ib_1)|u|^2u_x + (a_2 + ib_2)u^2u_x - (1 + iv)|u|^4u - (a_0 + ib_0)|u|^2u + \gamma u. \tag{1.2}
\]
If \( 4 > (b_1 - b_2)^2 \), then (1.2) with initial data \( u(x, 0) \in H^4_{\text{per}}[0, L] \) possesses a global classical solution in \( C([0, \infty); H^4_{\text{per}}[0, L]) \cap C^1([0, \infty); H^1_{\text{per}}[0, L]) \) [4]. It also has been found that the cubic terms involving partial derivatives can significantly slow the propagation speed of moving fronts and pulses [2] and must be balanced by the fifth-order term and the second derivative term.

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There are many papers in the literature regarding the GL equation and its generalized versions in one and two dimensions [5-14]. However, most of them study the initial value or periodic boundary value problems. Yet, in many cases of physical interest, the mathematical models lead to the forced problems when a nonzero boundary condition is imposed [15,16]. Forced problems occur when an external force is applied to the time evolution of systems governed by nonlinear partial differential equations. The forcing is often put in as a boundary condition of Dirichlet or Neumann type when \( x = 0 \). To establish the model for the forced GL equation, one considers the unstable waves specified for all times at the reflection point, and asks for the time development if the nonlinear phenomena at that point. The relevant mathematical model reads

\[
\begin{align*}
\frac{\partial u}{\partial t} &= (1 + i\mu)\frac{\partial^2 u}{\partial x^2} - (1 + i\nu)|u|^2u + \gamma u, \\
u(x, 0) &= h(x), \quad u(0, t) = Q(t).
\end{align*}
\]

For (1.3),(1.4), a global classical solution is available when \( \nu \mu > 0 \) or \( |\nu| \leq \sqrt{3} \) [17]. For the GL equation posed in the finite domain \( 0 < x < L, \quad 0 < t < \infty \) with Dirichlet or Neumann boundary condition, a unique weak solution in \( H^1 \) can be obtained by using the Galerkin-Vishik method [18,19].

However, we are not aware of any results regarding a generalized GL equation (1.2) with initial condition and either nonzero Dirichlet or Neumann boundary conditions.

In this paper, we will study (1.2) with initial and nonzero Dirichlet boundary data. For simplicity, we assume that \( a_0 = b_0 = \gamma = 0 \). Under suitable conditions, we prove the global existence.

2. LOCAL EXISTENCE THEOREM

We study the following initial-boundary value problem for a generalized Ginzburg-Landau equation:

\[
\begin{align*}
\frac{\partial u}{\partial t} &= (1 + i\mu)\frac{\partial^2 u}{\partial x^2} + (a_1 + ib_1)|u|^2\frac{\partial u}{\partial x} + (a_2 + ib_2)u^2\frac{\partial u}{\partial x} - (1 + i\nu)|u|^4u, \\
u(x, 0) &= h(x), \quad x \geq 0, \\
u(0, t) &= Q(t), \quad t \geq 0,
\end{align*}
\]

with compatibility condition \( h(0) = Q(0) \). We have the following local existence-uniqueness theorem.

**Theorem 2.1.** For \( h \in H^1 = H^1(\mathbb{R}^+), \ Q \in C^1[0, \infty) \), there exists a unique local solution \( u(t) \in C([0, T); H^1) \cap C^1([0, T); H^1) \) for (2.1)-(2.3). Here \( T \) depends on \( \|h\|_{H^1} \) and \( \|Q\|_{C^1} \).

**Proof.** We first use the standard technique of change of variables via \( v = u + Q(t)e^{-x} \). This substitution yields \( \frac{\partial v}{\partial t} + (1 + i\nu)v = A\frac{\partial v}{\partial x} + B \sum_{i=0}^3 |v|^i, \quad x, t > 0 \), with \( A = -(1 + i\mu)\partial_x + 1 \), and \( B = \sum_{i=0}^3 (a_i + ib_i) \).

