Determining Merged Relative Scores

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1. INTRODUCTION

Our starting point in this paper is the following type of situation. Performances of people (students, workers, job candidates, etc.) or technologies (for instance computer systems) are measured by their scores with respect to different benchmarks relative to those of some base performance (say that of a “typical” worker, student, or of the presently used technology). In order to arrive at a decision we want to merge (average, aggregate) the scores of each individual or technology into a single number which can then be compared with that of another individual or technology. It has been pointed out recently (e.g. [7, 9]) that the use of arithmetic means for averaging can lead to inappropriate conclusions.

The situation is similar, at least technically, to price indices where present prices of different products are considered relative to prices of these products in a base year and we again want to merge these into one number.

In order to find appropriate merging functions, Roberts [9] listed a number of properties one can reasonably expect from such functions, proved theorems establishing the relative strength of these properties as postulates, and then posed the open problem to determine, for each such property, all merging functions having that property (in one case he answered such a question).

In this paper we solve most of these problems and use the results to find alternative proofs and generalizations of some theorems in [9]. We also state a few problems which are still open.

2. NOTATIONS, DEFINITIONS, AND EXPLANATIONS

We unite the n measures of performances (or prices) into vectors written bold face, \( \mathbf{x} = (x_1, \ldots, x_n) \), for the actual performances (or prices),
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$z = (z_1, ..., z_n)$ for the base performances (prices); $x_k, z_k$ ($k = 1, 2, ..., n$) are, not unreasonably, supposed to be positive numbers, so $x \in \mathbb{R}_+^n$, $z \in \mathbb{R}_+^n$. The merged score is denoted by

$$U(x, z) = U(x_1, ..., x_n, z_1, ..., z_n)$$

and $U$ (v in [9]) is called a generalized relative merging function (GRMF). Often the merged score depends only upon $(x_1/z_1, ..., x_n/z_n) =: x/z$, so $U(x, z) = u(x/z)$. In this case $u$ is called a relative merging function (RMF; $x_1/z_1, ..., x_n/z_n$ are the relative scores). It is supposed that $u$ and $U$ are defined on the entire $\mathbb{R}_+^n$ or $\mathbb{R}_+^n \times \mathbb{R}_+^n$, respectively. As with $x/z$, other operations are also carried out componentwise, for instance $xy = (x_1y_1, ..., x_my_m)$. When a scalar, say $c$ (not bold face), is used in place of a vector then the vector $(c, c, ..., c)$ is meant. So $cx = (cx_1, ..., cx_n)$, in accordance with the usual notation. If not stated otherwise, it will be supposed that $u$ and $U$ are positive valued.

A GRMF is *equationally invariant* if

$$U(x, z) = U(y, z) \Rightarrow U(x, w) = U(y, w). \quad (EI)$$

Clearly for RMFs this means

$$u(p) = u(q) \Rightarrow u(rp) = u(rq). \quad (ei)$$

A GRMF is *ordinally invariant*, if

$$U(x, z) > U(y, z) \Rightarrow U(x, w) > U(y, w). \quad (OI)$$

Again, for RMFs this reduces to

$$u(p) > u(q) \Rightarrow u(xp) > u(q). \quad (oi)$$

Actually, (OI) and (oi) stand for the more exact

$$\forall (x, y) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : (\exists z \in \mathbb{R}_+^n : U(x, z) > U(y, z)) \Rightarrow (\forall w \in \mathbb{R}_+^n : U(x, w) > U(y, w))$$

or

$$\forall (p, q) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : [u(p) > u(q) \Rightarrow (\forall r \in \mathbb{R}_+^n : u(rp) > u(rq))],$$

respectively, and similarly for (EI), (ei) and for the definitions which follow. Clearly $>$ in (OI) or (oi) can be replaced by $<$ and, as a consequence,

$$(EI) \Rightarrow (EI), \quad (oi) \Rightarrow (ei). \quad (1)$$
Here and in what follows we give interpretation (explanation, motivation) of our definitions either for merged relative performance scores or for price indices. Ordinal (equational) invariance means the following, for instance, for price indices. If, relative to year $C$, the price index in year $A$ is greater than (equal to) the price index in year $B$, then the same is true relative to year $D$.

