Subgraph isomorphism in graph classes
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A B S T R A C T
We investigate the computational complexity of the following restricted variant of
Subgraph Isomorphism: given a pair of connected graphs \( G = (V_G, E_G) \) and \( H = (V_H, E_H) \),
determine if \( H \) is isomorphic to a spanning subgraph of \( G \). The problem is NP-complete in
general, and thus we consider cases where \( G \) and \( H \) belong to the same graph class such as
the class of proper interval graphs, of trivially perfect graphs, and of bipartite permutation
graphs. For these graph classes, several restricted versions of Subgraph Isomorphism
such as Hamiltonian Path, Clique, Bandwidth, and Graph Isomorphism can be solved
in polynomial time, while these problems are hard in general.

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1. Introduction

A graph \( H = (V_H, E_H) \) is subgraph-isomorphic to a graph \( G = (V_G, E_G) \) if there exists an injective map \( \eta \) from \( V_H \) to \( V_G \) such that \( \{\eta(u), \eta(v)\} \in E_G \) holds for each \( \{u, v\} \in E_H \). The problem Subgraph Isomorphism is a fundamental problem in graph
theory: given a pair of graphs \( G \) and \( H \), determine if \( H \) is subgraph-isomorphic to \( G \). We call \( G \) a base graph and \( H \) a pattern
graph. The problem generalizes many other combinatorial problems such as Hamiltonian Path, Clique, and Bandwidth.
Hence Subgraph Isomorphism is NP-complete in general [10]. For instance, the problem is NP-complete even in the case
where the base graph is a tree and the pattern graph is a set of paths [10].

By a slight modification of Damaschke’s proof in [7], Subgraph Isomorphism is hard when \( G \) and \( H \) are disjoint unions of
paths. Since the classes of proper interval graphs and of bipartite permutation graphs contain the disjoint unions of paths,
Subgraph Isomorphism is hard on proper interval graphs and bipartite permutation graphs. The construction strongly relies
on the disconnectedness. We ask whether the hardness of the problem also relies on the disconnectedness. Thus we consider
the following problem in which both \( G \) and \( H \) are connected.

Problem 1 (Spanning Subgraph Isomorphism).
Instance: A pair of connected graphs \( G = (V_G, E_G) \) and \( H = (V_H, E_H) \), where \( |V_G| = |V_H| \).
Question: Is \( H \) subgraph-isomorphic to \( G \)?

Spanning Subgraph Isomorphism is NP-complete even for bipartite graphs and for chordal graphs, since Hamiltonian
Path on these classes is NP-complete [10,1]. Subgraph Isomorphism on cographs is also NP-complete (see [6]). Meanwhile,
the computational complexity of SUBGRAPH ISOMORPHISM on interval graphs seems still not to be known since Johnson posed the question [16].

Our contributions. We study SPANNING SUBGRAPH ISOMORPHISM for the classes of proper interval graphs, of bipartite permutation graphs, and of trivially perfect graphs. For these classes, HAMILTONIAN PATH, CLIQUE, and BANDWIDTH are known to be solvable in polynomial time [17,3,11,19,14,30]. It is also known that the graph isomorphism for them can be solved in polynomial time [5,21].

We first show that SPANNING SUBGRAPH ISOMORPHISM is NP-complete even for proper interval graphs, bipartite permutation graphs, and of threshold graphs. Thus our results answer the question by Johnson [16]. Next we study SPANNING SUBGRAPH ISOMORPHISM for subclasses of the classes above such as the classes of chain graphs, of cochain graphs, and of threshold graphs. We show that SPANNING SUBGRAPH ISOMORPHISM is NP-complete even if the base graphs are in these subclasses. We finally show that SUBGRAPH ISOMORPHISM, a generalization of SPANNING SUBGRAPH ISOMORPHISM, can be solved in linear time if both the base and pattern graphs are in these subclasses. The algorithms are simple and consider only the degree sequences of graphs.

Fig. 1 presents a summary of our results. See standard textbooks [4,27] for graph classes and relationships among them.

Related topics. SUBGRAPH ISOMORPHISM for connected outerplanar graphs is NP-complete [28], while it can be solved in polynomial time for two-connected outerplanar graphs [28,20]. More generally, it is known that SUBGRAPH ISOMORPHISM for k-connected partial k-trees can be solved in polynomial time [23,12]. Eppstein [9] gave a kO(k)n-time algorithm for SUBGRAPH ISOMORPHISM on planar graphs, where k is the number of vertices in the pattern graph. Recently, Dorn [8] has improved the running time to 2O(k)n.

Another related topic may be the induced subgraph isomorphism problem. Damaschke [7] showed that INDUCED SUBGRAPH ISOMORPHISM on cographs is NP-complete. He also showed that INDUCED SUBGRAPH ISOMORPHISM is NP-complete for graph classes that include the disjoint unions of paths, and thus for proper interval graphs. Marx and Schlotter [22] showed that INDUCED SUBGRAPH ISOMORPHISM on interval graphs is W[1]-hard when parameterized by the number of vertices in the pattern graph, but fixed-parameter tractable when parameterized by the numbers of vertices to be removed from the base graph. Heggernes et al. [15] showed that INDUCED SUBGRAPH ISOMORPHISM on proper interval graphs is NP-complete even if the base graph is connected, while the problem can be solved in polynomial time if the pattern graph is connected. Furthermore, they showed that INDUCED SUBGRAPH ISOMORPHISM is fixed-parameter tractable when parameterized by the number of the connected components in the pattern graph, if the base graph is an interval graph and the pattern graph is a proper interval graph.

Kilpelainen and Mannila [18] showed the NP-completeness of the unordered tree inclusion problem. In the problem, we are given two rooted unordered trees T1, T2 with labels for each vertex, and asked that “By contracting some edges, can T2 be obtained from T1?”. (After a contraction, the vertex that corresponds to the contracted edge {u, v}, where u is the parent of v, has the label of u.) This fact is interesting since SUBGRAPH ISOMORPHISM for trees can be solved in polynomial time [10].