Since \( Q(t) \in C^1([0, \infty)), \) we have \( \|G_0\|_{L^2} \) is bounded and \( G_1 \) is local Lipschitz from \( H^1 \) to \( L^2 \). Let

\[
\text{Re}(Au, u) = \|u_x\|^2 + \|u\|^2 = \|u\|^2_{H^1},
\]

(2.6)
for \( u \in D(A) \), here \( \| \cdot \| \) denotes the norm of \( L^2 = L^2(\mathbb{R}^+) \). \( A \) is strongly elliptic and \( -A \) generates an analytic semigroup of contraction in \( L^2 \) [20, Chapter 7, Definition 2.1 and Theorem 2]. Meanwhile, the eigenvalue of \( A \) is positive for any \( \mu \). Thus, we can define the fractional power \( A^\alpha, 0 \leq \alpha \leq 1 \). Now \( X_0 = D(A^0) \) becomes a dense subspace of \( X_0 = L^2 \) with graph norm. In particular, let \( X_{1/2} = \{ u \in H^1(\mathbb{R}^+), u(0) = 0 \} \), \( X_1 = D(A) \) with graph norms on \( X_{1/2}, X_1 \) equivalent to the usual norms on \( H^1 \) and \( L^2 \), respectively. Since \( H^1 \subset L^\infty \), the nonlinear operator \( N(u) = G_1 - u \) satisfies

\[
\| N(u) - N(v) \| \leq c\| u - v \|_{H^1}, \quad u, v \in X_{1/2},
\]

where the Lipschitz constant \( c \) depends continuously on \( \| u \|_{H^1}, \| v \|_{H^1} \). Hence, \( N : H^1 = X_{1/2} \subset L^2 = X_0 \) is Lipschitz continuous and Assumption F of [20, Chapter 6] is satisfied on any open subset \( U \) of \( X_{1/2} \). We may therefore apply Theorem 6.3.1 of [20] to conclude that the initial value problem

\[
\frac{dv}{dt} + Av = N(v), \quad v(0, t) = 0, \quad v(x, 0) = g(x) \in H^1,
\]

has a unique local solution \( v(t) \in C([0, \tau); L^2) \cap C^1([0, \tau); L^2) \) for some \( \tau > 0 \) depending on \( g \). In fact, short-time existence holds in \( H^1 \), as we now indicate via the method of Henry [21]. Note that \( A \) defined above is a sectorial operator [21, Definition 1.3.1], we can therefore apply Theorems 3.3.3 and 3.5.2. of [21] with \( \tau = \alpha = 1/2 \) to conclude that \( v(t) \in C([0, T); H^1) \cap C((0, T); H^1) \) for some \( T > 0 \), possibly smaller than \( \tau \). Thus, the local existence theorem is proved.

### 3. GLOBAL EXISTENCE THEOREM

In this section, we will prove the following global existence theorem. For this, we need to obtain the bound of \( \| u \|_{H^1} \) for any given \( T > 0 \).

**Theorem 3.1.** If \( |\nu| < \sqrt{5}/2 \) and \((|a_1| + |a_2| + |b_1| + |b_2|)^2 < 2(3 - 2\sqrt{1 + \nu^2}) \), then the local solution in Theorem 2.1 is a global solution.

**Proof.** Let \( P = u_x(0, t) \). The following identities can be easily verified by integrating by parts and substituting the original equation. First,

\[
\frac{d}{dt}\| u \|^2 = -2 \text{Re}(1 + i\mu)PQ - 2\| u_x \|^2 - (a_1 + a_2)|Q|^4 + 2(b_1 - b_2) \int_0^\infty |u|^2u\bar{u}_x dx - 2 \int_0^\infty |u|^6 dx.
\]

Second,

\[
\frac{d}{dt}\| u_x \|^2 = -2 \text{Re} PQ' - 2\| u_{xx} \|^2 - 2 \text{Re}(a_1 + ib_1) \int_0^\infty |u|^2u_xu_{xx} dx - 2 \text{Re}(a_2 + ib_2) \int_0^\infty u^2\bar{u}_xu_{xx} dx + 2 \text{Re}(1 + i\nu) \int_0^\infty |u|^4u\bar{u}_{xx} dx.
\]