A GRMF is *equationally interval invariant* if

$$U(x, z) - U(s, z) = U(y, z) - U(t, z) \Rightarrow U(x, w) - U(s, w)$$

$$= U(y, w) - U(t, w),$$

(EII)

which for RMFs reduces to

$$u(p) - u(P) = u(q) - u(Q) \Rightarrow u(rp) - u(rP = u(rq) - (rQ).$$

(eii)

Similarly, *ordinal interval invariance* means for GRMFs or RMFs

$$U(x, z) - U(s, z) > U(y, z) - U(t, z)$$

$$\Rightarrow U(x, w) - U(s, w) > U(y, w) - U(t, w)$$

(OII)

and

$$u(p) - u(P) > u(q) - u(Q) \Rightarrow u(rp) - u(rP > u(rq) - u(rQ),$$

(oi)

respectively. Again

$$(OII) \Rightarrow (EII), \quad (oi) \Rightarrow (eii).$$

(2)

Ordinal (equational) interval invariance can be interpreted for merged performance scores as follows. If the difference between merged scores of two performances is greater than (equal to) the difference between two others relative to one basic performance, then the same holds relative to any other basic performances.

*Ratio invariance* for GRMFs and RMFs means that, for $c \in \mathbb{R}_+$,

$$U(x, z) = cU(y, z) \Rightarrow U(x, w) = cU(y, w)$$

(RI)

and

$$u(p) = cu(q) \Rightarrow u(rp) = cu(rq),$$

(ri)

respectively. Roberts' definition of *ratio interval invariance* includes ordinal invariance and, for any $c > 0$,

$$U(x, z) - U(s, z) = c[U(y, z) - U(t, z)] \Rightarrow U(x, w) - U(s, w)$$

$$= c[U(y, w) - U(t, w)].$$

(RII)

$$u(p) - u(P) = c[u(q) - u(Q)] \Rightarrow u(rp) - u(rP) = c[u(rq) - u(rQ)].$$

(rii)
For interpretations, just replace in the above interpretations of ordinal (equational) invariance and interval invariance "greater (equal)" by "c times greater."

Since (EII), (eii), (OII), (oii), (RII), (rii) contain only differences of merged relative scores, we do not suppose in these cases that U and u are positive valued.

Following the established terminology for price indices in economics (e.g. [5, p. 156], "circularity test"), circularity is defined for GRMFs by

\[ U(x, y) U(y, z) = U(x, z) \]

(the price index in year A relative to year C equals the price index in year A relative to year B times the price index in year B relative to year C). Interestingly, circularity for RMFs reduces to multiplicativity:

\[ u(sw) = u(s)u(w). \]

The technical postulate of generalized multiplicativity has been used (see e.g. [5, p. 141], "multiplicativity test") for price levels (not price indices). For RMFs it is defined by

\[ u(sw) = v(s)u(w) \]

(clearly a generalization of (m)). We extend the definition to GRMFs:

\[ U(rx, tz) = R(r, t) U(x, z). \]

Note that if, according to our definition, \( U(x, z) = u(x/z) \) (RMFs), then (GM) becomes

\[ u[(r/t)(x/z)] = R(r, t)u(x/z). \]

Putting here \( z = x \) gives \( 1 = (1, ..., 1) \)

\[ R(r, t) = u(r/t)/u(1) = v(r/t) \]

so that \( R \) necessarily can depend only upon \( r/t \) and, from (3) and (4) we get (gm). However, while (gm) is a generalization of (m) for RMFs, the generalized multiplicativity (GM) is unrelated to (C), which corresponded to (m) for GRMFs [9].

The remaining conditions are quite weak and will be used in addition to some of the previous ones.

For GRMFs linear homogeneity with respect to the first vector means

\[ U(\rho x, z) = \rho U(x, z), \quad \text{for all } \rho > 0, \quad x, z \in \mathbb{R}^n \]

(LH)
while homogeneity of degree $-1$ with respect to the second vector means, of course,

$$U(x, \rho z) = \rho^{-1}U(x, z), \quad \text{for all } \rho > 0, \ x, z \in \mathbb{R}_+^n. \quad (H^{-1})$$

In the case of RMFs both reduce to the linear homogeneity

$$u(\rho p) = \rho u(p) \quad \text{for all } \rho > 0, \ p \in \mathbb{R}_+^n. \quad (1h)$$

The following is an interpretation of (LH) and of $(H^{-1})$ for price indices. If all prices in year $A$ are multiplied by $\rho$ while those in year $B$ are unchanged or the latter are multiplied by $(1/\rho)$ while the former remain unchanged then the price index of year $A$ relative to year $B$ is also multiplied by $\rho$.