For proper interval graphs and bipartite permutation graphs, there are enumeration and random generation algorithms that involve their characterizations by balanced parentheses [26,25]. These characterizations can be used to efficiently solve the graph isomorphism problem for them. One might think that these characterizations can be used also for SPANNING SUBGRAPH ISOMORPHISM. By our results in this paper, this is not the case unless P = NP.
Problem 2
Partition set of positive integers.

Instance: Positive integers


Note: By the assumption \( a \) is the set \( N \)

Terminologies and notations. All graphs in this paper are undirected and simple. Recall that we say a graph \( H = (V_H, E_H) \) is subgraph-isomorphic to \( G = (V_G, E_G) \) if there exists an injective map \( \eta \) from \( V_H \) to \( V_G \) such that \( \{ \eta(u), \eta(v) \} \in E_G \) holds for each \( \{ u, v \} \in E_H \). We call such a map \( \eta \) a subgraph-isomorphism from \( H \) to \( G \). For a map \( \eta : V \to V' \) and \( S \subseteq V \), let \( \eta(S) \) denote the set \( \{ \eta(s) \mid s \in S \} \). Let \( G[U] \) denote the subgraph of \( G = (V, E) \) induced by \( U \subseteq V \). The disjoint union of two graphs \( G = (V_G, E_G) \) and \( H = (V_H, E_H) \) is the graph \( (V_G \cup V_H, E_G \cup E_H) \), where \( V_G \cap V_H = \emptyset \). The neighborhood of a vertex \( v \) is the set \( N(v) = \{ u \in V \mid \{u, v\} \in E \} \), and the degree of \( v \) is \( |N(v)| \) denoted by \( d(v) \). The closed neighborhood of \( v \) is the set \( N(v) \cup \{ v \} \), and denoted by \( \overline{N}(v) \). A set \( S \subseteq V \) in \( G = (V, E) \) is an independent set if any two vertices in \( S \) are not adjacent in \( G \). A set \( S \subseteq V \) in \( G = (V, E) \) is a clique if any two vertices in \( S \) are adjacent in \( G \). A pair \( (X, Y) \) of sets of vertices of a bipartite graph \( H \) is a biclique if any two vertices \( x \in X \) and \( y \in Y \) are adjacent in \( H \). A biclique \( (X, Y) \) is balanced if \( |X| = |Y| \). The complement of a graph \( G = (V, E) \) is the graph \( \overline{G} = (V, \overline{E}) \) such that \( \{u, v\} \notin \overline{E} \) if and only if \( \{u, v\} \notin E \). Let \( \mathbb{Z}^+ \) denote the set of positive integers.

2. NP-completeness

We show that Spanning Subgraph Isomorphism (Problem 1) is NP-complete for proper interval graphs, bipartite permutation graphs, and trivially perfect graphs. Since the problem is in NP for arbitrary graphs, we need to show that the problem is NP-hard for these three graph classes. We show this by reducing from 3-Partition defined as follows.

Problem 2 (3-Partition (cf. [10])).

Instance: Positive integers \( a_1, \ldots, a_{3m} \) and a bound \( B \in \mathbb{Z}^+ \) such that \( \sum_{j \in \{1, \ldots, 3m\}} a_j = mB \) and \( B/4 < a_j < B/2 \) for each \( j \in \{1, \ldots, 3m\} \). An instance is represented as \( (a_1, \ldots, a_{3m}; B) \).

Question: Can \( \{1, \ldots, 3m\} \) be partitioned into \( m \) disjoint sets \( A^{(1)}, A^{(2)}, \ldots, A^{(m)} \) such that, for \( 1 \leq i \leq m \), \( \sum_{j \in A^{(i)}} a_j = B \)?

Note: By the assumption \( B/4 < a_j < B/2 \), each \( A^{(i)} \) must contain exactly three elements.

It is well known that 3-Partition is strongly NP-hard, meaning that it is NP-hard even when every \( a_j (j \in \{1, \ldots, 3m\}) \) is bounded by a polynomial in \( m \); namely, \( a_j \leq 2^{10}m^4 + 10 \) for each \( j \mid 10 \).

2.1. Proper interval graphs

A proper interval graph is the intersection graph of a family of closed intervals of the real line where no interval is properly contained in another [24]. The class of proper interval graphs is known to be equivalent to the class of unit interval graphs [24, 2], where a graph is unit interval if it is the intersection graph of a family of closed unit intervals of the real line.

Theorem 1. Spanning Subgraph Isomorphism is NP-complete for proper interval graphs.

Proof. Let \( (a_1, \ldots, a_{3m}; B) \) be an instance of 3-Partition. We construct proper interval graphs \( G \) and \( H \) in polynomial time.

Construction of \( G \) and \( H \). Let \( M = 7m^2 \). Roughly speaking, graph \( H = (V_H, E_H) \) consists of \( 3m \) disjoint cliques, each of which has \( ma_i \) vertices, and these cliques are sequentially connected by paths on \( m - 1 \) vertices (see Fig. 2). More precisely, let

\[
V_H = \bigcup_{i=1}^{3m} X^{(i)} \cup \bigcup_{i=1}^{3m-1} Y^{(i,i+1)} \cup Z,
\]

where \( |X^{(i)}| = ma_i \) for \( i \in \{1, \ldots, 3m\} \), \( |Y^{(i,i+1)}| = m - 1 \) for \( i \in \{1, \ldots, 3m - 1\} \), and \( |Z| = 3m^2(m - 1) - (3m - 1)(m - 1) \).