Third, the difficulty in the nonzero Dirichlet problem is that we could not get the estimate for \( \| u \| \) by (3.1) like the paper [4] under the condition \( 4 > (b_1 - b_2)^2 \). So, we need the following identity:

\[
\frac{d}{dt} \int_0^\infty u\bar{u}_x dx = -QQ' - i\mu|P|^2 + 2i \text{Im} \int_0^\infty u_x\bar{u}_x + 2i \text{Im} \int_0^\infty (a_1 + ib_1)|u|^2u_x^2 dx + 2i \text{Im} \int_0^\infty (a_2 + ib_2)u^2\bar{u}_x dx - 2i \text{Im} \int_0^\infty (1 + i\nu)|u|^4u\bar{u}_x dx.
\]
By taking the imaginary part on both sides of (3.3) and integrating in $t$ variable, we obtain

$$
\mu \int_0^t |P|^2 \, dt = - \text{Im} \int_0^\infty u \bar{u}_x \, dx + \text{Im} \int_0^\infty h \dot{h} \, dx
$$
$$
- \text{Im} \int_0^t Q \dot{Q}' \, dt + 2 \text{Im} \int_0^t \int_0^\infty u_{xx} \bar{u}_x \, dx \, d\tau - 2 \text{Im} \int_0^t \int_0^\infty (1 + iv)|u|^4u \bar{u}_x \, dx \, d\tau
$$
$$
+ 2 \text{Im} \int_0^t \int_0^\infty (a_1 + ib_1)|u|^2|u_x|^2 \, dx \, d\tau + 2 \text{Im} \int_0^t \int_0^\infty (a_2 + ib_2)u^2 \bar{u}_x^2 \, dx \, d\tau.
$$

Therefore,

$$
\int_0^t |P|^2 \, dt \leq \frac{1}{|\mu|} \left( ||u||^2 + ||u_x||^2 \right) + c' + \int_0^t \left( ||u||^2 + ||u_{xx}||^2 \right) \, dt
$$
$$
+ c' \int_0^t \int_0^\infty \left( |u|^8 + |u|^4|u_x|^2 + |u_x|^2 \right) \, dx \, d\tau,
$$

for some $c' > 0$ depends only on $||Q||_{C^1([0, \infty))}$, $||h||_{H^1}$, $a_1, a_2, b_1,$ and $b_2$.

We need to estimate the following term in (3.2) (where $|u|^4u \bar{u}_{xx} = u^3 \bar{u}^2 u_{xx}$):

$$
\text{Re}(1 + iv) \int_0^\infty u^3 \bar{u}^2 \bar{u}_{xx} \, dx = \text{Re}(1 + iv)u^3 \bar{u}^2 \bar{u}_{xx}|_0^\infty
$$
$$
- \text{Re}(1 + iv) \int_0^\infty 3|u|^4|u_x|^2 \, dx - \text{Re}(1 + iv) \int_0^\infty 2|u|^2u^2 \bar{u}_x^2 \, dx
$$
$$
= - \text{Re}(1 + iv)|Q|^4Q \bar{P} - 3 \int_0^\infty |u|^4|u_x|^2 \, dx - 2 \text{Re}(1 + iv) \int_0^\infty |u|^2u^2 \bar{u}_x^2 \, dx
$$
$$
\leq -(3 - 2\sqrt{1 + v^2}) \int_0^\infty |u|^4|u_x|^2 \, dx - \text{Re}(1 + iv)|Q|^4Q \bar{P}.
$$

Let $T$ be any positive number, $0 \leq t < T$ and $|a_1| + |a_2| + |b_1| + |b_2| = ta$. Since $|v| < \sqrt{5}/2$, there is a number $\delta = 3 - 2\sqrt{1 + v^2} > 0$. By the condition of Theorem 3.1, we have $\beta^2 < 2\delta$. In our attempt to establish an upper bound for $u$ in $H^4$, $c$ is treated as a generic constant. From (3.1), (3.2), and (3.6), we get