Proportionality (cf. [5, p. 155], “proportionality test”) is defined for GRMFs by

$$U(\rho x, x) = \rho \quad (P)$$

(for all $\rho \in \mathbb{R}_+, \ x \in \mathbb{R}_+^n$). A particular case ($\rho = 1$) is the identity requirement

$$U(x, x) = 1. \quad (I)$$

Interpretation: If the performance scores with respect to all benchmarks are $\rho$ times the scores of the base performance, then the merged relative score is $\rho$. In particular, if each score equals the respective base score, then the merged relative score is 1. For RMFs, $(P)$ reduces to the agreement property

$$u(\rho) = u(\rho, ..., \rho) = \rho \quad (a)$$

(if all relative scores agree, $x_1/z_1 = x_2/z_2 = \cdots = x_n/z_n = \rho$, then the merged relative score agrees with them and is also $\rho$). Note that

$$[[\text{(LH) and (I)}] \Rightarrow (P)], \quad [[(H^{-1}) \text{ and (I)}] \Rightarrow (P)], \quad \quad \quad [[(m) \text{ and (a)}] \Rightarrow (lh)], \quad [[(m) \text{ and (lh)}] \Rightarrow (a)]$$

$$[U(\rho x, x) = \rho U(x, x) = \rho, \quad \text{or, with } y = \rho x, \ U(\rho x, x) = U(y, (1/\rho)y) = \rho U(y, y) = \rho; \ u(\rho p) = u(\rho)u(p) = \rho u(p); \ \rho u(p) = u(\rho p) = u(\rho)u(p)].$$

These were somewhat restrictive but still quite reasonable postulates. The symmetry conditions

$$U(x_1, ..., x_n, z_1, ..., z_n) = U(x_{\pi(1)}, ..., x_{\pi(n)}, z_{\pi(1)}, ..., z_{\pi(n)}) \quad (S)$$

$$u(p_1, ..., p_n) = u(p_{\pi(1)}, ..., p_{\pi(n)}) \quad (s)$$
for all permutations \( \pi \) of \( \{1, \ldots, n\} \), though supposed in [7] and (in some places) in [9], may be more controversial: It would mean that interchanging scores on benchmarks (or prices of goods) should not change the merged scores. However, the benchmarks (goods) may not be equally important. Nevertheless, this assumption has been used as "first approximation."

3. Determining All Merging and Generalized Merging Functions Satisfying a Single Postulate

**Proposition 1.** The general circular (C) GRMFs are given by

\[
U(x, y) = \frac{F(x)}{F(y)},
\]

where the function \( F: \mathbb{R}_+^n \to \mathbb{R}_+ \) is arbitrary. As a consequence, (C) \( \Rightarrow \) (I), the identity property.

**Proof** (cf., e.g., [1, 223–225]). Putting \( z = z_0 \) (constant) and \( F(x) := U(x, z_0) \) into (C), we get (6)

\[
U(x, y) = U(x, z_0)/U(y, z_0) = F(x)/F(y).
\]

Conversely, (C) is satisfied by (6), whatever \( F \) is. Also \( U(x, x) = F(x)/F(x) = 1 \), as asserted. 1

The following proposition is contained in [9]. We repeat it here for completeness and for the sake of its consequences.

**Proposition 2.** The general (GM) generalized multiplicative GRMFs are given by

\[
U(x, z) = aM(x) \tilde{M}(z),
\]

where \( a > 0 \) is an arbitrary constant and \( M, \tilde{M}: \mathbb{R}_+^n \to \mathbb{R}_+ \) are arbitrary positive valued multiplicative functions:

\[
M(xy) = M(x)M(y), \quad \tilde{M}(xy) = \tilde{M}(x)\tilde{M}(y) \quad \text{for all} \quad x, y \in \mathbb{R}_+^n.
\]

**Proof.** Choosing \( x = z = 1 = (1, \ldots, 1) \), \( a := U(1, 1) > 0 \) in (GM), we get

\[
U(r, t) = aR(r, t)
\]

and putting this back into ((GM))

\[
R(rx, tz) = R(r, t)R(x, z).
\]
Consequently,
\[ R(r, z) = R(r, 1)R(1, z) = M(r)\tilde{M}(z) \]  \hspace{1cm} (10)

and
\[ M(rs) = R(r, 1)R(s, 1) = M(r)M(s), \]
\[ \tilde{M}(xy) = R(1, x)R(1, y) = \tilde{M}(x)\tilde{M}(y), \]
which is (8). On the other hand, (10) and (9) give (7).

Conversely, (7) satisfies \((GM)\) if \(M\) and \(\tilde{M}\) satisfy (8).