Hence

\[
|V_H| = \sum_{i=1}^{3m} |X^{(i)}| + \sum_{i=1}^{3m-1} |Y^{(i,i+1)}| + |Z| = MmB + 3m^2(m - 1).
\]

The edge set \( E_H \) is described as follows. Every \( X^{(i)} (i \in \{1, \ldots, 3m\}) \) is a clique. Every \( Y^{(i,i+1)} (i \in \{1, \ldots, 3m - 1\}) \) induces a path \( y_1^{(i,i+1)}, y_2^{(i,i+1)}, \ldots, y_{m-1}^{(i,i+1)} \), where the end \( y_1^{(i,i+1)} \) is linked to \( x_0^{(i)} \in X^{(i)} \) and the other end \( y_{m-1}^{(i,i+1)} \) is linked to
$x^{(i+1)}_i \in X^{(i+1)}_i$ such that $x^{(i)}_i \neq x^{(i)}_j$ for $i \in \{2, \ldots, 3m - 1\}$. Also, $Z$ induces a path $z_1, z_2, \ldots, z_{|Z|}$, where $z_1$ is linked to $x^{(3m)}_i \in X^{(3m)}_i$ and $x^{(3m)}_j \neq x^{(3m)}_j$. There is no other edge in $H$. It is not difficult to see that $H$ is a proper interval graph (see Figs. 2 and 3).

Next we define graph $G = (V_G, E_G)$ with $|V_G| = |V_H|$ and $|E_G| > |E_H|$. Roughly speaking, $G$ consists of $m$ disjoint cliques, each of which has $MB$ vertices, and these cliques are sequentially connected by cliques of size $3m^2$ (see Fig. 2). More precisely, let

$$V_G = \bigcup_{i=1}^{m} U^{(i)} \cup \bigcup_{i=1}^{m-1} T^{(i)},$$

where $|U^{(i)}| = MB$ for $i \in \{1, \ldots, m\}$, and $|T^{(i)}| = 3m^2$ for $i \in \{1, \ldots, m-1\}$. Thus

$$|V_G| = \sum_{i=1}^{m} |U^{(i)}| + \sum_{i=1}^{m-1} |T^{(i)}| = mMB + 3m^2(m - 1) = |V_H|.$$

Every $T^{(i-1)} \cup U^{(i)} \cup T^{(i)}$ is a clique for $i \in \{1, \ldots, m\}$, where we define $T^{(0)} = T^{(m)} = \emptyset$ for convenience. Those edges are all of $E_G$. It is not difficult to see that $G$ is a proper interval graph and $|E_G| > |E_H|$ (see Figs. 2 and 3).

If $3$-PARTITION has a solution then $H$ is subgraph-isomorphic to $G$. Suppose $A^{(1)}, \ldots, A^{(m)}$ is a solution of $3$-PARTITION; that is, $\bigcup_{i=1}^{m} A^{(i)} = \{1, \ldots, 3m\}$ and $\sum_{j \in A^{(i)}} a_j = B$ for $i \in \{1, \ldots, m\}$. Now we construct a bijective map $\eta$ from $V_H$ to $V_G$ as follows.

For each $A^{(i)}$, let $\eta(\bigcup_{j \in A^{(i)}} X^{(j)}) = U^{(i)}$. It is easy to see that $H[\bigcup_{j \in A^{(i)}} X^{(j)}]$ is subgraph-isomorphic to $G[U^{(i)}]$ since $U^{(i)}$ is a clique and $|\bigcup_{j \in A^{(i)}} X^{(j)}| = |U^{(i)}| = MB$.

Now let $W = \bigcup_{i=1}^{m-1} T^{(i)}$. We embed $H[\bigcup_{j=1}^{3m-1} Y^{(j,j+1)}]$ into $G[W]$. Assume that $j \in A^{(k)}$ and $j+1 \in A^{(k')}$. By symmetry, it suffices to consider the case $k \leq k'$. The map $\eta$ embeds $H[Y^{(j,j+1)}]$, a path on $m - 1$ vertices, into $G[W]$ so that the following conditions are satisfied:

- $\eta(Y^{(j,j+1)}_{1}) = T^{(k-1)} \cup T^{(k)}$ and $\eta(Y^{(j,j+1)}_{3}) = T^{(k'-1)} \cup T^{(k')}$. If $\eta(Y^{(j,j+1)}_{2}) = T^{(k)}$ and $\eta(Y^{(j,j+1)}_{2}) = T^{(k')}$, then $|a - b| \leq 1$.

Clearly, $\eta(e) \in E_G$ for each edge $e$ in $H[Y^{(j,j+1)}]$. For example, if $k = k'$, then $\eta$ embeds $H[Y^{(j,j+1)}]$ into $G[T^{(k-1)}]$. Otherwise, $\eta$ embeds $Y^{(j,j+1)}_{1}$ into $T^{(k-1)}$ for $1 \leq i \leq k' - k$, embeds $Y^{(j,j+1)}_{2}$ into $T^{(k'-1)}$ for $k' - k + 1 \leq i \leq m - 1$. Such $\eta$ that simultaneously embeds $Y^{(j,j+1)}$ into $G[W]$ for all $j \in \{1, \ldots, 3m - 1\}$ exists, since $|Y^{(j,j+1)}| = m - 1 \geq k' - k$ for all $j \in \{1, \ldots, 3m - 1\}$, and

$$\left| \bigcup_{j=1}^{3m-1} Y^{(j,j+1)} \right| = (3m - 1)(m - 1) \leq 3m^2 - 2 = |T^{(0)}| - 2 \ (1)$$

for all $i \in \{1, \ldots, m\}$.

Finally, let $W' = W \setminus \eta(\bigcup_{j=1}^{3m-1} Y^{(j,j+1)})$, and we embed $H[Z]$ into $G[W']$ so that $\eta(\{z_1, x^{(3m)}_i\}) \in E_G$. The inequality (1) implies that $|W' \cap T^{(0)}| \geq 2$ for every $i \in \{1, \ldots, m\}$. Thus $G[W']$ contains a Hamiltonian cycle, which implies that a desirable embedding exists. Therefore, $H$ is subgraph-isomorphic to $G$.

If $H$ is subgraph-isomorphic to $G$ then $3$-PARTITION has a solution. Suppose that $\eta$ is a subgraph-isomorphism from $H$ to $G$. Let $A^{(i)} = \{j \mid \eta(x^{(j)}_{X}) \cap U^{(i)} \neq \emptyset\}$.