$$
\frac{d}{dt} \left( ||u||^2 + ||u_x||^2 \right) \leq c - 2||u_{xx}||^2 - 2||u_x||^2
$$
$$
- 2 \int_0^\infty |u|^6 \, dx + c||P|| + 2(b_1 - b_2) \int_0^\infty |u|^3|u_x| \, dx
$$
$$
+ 2\beta \int_0^\infty |u|^2|u_xu_{xx}| \, dx - 2 \left( 3 - 2\sqrt{1 + v^2} \right) \int_0^\infty |u|^4|u_x|^2 \, dx
$$
$$
\leq c - 2||u_{xx}||^2 - 2||u_x||^2 - 2 \int_0^\infty |u|^6 \, dx + c||P|| + \frac{\delta}{2} \int_0^\infty |u|^4|u_x|^2 \, dx + \frac{2(b_1 - b_2)^2}{2} ||u||^2
$$
$$
+ \frac{\beta^2}{3} \int_0^\infty |u|^4|u_x|^2 \, dx + \frac{3}{2} ||u_{xx}||^2 - 2\delta \int_0^\infty |u|^4|u_x|^2 \, dx
$$
$$
\leq c - \frac{1}{2}||u_{xx}||^2 - 2||u_x||^2 - \frac{1}{\delta} ||u||^2 + c||P|| + \frac{2(b_1 - b_2)^2}{\delta} ||u||^2 - \frac{\delta}{6} \int_0^\infty |u|^4|u_x|^2 \, dx.
$$

By Cauchy inequality, (3.7) becomes

$$
\frac{d}{dt} \left( ||u||^2 + 2||u_x||^2 \right) \leq - 2||u_x||^2 - \frac{1}{2}||u_{xx}||^2 - \frac{1}{\delta} ||u||^2
$$
$$
+ m||P||^2 + \frac{c^2}{4m} + c_0||u||^2 - \frac{\delta}{6} \int_0^\infty |u|^4|u_x|^2 \, dx,
$$

(3.8)
for some $c_0 > 0$ and any $m > 0$ (to be determined later). Hence,

$$
\|u\|_{H^1}^2 + \int_0^t \left( 2\|u_x\|^2 + 2\|u\|_6^6 + \frac{1}{2}\|u_{xx}\|^2 + \frac{\delta}{6} \int_0^\infty |u|^4 |u_x|^2 \, dx \right) \, d\tau
\leq \tilde{c} + m \int_0^t \|P\|^2 \, d\tau + c_0 \int_0^t \|u\|^2 \, d\tau.
$$

(3.9)

Now we substitute (3.5) in (3.9) to get

$$
\|u\|_{H^1}^2 + \int_0^t \left( 2\|u_x\|^2 + 2\|u\|_6^6 + \frac{1}{2}\|u_{xx}\|^2 + \frac{\delta}{6} \int_0^\infty |u|^4 |u_x|^2 \, dx \right) \, d\tau
\leq \tilde{c} + c_0 \int_0^t \|u\|^2 \, d\tau + m \|u\|_{H^1}^2 + \frac{m}{|\mu|} \|u\|_{H^1}^2 + c'm + m \int_0^t \left( \|u\|^2 + \|u_{xx}\|^2 \right) \, d\tau
\leq \tilde{c} + c_0 \int_0^t \|u\|^2 \, d\tau + m \|u\|_{H^1}^2 + \frac{m}{|\mu|} \|u\|_{H^1}^2 + c'm + m \int_0^t \left( |u|^6 + |u|^4 |u_x|^2 + |u_x|^2 \right) \, dx \, d\tau.
$$

(3.10)

We choose $m$ such that $m/|\mu| < 1$, $m \leq 1/2$, and $c'm \leq \min\{2,(\delta/6)\}$, then (3.10) can be rewritten as

$$
\|u\|_{H^1}^2 \leq M_0 + M \int_0^t \|u\|_{H^1}^2 \, d\tau.
$$

(3.11)

By Gronwall’s lemma, $\|u\|_{H^1}^2$ is bounded when $0 \leq t \leq T$. By the semigroup theory and since $T$ is arbitrarily given, the solution $u$ to (2.1)-(2.3) is global. The proof of Theorem 3.1 is finished.

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