The following can be obtained from Proposition 2 or directly by a reduction of the above argument to this special case.

**Corollary 3.** The general \((gm)\) generalized multiplicative RMFs are given by

\[ u(p) = aM(p) \]

with arbitrary constant \(a > 0\) and arbitrary positive valued multiplicative \(M\).

This follows also from [4], where \((gm)\) is listed as Case 4.

In what follows, multiplicative functions will always be denoted by \(M\). Note that the general multiplicative \((m)\) RMFs can also be written as

\[ u(p) = M(p) = M(p_1, 1, ..., 1)M(1, p_2, 1, ..., 1) \cdots M(1, ..., 1, p_n) \]
\[ = M_1(p_1)M_2(p_2) \cdots M_n(p_n), \]  \hspace{1cm} (11)

where \(M_k: \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) \((k = 1, ..., n)\) are multiplicative functions. We also have (cf. [1, pp. 31-35, 37-42, 213-216; 2, pp. 12-15, 26-28, 55-57]) the following.

**Corollary 4.** The general continuous (at a point; or just bounded from above on an \(n\)-dimensional proper interval or set of positive measure) generalized multiplicative GRMFs and RMFs \(((GM), (gm))\) are given by

\[ U(x, y) = U(x_1, ..., x_n, y_1, ..., y_n) = a \prod_{k=1}^{n} (x_k^{b_k} y_k^{c_k}) \]

or

\[ u(p) = u(p_1, ..., p_n) = a \prod_{k=1}^{n} p_k^{b_k} \]  \hspace{1cm} (12)
respectively, while the general locally continuous (locally bounded) multiplicative (m) RMFs are

\[ u(p) = \prod_{k=1}^{n} p_k^{b_k}, \]

where \( a > 0, b_1, \ldots, b_n, c_1, \ldots, c_n \) are arbitrary constants.

(Locally continuous will mean continuous at a point, locally bounded will stand for bounded on a proper \( n \)-dimensional interval or on a set of positive \( n \)-dimensional measure).

Now we start solving the problems posed by Roberts [9].

**Theorem 5.** The general ratio invariant (RI) GRMFs are given by

\[ U(x, z) = F(x)G(z), \]

where \( F, G : \mathbb{R}^n_+ \to \mathbb{R}_+ \) are arbitrary functions.

**Proof.** Put \( y = y_0, z = z_0 \) into (RI). The implication

\[ U(x, z_0) = cU(y_0, z_0) \Rightarrow U(x, w) = cU(y_0, w) \]

can be written as

\[ U(x, w) = \frac{U(x, z_0)}{U(y_0, z_0)} U(y_0, w) = F(x)G(w), \]

that is, (13). Conversely, (13) satisfies (RI) with arbitrary \( F, G : \mathbb{R}^n_+ \to \mathbb{R}_+ : \)

\[ U(x, z) = cU(y, z) \Leftrightarrow F(x)G(z) = cF(y)G(z) \Leftrightarrow F(x) = cF(y) \]
\[ \Rightarrow F(x)G(w) = cF(y)G(w) \Leftrightarrow U(x, w) = cU(y, w). \]

Comparison with Propositions 1 and 2 gives

\[ (C) \Rightarrow (RI), (GM) \Rightarrow (RI) \quad \text{but} \quad (RI) \not\Rightarrow (C), (RI) \not\Rightarrow (GM). \]

For RMFs we have the following.

**Corollary 6.** The general ratio invariant (ri) RMFs are given by

\[ u(p) = aM(p) = a \prod_{k=1}^{n} M_k(p_k), \]

where \( a > 0 \) is an arbitrary constant and \( M : \mathbb{R}^n_+ \to \mathbb{R}_+, M_k : \mathbb{R}_+ \to \mathbb{R}_+ \)
\((k = 1, \ldots, n)\) are arbitrary positive valued multiplicative functions.
**Proof.** By (13) and the definition of RMFs

\[ u(x/z) = F(x)G(z) \]  

must hold (cf. [1, pp. 141–143]). The substitutions \( z = 1, G(1) = \alpha \) or \( x = 1, F(1) = \beta \) give

\[ F(x) = \frac{1}{\alpha} u(x) \quad \text{and} \quad G(z) = \frac{1}{\beta} u(1/z), \]

respectively, so that (16) becomes, with \( y = 1/z \),

\[ u(xy) = \frac{1}{\alpha \beta} u(x) u(y). \]

Therefore \( M(p) := (1/\alpha \beta)u(p) \) is multiplicative which, with \( a = \alpha \beta \), proves (15).