We first show that $A^{(i)}$, $\ldots$, $A^{(m)}$ is a partition of $\{1, \ldots, 3m\}$. Observe that $A^{(i)} \cap A^{(i')} = \emptyset$ for $i \neq i'$, since $X^{(i)}$ is a clique and there is no edge between $U^{(i)}$ and $U^{(i')}$. Also, it can be observed that for each $j \in \{1, \ldots, 3m\}$ there is an index $i$ such that $\eta(X^{(j)}_{X}) \cap U^{(i)} \neq \emptyset$. Otherwise, we have to embed $H[X^{(j)}]$ into $G[T^{(i)} \cup T^{(i')}^{(i')}]$ for some $i'$. This is impossible because $|X^{(j)}| > |T^{(i)} \cup T^{(i')}^{(i')}|$.

Now we show that $\sum_{j \in A^{(i)}} a_j = B$ for each $i$. If there exists $i' \in \{1, \ldots, m\}$ satisfying that $\sum_{j \in A^{(i')}} a_j \neq B$, then there exists $i \in \{1, \ldots, m\}$ such that $\sum_{j \in A^{(i)}} a_j > B$. Now we have $|\bigcup_{j \in A^{(i)}} X^{(j)}| = M \sum_{j \in A^{(i)}} a_j \geq M(B + 1) > MB + 6m^2 = |U^{(i)} \cup T^{(i-1)} \cup T^{(i)}|$, a contradiction. □
2.2. Bipartite permutation graphs

A graph \( G = (V, E) \) with \( V = \{1, 2, \ldots, n\} \) is a permutation graph if there is a permutation \( \pi \) over \( V \) such that \( [i, j] \in E \) if and only if \( (i - j)(\pi(i) - \pi(j)) < 0 \) [4]. Equivalently, each vertex \( i \) in a permutation graph corresponds to a segment \( s_i \), joining two points on two parallel lines \( L_1 \) and \( L_2 \). Then two vertices \( i \) and \( j \) are adjacent if and only if the corresponding segments \( s_i \) and \( s_j \) intersect. We call the intersection model a permutation diagram of the permutation graph. A bipartite permutation graph is a permutation graph that is bipartite.

**Theorem 2.** **SPANNING SUBGRAPH ISOMORPHISM** is NP-complete for bipartite permutation graphs.

**Proof (Sketch).** Since the proof is just a bipartite analogue of the one of Theorem 1, we only sketch the difference of the construction and the main steps of the proof of correctness. Let \((a_1, \ldots, a_{3m}; 8)\) be an instance of 3-PARTITION. We construct bipartite permutation graphs \( G \) and \( H \) in polynomial time.

Construction of \( G \) and \( H \). Recall that \( M = 7m^2 \). We only describe the differences from the case of proper interval graphs. See Figs. 4 and 5 for the illustration of the graphs.

In \( H \), each clique \( X^{(i)} \) of \( M_a \) vertices is replaced by a balanced biclique \( (X^{(i)}, Y^{(i)}) \) of \( 2M_a \) vertices, each path \( H[Y^{(i)}] \) of \( m - 1 \) vertices connected to \( x_1^{(i)} \) and \( x_1^{(i+1)} \) is replaced by a path \( H[p^{(i+1)}] \) of \( 2m \) vertices connected to \( x_p^{(i)} \) in \( X^{(i)} \) and \( y_p^{(i+1)} \) in \( Y^{(i+1)} \), and the path \( H[Z] \) of \( 3m^2(m - 1) - (3m - 1)(m - 1) \) vertices connected to \( x_1^{(3m)} \) is replaced by a path \( H[Q] \) of \( 6m^3 - 12m^2 + 2m \) vertices connected to \( x_q^{(3m)} \).

In \( G \), each clique \( U^{(i)} \) of \( M_b \) vertices is replaced by a balanced biclique \( (U^{(i)}, Y^{(i)}) \) of \( 2M_b \) vertices, each clique \( T^{(i)} \) of \( 3m^2 \) vertices is replaced by a balanced biclique \( (T_X^{(i)}, T_Y^{(i)}) \) of \( 6m^2 \) vertices, and \( T_X^{(i)} \cup U^{(i)} \cup T_X^{(i)} \cup T_Y^{(i)} \cup U_Y^{(i)} \) is a biclique for \( i \in \{1, \ldots, m\} \), where \( T_X^{(i)} = T_Y^{(i)} = T_X^{(m)} = T_Y^{(m)} = \emptyset \).

If 3-PARTITION has a solution then \( H \) is subgraph-isomorphic to \( G \). Let \( A^{(1)}, \ldots, A^{(m)} \) be a solution of 3-PARTITION. Similar to the case of proper interval graphs, we construct a bijective map \( \eta \) from \( V_H \) to \( V_G \) as follows: \( \eta \) maps each \( A^{(i)} \) to \( U^{(i)} \) and \( A^{(i+1)} \) to \( Y^{(i)} \); \( \eta \) maps each \( p^{(i+1)} \) with \( j \in A^{(k)} \) and \( j + 1 \in A^{(k')} \) to a subset of \( \bigcup_{k=1}^{k'} (T_X^{(i)} \cup T_Y^{(i)}) \); \( \eta \) maps \( Q \) to the remaining vertices \( \bigcup_{j=1}^{3m-1} (T_X^{(i)} \cup T_Y^{(i)}) \setminus \eta(\bigcup_{j=1}^{3m-1} p^{(j+1)}) \). Like the inequality (1), the following is the key inequality for showing the existence of such a map:

\[
\frac{1}{2} \left| \bigcup_{j=1}^{3m-1} p^{(j+1)} \right| = (3m - 1)m \leq 3m^2 - 2 = |T_X^{(i)}| - 2 = |T_Y^{(i)}| - 2.
\]

If \( H \) is subgraph-isomorphic to \( G \) then 3-PARTITION has a solution. Similar to the case of proper interval graphs, let \( \eta \) be a subgraph-isomorphism from \( H \) to \( G \), and let \( A^{(i)} = \{ j \mid \eta(X^{(i)} \cup Y^{(i)}) \cap (U_X^{(i)} \cup U_Y^{(i)}) \neq \emptyset \} \). The remaining steps are almost
the same with the facts \((X^{(i)}, Y^{(i)})\) is a biclique for each \(j\), there is no edge between \(U^{(i)}_X \cup U^{(i)}_Y\) and \(U^{(i)}_X \cup U^{(i)}_Y\) if \(i \neq i',\)
\(|X^{(i)} \cup Y^{(i)}| > |(T^{(i)}_X \cup T^{(i)}_X) \cup (T^{(i)}_Y \cup T^{(i)}_Y)|\) for any \(i \text{ and } j\), and \(|\bigcup_{i \in A^{(i)}} X^{(i)} \cup Y^{(i)}| \leq |(T^{(i)}_X \cup T^{(i)}_X) \cup (T^{(i)}_Y \cup T^{(i)}_Y)|\)
for any \(i\). □

2.3. Trivially perfect graphs

A graph is \textit{trivially perfect} if the size of the maximum independent set is equal to the number of maximal cliques for every induced subgraph [11]. Trivially perfect graphs can be characterized as the intersection graphs of families of nested, i.e., non-overlapping, closed intervals of the real line [4].

\textbf{Theorem 3.} SPANNING SUBGRAPH ISOMORPHISM is \textit{NP-complete} for trivially perfect graphs.

\textbf{Proof.} Let \((a_1, \ldots, a_{3m}; B)\) be an instance of 3-PARTITION. We construct trivially perfect graphs \(G\) and \(H\) in polynomial time as follows.

A vertex of a graph is \textit{universal} if it is adjacent to every other vertex in the graph. The graph \(H = (V_H, E_H)\) consists of a universal vertex \(u\) and 3m disjoint cliques \(X^{(i)}, 1 \leq i \leq 3m\), each of which has \(a_i\) vertices. The graph \(G = (V_G, E_G)\) consists of a universal vertex \(v\) and \(m\) disjoint cliques \(U^{(i)}, 1 \leq i \leq m\), of the same size \(B\). Fig. 6 shows representations of \(H\) and \(G\) by nested intervals, and hence \(H\) and \(G\) are trivially perfect. It is not difficult to see that \(|V_G| = |V_H| = mB + 1\) and \(|E_G| > |E_H|\). This reduction can be done in polynomial time. A subgraph-isomorphism \(\eta\) from \(H\) to \(G\), if any, must satisfy \(\eta(u) = v\) because \(d(u) > d(w)\) for any \(w \in V_G \setminus \{v\}\). Now, it is not difficult to see that an embedding \(H[X^{(i)}]\) into \(G[U^{(i)}]\) corresponds to \(j \in A^{(i)}\) in 3-PARTITION. We omit the detail. □

3. More on the hardness of \textbf{SPANNING SUBGRAPH ISOMORPHISM}

In this section, we consider the cases where the base graph has a very simple structure. More precisely, we study the problems for deciding whether a connected graph \(H\) is subgraph-isomorphic to a connected graph \(G\), where \(G\) is a chain graph, a cochain graph, or a threshold graph. As we will show, the problems are \textit{NP}-hard even if \(H\) has some restrictions.

3.1. Definitions of subclasses

Let \(G = (V, E)\) be a (not necessarily bipartite) graph with a partition \((X, Y)\) of \(V\). An ordering \((x_1, x_2, \ldots, x_p)\) on \(X\) is an \textit{inclusion ordering} if

\[ N_C(x_1) \cap Y \supseteq \cdots \supseteq N_C(x_p) \cap Y. \]

It is easy to see that if there exists such an ordering on \(X\), then there exists an ordering \((y_1, y_2, \ldots, y_q)\), also called an \textit{inclusion ordering}, on \(Y\) such that

\[ N_C(y_1) \cap X \supseteq \cdots \supseteq N_C(y_q) \cap X. \]

Threshold graphs, chain graphs, and cochain graphs are defined with inclusion orderings as follows. A graph is a \textit{threshold graph} if its vertex set can be partitioned into \(X\) and \(Y\) so that \(X\) is a clique, \(Y\) is an independent set, and \(X\) has an inclusion ordering [11]. A bipartite graph \(G = (X, Y; E)\) is a \textit{chain graph} if there is an inclusion ordering on \(X\) [4]. A graph is a \textit{cochain graph} if its complement is a chain graph; in other words, a graph is a cochain graph if its vertex set can be partitioned into \(X\) and \(Y\) so that \(X\) and \(Y\) are cliques and \(X\) has an inclusion ordering. We write \(G = (X, Y; E)\) if its complement is a chain graph \(G = (X, Y; \overline{E})\).

3.2. Hardness results

Throughout this subsection, let \(I\) be an instance \((a_1, \ldots, a_{3m}; B)\) of 3-PARTITION. Two bipartite graphs \(G_I = (X_I, Y_I; E_I)\) and \(H_I = (X_{H_I}, Y_{H_I}; E_{H_I})\) are defined as follows (see Fig. 7). Both sets \(X_I\) and \(Y_I\) of \(G\) consist of \(m\) disjoint parts \(X_I^{(1)} , \ldots, X_I^{(m)}\)
and \(Y_I^{(1)} , \ldots, Y_I^{(m)}\) such that each part is of size \(B\), and two vertices are adjacent if and only if one is in \(X_I^{(j)}\) and the other
is in $Y_G^{(i)}$ for some $i$ and $j$ with $i \geq j$. Similarly, both sets $X_H$ and $Y_H$ of $H$ consist of $3m$ disjoint parts $X_H^{(1)}, \ldots, X_H^{(3m)}$ and $Y_H^{(1)}, \ldots, Y_H^{(3m)}$ such that $|X_H^{(i)}| = |Y_H^{(i)}| = a_i$ for each $i$, and two vertices are adjacent if and only if one is in $X_H^{(i)}$ and the other is in $Y_G^{(j)}$ for some $j$.

**Lemma 4.** $I$ is a yes-instance of 3-PARTITION if and only if there exists a subgraph-isomorphism $\eta$ from $H_i$ to $G_i$ such that $\eta(X_{G_i}) = X_G$ (and thus $\eta(Y_{H_i}) = Y_G$).

**Proof.** To prove the only-if part, let $A(1), \ldots, A(m)$ be a partition with required properties. Let $\eta : V(H_i) \rightarrow V(G_i)$ be any mapping satisfying $\eta \left( \bigcup_{j \in A(i)} X_{H_i}^{(j)} \right) = X_{G_i}^{(i)}$ and $\eta \left( \bigcup_{j \in A(i)} Y_{H_i}^{(j)} \right) = Y_{G_i}^{(i)}$ for each $i$. Such a mapping exists since $|\bigcup_{j \in A(i)} X_{H_i}^{(j)}| = |\bigcup_{j \in A(i)} Y_{H_i}^{(j)}| = \sum_{j \in A(i)} a_j = B$ for each $i$. It is easy to see that $\eta$ is a subgraph-isomorphism with $\eta(X_{H_i}) = X_G$.

Next we prove the if part. Assume that $\eta$ is a subgraph-isomorphism from $H_i$ to $G_i$ such that $\eta(X_{H_i}) = X_G$. Let $S$ denote the set $\{ j \mid \eta(X_{H_i}^{(j)}) \cap X_G^{(i)} \neq \emptyset \}$. Hence $\eta \left( \bigcup_{j \in S} X_{H_i}^{(j)} \right) \supseteq X_G^{(i)}$, and thus $\sum_{j \in S} a_j \geq B$. On the other hand, if $\eta(v) \in X_G^{(i)}$ for some vertex $v \in X_{H_i}^{(j)}$, then $\eta(N_H(v)) = \eta(Y_{H_i}^{(j)}) \subseteq Y_G^{(i)}$ from the definition of $G_i$. Thus $\eta \left( \bigcup_{j \in S} Y_{H_i}^{(j)} \right) \subseteq Y_G^{(i)}$ and $\sum_{j \in S} a_j \leq B$. Now we have $\sum_{j \in S} a_j = B$. These discussions imply that $\eta \left( \bigcup_{j \in S} X_{H_i}^{(j)} \right) = X_G^{(i)}$ and $\eta \left( \bigcup_{j \in S} Y_{H_i}^{(j)} \right) = Y_G^{(i)}$. We set $A(1) = S$. Next we apply the same argument to $X_G^{(i)}$ and $Y_G^{(i)}$, and obtain the set $A(2)$. Repeating this process $m$ times, we can obtain the desired partition $A(1), \ldots, A(m)$.

A bipartite graph $H = (X, Y; E)$ is a convex graph if one of $X$ and $Y$ can be ordered such that the neighborhood of each vertex in another side is consecutive in the ordering [29]. It is easy to see that every bipartite permutation graph is a convex graph.

**Theorem 5.** **Spanning Subgraph Isomorphism** is NP-complete if the base graphs are chain graphs and the pattern graphs are convex graphs.

**Proof.** We add two vertices $x_G$ and $y_G$ to $G_i$, and make $y_G$ adjacent to all vertices in $X_G$ and $x_G$. Similarly, we add two vertices $x_{H_i}$ and $y_{H_i}$ to $H_i$, and make $y_{H_i}$ adjacent to all vertices in $X_{H_i}$ and $x_{H_i}$. We call the resultant graphs $G_i'$ and $H_i'$, respectively. This reduction can be done in polynomial time. It is easy to see that $G_i'$ is a chain graph and $H_i'$ is a convex graph.

We now show that $H_i'$ is subgraph-isomorphic to $G_i$ if and only if there is a subgraph-isomorphism $\eta'$ from $H_i$ to $G_i$ such that $\eta'(X_{H_i}) = X_G$. The if part is obvious. To prove the only-if part, we assume $\eta'$ is a subgraph-isomorphism from $H_i'$ to $G_i$. Since $y_G$ and $y_{H_i}$ are the unique maximum degree vertices of degree $mB + 1$ in $G_i$ and $H_i'$, it holds that $\eta'(y_{H_i}) = y_G$. Also, it is easy to see that $\eta'(X_{H_i}) = X_G$. Thus $\eta'(X_{H_i}) = X_G$. Therefore, by restricting $\eta'$ to $V(H_i)$, we have a subgraph-isomorphism from $H_i$ to $G_i$ with $\eta(X_{H_i}) = X_G$. Now, by Lemma 4, $I$ is a yes-instance if and only if $H_i'$ is subgraph-isomorphic to $G_i'$. This proves the NP-hardness.

A split graph is a graph whose vertex set can be partitioned into a clique and an independent set. Clearly, every threshold graph is a split graph, and every split graph is a chordal graph.

**Theorem 6.** **Spanning Subgraph Isomorphism** is NP-complete if the base graphs are threshold graphs and the pattern graphs are split graphs.

**Proof.** We add a set $X_G^{(0)}$ of $2B$ vertices to $G_i$, and make $X_G^{(0)} \cup X_G$ to be a clique. Similarly, we add a set $X_{H_i}^{(0)}$ of $2B$ vertices to $H_i$, and make $X_{H_i}^{(0)} \cup X_{H_i}$ to be a clique. We call the resultant graphs $G_i''$ and $H_i''$, respectively. From the construction, $G_i''$ is a threshold graph and $H_i''$ is a split graph. By a simple argument similar to one in the proof of Theorem 5, we can show that $H_i''$ is subgraph-isomorphic to $G_i''$ if and only if there is a subgraph-isomorphism $\eta''$ from $H_i$ to $G_i$ such that $\eta''(X_{H_i}) = X_G$. To see this, observe that every subgraph-isomorphism maps $X_G^{(0)} \cup X_G$ to $X_{H_i}^{(0)} \cup X_{H_i}$, since they are the unique maximum cliques of size $(m + 1)B$ in $G_i''$ and $H_i''$. The remaining steps are almost the same.
A graph is cobipartite if its complement is bipartite. From the definition, every cochain graph is cobipartite.

**Theorem 7.** SPANNING SUBGRAPH ISOMORPHISM is NP-complete if the base graphs are cochain graphs and the pattern graphs are cobipartite graphs.

**Proof.** We construct \( G'' \) and \( H'' \) as described above, and then make \( Y_C \) and \( Y_H \) to be cliques. The new graphs are \( G'' \) and \( H'' \). Clearly, \( G'' \) is a cobipartite graph and \( H'' \) is cobipartite. It is easy to see that \( H'' \) is subgraph-isomorphic to \( G'' \) if and only if \( H'' \) is subgraph-isomorphic to \( G'' \). This implies the hardness. \( \square \)

4. Linear-time algorithms

In this section, we show that SUBGRAPH ISOMORPHISM can be solved in linear time for chain graphs, cochain graphs, and threshold graphs. Note that here we study the original problem SUBGRAPH ISOMORPHISM, and thus two input graphs may have different numbers of vertices and could be disconnected. Since SUBGRAPH ISOMORPHISM is a generalization of SPANNING SUBGRAPH ISOMORPHISM, we can conclude that SPANNING SUBGRAPH ISOMORPHISM for these graph classes can be solved in linear time as well.

For the classes of threshold graphs, of chain graphs, and of cochain graphs, linear-time certifying recognition algorithms are presented by Heggernes and Kratsch [13].

4.1. Threshold graphs

Recall that a graph is a threshold graph if its vertex set can be partitioned into \( X \) and \( Y \) so that \( X \) is a clique, \( Y \) is an independent set, and \( X \) has an inclusion ordering [11]. The degree sequence \( (d(v_1), \ldots, d(v_n)) \) of \( G \) is the nonincreasing sequence of the degree of the vertices of \( G \). Given a degree sequence, we define \( N^*(v_i) \) as follows:

\[
N^*_C(v_i) = \begin{cases} 
N_C[v_i] & \exists j > i, (v_i, v_j) \in E_C, \\
N_C(v_i) & \text{otherwise.}
\end{cases}
\]

This notation gives the following nice property.

**Lemma 8 ([11]).** Let \( G \) be a threshold graph with the vertex set \( \{v_1, \ldots, v_n\} \). If \( (d(v_1), \ldots, d(v_n)) \) is a degree sequence of \( G \), then \( N^*_C(v_i) \supseteq \cdots \supseteq N^*_C(v_n) \).

**Lemma 9.** Let \( G \) and \( H \) be threshold graphs with degree sequences \( (d(v_1), \ldots, d(v_n)) \) and \( (d(u_1), \ldots, d(u_n)) \), respectively, and \( n \geq n' \). Then \( H \) is subgraph-isomorphic to \( G \) if and only if \( d(u_i) \leq d(v_i) \) for \( i \in \{1, \ldots, n'\} \).

**Proof.** The only-if part is trivial. For the if-part, assume \( d(u_i) \leq d(v_i) \) for \( i \in \{1, \ldots, n'\} \). Let \( \eta : V_H \to V_G \) be the mapping such that \( \eta(u_i) = v_i \). Suppose \( \eta \) is not a subgraph-isomorphism. Thus there exists a pair \( (i, j) \) of indices with \( i < j \) such that \( \{u_i, u_j\} \notin E_H \) and \( \{v_i, v_j\} \notin E_C \). By **Lemma 8**, \( N^*_H(u_i) \subseteq N^*_H(u_j) \) for any \( i \in \{1, \ldots, i\} \). Since \( u_i \notin N_H(u_j) \), we have \( u_i \notin N_H(u_r) \) for any \( r \in \{1, \ldots, i\} \). Thus \( \{u_1, \ldots, u_i\} \subseteq N_H(u_j) \). On the other hand, since \( \{v_i, v_j\} \notin E_C \), there exists an index \( i' \) such that \( i' < i \) and \( \{v_1, \ldots, v_i'\} \subseteq N_C(v_j) \). This fact contradicts \( d(u_i) \leq d(v_i) \). \( \square \)

By **Lemma 9**, given two threshold graphs \( G \) and \( H \), it suffices to compare their degree sequences for solving SUBGRAPH ISOMORPHISM.

**Theorem 10.** SUBGRAPH ISOMORPHISM for threshold graphs can be solved in linear time.

4.2. Chain graphs

Recall that a bipartite graph \( G = (X, Y; E) \) is a chain graph if there is an inclusion ordering on \( X \) [4]. By the definition, a chain graph consists of a connected chain graph and a (possibly empty) set of isolated vertices.

**Lemma 11.** Let \( G = (X_G, Y_G; E_G) \) and \( H = (X_H, Y_H; E_H) \) be chain graphs, where \( |X_H| \leq |X_G| \) and \( |Y_H| \leq |Y_G| \). If \((x_1, \ldots, x_p)\) and \((y_1, \ldots, y_p)\) are inclusion orderings on \( X_G \) and \( X_H \), respectively, then there is a subgraph-isomorphism \( \eta \) from \( H \) to \( G \) such that \( \eta(x_i) \subseteq X_G \) and \( \eta(y_i) \subseteq Y_G \) if and only if \( d(x_i) \leq d(y_i) \) for \( i \in \{1, \ldots, p\} \).

**Proof.** For the only-if part, suppose that \( d(x_i) > d(y_i) \) for some \( i \in \{1, \ldots, p\} \). Since \( d(x_j) \geq d(x_i) \) for \( j \in \{1, \ldots, i\} \), it follows that \( X_H \) contains at least \( i \) vertices of degree at least \( d(x_i) \). On the other hand, since \( d(x_i) \leq d(y_j) \) for \( j \in \{1, \ldots, p\} \), there are at most \( i-1 \) vertices of degree at least \( d(x_i) \) in \( X_G \). Thus there is no subgraph-isomorphism \( \eta \) from \( H \) to \( G \) such that \( \eta(x_i) \subseteq X_G \).

For the if-part, assume that \( |X_H| \leq |X_G|, |Y_H| \leq |Y_G| \), and \( d(x_i) \leq d(x_j) \) for \( i \in \{1, \ldots, p\} \). Let \( (y_1, \ldots, y_{|Y_G|}) \) and \((y_1, \ldots, y_{|Y_H|})\) be inclusion orderings on \( Y_G \) and \( Y_H \), respectively. Let \( \eta : V_H \to V_G \) be the mapping such that \( \eta(x_i) = x_i \) and \( \eta(y_i) = y_i \). If \( \eta \) is not a subgraph-isomorphism, then there exists a pair \( (i, j) \) of indices such that \( (x_i, y_j) \notin E_H \) and \( (x_i, y_j) \notin E_G \). By the definition, \( N(y_j) \subseteq N(y_i) \) if \( h < j \). Hence \( (x_i, y_j) \in E_H \) implies \( (y_i, \ldots, y_j) \notin N(x_i) \). However, \( (x_i, y_j) \notin E_G \) implies \( N(x_i) \subseteq \{y_i, \ldots, y_{j-1}\} \). Thus we have \( d(x_i) > d(x_j) \), which contradicts the assumption. \( \square \)
Theorem 12. **Subgraph Isomorphism for chain graphs can be solved in linear time.**

**Proof.** Let $G = (X_C, Y_C; E_C)$ and $H = (X_H, Y_H; E_H)$ be chain graphs. We first compute the degrees of vertices and inclusion orderings in linear time. Let $G'$ and $H'$ be the graphs obtained from $G$ and $H$, respectively, by removing all isolated vertices. Obviously, $H$ is subgraph-isomorphic to $G$ if and only if $|V(H)| \leq |V(G)|$ and $H'$ is subgraph-isomorphic to $G'$. Checking the condition $|V(H)| \leq |V(G)|$ and removing isolated vertices can be done in $O(n)$ time, where $n = |V(G)|$. Also, computing the degrees of vertices and inclusion orderings for new graphs can be done in $O(n)$ time, by using the information of the original graphs. Hence we assume $G$ and $H$ are connected in what follows.

Suppose that there is a subgraph-isomorphism $\eta$ from $H$ to $G$. Since $G$ and $H$ are connected, it follows that either $\eta(X_H) \subseteq X_C$ and $\eta(Y_H) \subseteq Y_C$, or $\eta(X_H) \subseteq Y_C$ and $\eta(Y_H) \subseteq X_C$. Otherwise, the parity of the distance between some vertices becomes invalid. The existence of such a map can be determined in $O(n)$ time by Lemma 11. □

4.3. **Cochain graphs**

Recall that a graph is a cochain graph if its complement is a chain graph. The following fact connects the concepts of subgraph-isomorphism and of complement graphs.

**Lemma 13.** If $|V_H| = |V_C|$, then $H$ is subgraph-isomorphic to $G$ if and only if $\bar{G}$ is subgraph-isomorphic to $\bar{H}$.

**Proof.** Let $\eta$ be a subgraph-isomorphism from $H$ to $G$. Now $(\eta(u), \eta(v)) \in E_C$ for any edge $(u, v)$ of $H$. In other words, $\{\eta^{-1}(u), \eta^{-1}(v)\} \notin E_H$ for any nonedge $(u, v)$ of $G$. Hence $\eta^{-1}$ is a subgraph-isomorphism from $\bar{G}$ to $\bar{H}$. The other direction can be shown in the same way. □

**Theorem 14.** **Subgraph Isomorphism** for cochain graphs can be solved in linear time.

**Proof.** Let $G = (X_C, Y_C; E_C)$ and $H = (X_H, Y_H; E_H)$ be cochain graphs. We want to find a subgraph-isomorphism from $H$ to $G$, or decide that there is no such mapping. We first compute the degrees of the vertices and inclusion orderings in linear time. Let $(x_1, \ldots, x_p)$ and $(y_1, \ldots, y_q)$ be inclusion orderings on $X_C$ and $Y_C$, respectively. Observe that if $\bar{H}$ is subgraph-isomorphic to $G$, then there exists a subgraph-isomorphism $\eta$ from $H$ to $G$ such that $\eta(X_H) = \{x_1, \ldots, x_t\} \cup \{y_1, \ldots, y_s\}$ for some $s$ and $t$, where $s + t = |X_H \cup Y_H|$. This is because of the inclusion property of neighborhoods.

We now guess $s$. Let $G' = (X_C, Y_C; E_C')$ be the subgraph of $G$ induced by $X_C \cup Y_C$, where $X_C = \{x_1, \ldots, x_t\}$ and $Y_C = \{y_1, \ldots, y_s\}$. Since we know the inclusion orderings and the degrees of vertices of the original graphs, we can compute this same information for the new graphs in $O(n)$ time. By Lemma 13, $H$ is subgraph-isomorphic to $G'$ if and only if $\bar{G}'$ is subgraph-isomorphic to $\bar{H}$. Now we use the algorithm for chain graphs presented above, since $\bar{G}'$ and $\bar{H}$ are chain graphs. Observe that in the algorithm for chain graphs, all the steps except for the computation of the degrees and inclusion orderings can be done in $O(n)$ time. Since we already know this information, we can check whether $\bar{G}'$ is subgraph-isomorphic to $\bar{H}$ in $O(n)$ time. We have $O(n)$ possible guesses for $s$. For each guess, we take $O(n)$ time. Thus this phase can be done in $O(n^2)$ time. Since the cochain graph has $\Omega(n^2)$ edges, this is still linear time. □

5. **Future work**

We showed several hardness and tractability results. For example, we showed that **Spanning Subgraph Isomorphism** is NP-complete for proper interval graphs, and that **Subgraph Isomorphism** is linear-time solvable for cochain graphs. An interesting open problem is the complexity of **Subgraph Isomorphism** where the base graph $G$ is a proper interval graph and the pattern graph $H$ is a cochain graph. Also it is interesting to study the case where $G$ is a bipartite permutation graph and $H$ is a chain graph, and the case where $G$ is a trivially perfect graph and $H$ is a threshold graph. The answers for these questions would provide more sharp contrasts of the complexity of **Subgraph Isomorphism** in graph classes.

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