Moreover, (15) with arbitrary positive valued \( M, M_1, \ldots, M_n \), and \( a > 0 \) satisfies (ri).

The result in Corollary 6 has been proved also in [9, Theorem 6]. We give here a third proof which is shorter than either of the previous two, by reduction to the simple Corollary 3 (or to Case 4 of [4]).

**Shorter Proof of Corollary 6.** Put into (ri) \( q = 1 \):

\[ u(p) = cu(1) \Rightarrow u(rp) = cu(r). \]

This can be written as

\[ u(rp) = \frac{u(p)}{u(1)} u(r) = v(p)u(r), \]

which is the generalized multiplicativity \( \text{(gm)} \) for RMFs. Corollary 3 then proves (15) and the converse is again obvious.

We get also from Corollaries 3 and 6 a slight improvement of the result \( \text{(gm)} \Rightarrow \text{(ri)} \) in [9], namely that

\[ (\text{gm}) \Leftrightarrow \text{(ri)}. \]  

(17)

This shows also that the locally bounded or continuous ratio invariant \( \text{(ri)} \) RMFs are given by (12).

**Theorem 7.** The general ratio interval invariant \( \text{(RII)} \) GRMFs, without supposing \( U > 0 \) or \( \text{(OI)} \) additionally, are given by

\[ U(x, z) = F(x)G(z) + H(z), \]  

(18)
where the functions $F, H: \mathbb{R}^n_+ \to \mathbb{R}, G: \mathbb{R}^n_+ \to \mathbb{R} \setminus \{0\}$ (the nonzero reals) are arbitrary.

Proof. If $U(x, z)$ is independent of $x$ for all $z$ then (RII) is trivially satisfied ($0 = 0 \Rightarrow 0 = 0$). This is contained in (18) with $F(x) \equiv 0$. Otherwise there exists a $z_0$ such that $U(\cdot, z_0)$ is not constant. Choose $a, b \in \mathbb{R}^n_+$ so that

$$U(a, z_0) > U(b, z_0).$$

Fix also an arbitrary $s_0 \in \mathbb{R}^n_+$. All $x \in \mathbb{R}^n_+$ are in one of the following three sets

$$S_+ := \{x \mid U(x, z_0) > U(s_0, z_0)\}, \quad S_- := \{x \mid U(x, z_0) < U(s_0, z_0)\}$$

$$S_0 := \{x \mid U(x, z_0) = U(s_0, z_0)\}.$$

If $(\alpha)x \in S_+$ or $(\beta)x \in S_-$ or $(\gamma)x \in S_0$, choose in (RII) $y = a$, $t = b$ or $y = b$, $t = a$ or $y = t = a$, respectively. In case $(\alpha)$, $x \in S_+$,

$$U(x, z_0) - U(s_0, z_0) = c(U(a, z_0) - U(b, z_0))$$

$$\Rightarrow U(x, w) - U(s_0, w) = c[U(a, w) - U(b, w)],$$

therefore,

$$U(x, w) = \frac{U(x, z_0) - U(s_0, z_0)}{U(a, z_0) - U(b, z_0)} [U(a, w) - U(b, w)] + U(s_0, w)$$

that is

$$U(x, w) = F(x)G(w) + H(w),$$

which is (18). On the other hand, in case $(\beta)$, when $x \in S_-$, we have

$$U(x, z_0) - U(s_0, z_0) = c[U(b, z_0) - U(a, z_0)]$$

$$\Rightarrow U(x, w) - U(s_0, w) = c[U(b, w) - U(a, w)]$$

(note that in both cases $c > 0$), so that

$$U(x, w) = \frac{U(x, z_0) - U(s_0, z_0)}{U(b, z_0) - U(a, z_0)} [U(b, w) - U(a, w)] + U(s_0, w)$$

$$= \frac{U(x, z_0) - U(s_0, z_0)}{U(a, z_0) - U(b, z_0)} [U(a, w) - U(b, w)] + U(s_0, w)$$

$$= F(x)G(w) + H(w)$$

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which is again (18). Note that in both cases $F$, $G$, $H$ are the same:

$$F(x) = \frac{U(x, z_0) - U(s_0, z_0)}{U(a, z_0) - U(b, z_0)},$$

$$G(w) = U(a, w) - U(b, w), \quad H(w) = U(s_0, w).$$

Finally, in case $(y)$, $x \in S_-$, (RII) reduces to

$$U(x, z_0) - U(s_0, z_0) = 0 = c[U(a, z_0) - U(a, z_0)]$$

$$\Rightarrow U(x, w) - U(s_0, w) = c[U(a, w) - U(a, w)] = 0,$$

so

$$U(x, w) = U(s_0, w) = H(w)$$

with the same $H$ as before but $F(x) = 0$ for all $x \in S_-$. Conversely, (18) satisfies (RII), whatever $F, H : W^+ \to [w]$ are, but only if $G$ is nowhere 0: If (18) holds, then

$$U(x, z) - U(s, z) = c[U(y, z) - U(t, z)] = [F(x) - F(s)] G(z)$$

$$= c[F(y) - F(t)] G(z).$$

If (and only if) $G$ is nowhere 0 does it follow for any particular $z$ (which stood on the left-hand side of (RII)), that $F(x) - F(s) = c[F(y) - F(t)]$ and so $[F(x) - F(s)] G(w) = c[F(y) - F(t)] G(w)$, that is, $U(x, w) - U(s, w) = c[U(y, w) - U(t, w)]$.  

Comparison with Theorem 5 gives

(RI) ⇒ (RII) but (RII) ≠ (RI).  

(19)

The next theorem, on RMF's, could again be proved with the aid of Theorem 7, but we give here an independent proof.

**Theorem 8.** The general ratio interval invariant (rii) RMFs (again without supposing $u > 0$ or (oi)) are

$$u(p) = aM(p) + b$$

and

$$u(p) = L(p) + b,$$

where $a \neq 0$, $b$ are arbitrary constants, $M : \mathbb{R}_+^n \to \mathbb{R}_+$ an arbitrary multiplicative function, and $L : \mathbb{R}_+^n \to \mathbb{R}$ is logarithmic, that is, an arbitrary solution of

$$L(pq) = L(p) + L(q) \quad (p, q \in \mathbb{R}_+^n).$$

(22)
Proof. If \( u \) is constant then (rii) is trivially satisfied and this is contained in (21) with \( L(p) \equiv 0 \).

If \( u \) is not constant, then there exist \( q_0, Q_0 \) such that \( u(q_0) \neq u(Q_0) \). We put into (rii) \( q = q_0, Q = Q_0, \) and \( P = P_0 \) (any constant):

\[
u(rp) = \frac{u(p) - u(P_0)}{u(q_0) - u(Q_0)} \left[u(rq_0) - u(rQ_0)\right] + u(rP_0)] = R(r)u(p) + S(r).
\]

This is Case 3 in [4]. There \( R(r) > 0 \) was supposed but the proof does not change if this supposition is dropped and neither does the result, which is exactly (20) and (21) with (22), except that \( M(r) > 0 \) does not follow from the supposition \( R(r) > 0 \). But \( M \) is either everywhere or nowhere 0 (if \( M(r_0) = 0 \) then \( M(r) = M([r/r_0]r_0) = M(r/r_0)M(r_0) = 0 \) for all \( r \in R^+ \)) and \( M(r) \equiv 0 \) gives again \( u = \) constant and so does \( a = 0 \), so we can suppose that \( a \neq 0 \) and \( M \) is nowhere 0. But then \( M(r) = M([\sqrt{r_1}^2, ..., \sqrt{r_n}^2]) = M(\sqrt{r_1}, ..., \sqrt{r_n})^2 > 0 \). So only those given in Theorem 8 can be solutions of (rii).

Conversely, it is easy to check that both (20) and (21) with any multiplicative \( M \), logarithmic \( L \), and with any \( a \neq 0, b \), satisfy (rii).

Comparison with Corollary 6 and (17) yields

\[(m) \Rightarrow (gm) \Leftrightarrow (ri) \Rightarrow (rii) \quad \text{but (rii)} \neq (ri), (rii) \neq (gm). \quad (23)\]

If the ratio interval invariant (rii) RMF \( u \) is supposed to be locally continuous or locally bounded, then (see [4]; [2, pp. 52-57])

\[
u(p) = u(p_1, ..., p_n) = a \prod_{k=1}^{n} p_k^{c_k} + b \quad (24a)
\]

or

\[
u(p) = u(p_1, ..., p_n) = \sum_{k=1}^{n} c_k \log p_k + b, \quad (24b)
\]

where \( a, b, c_1, ..., c_n \) are arbitrary constants. This follows from the general regular (locally continuous or locally bounded) solutions of (8) or (22), \( M(p) = \Pi p_k^{c_k} \) and \( L(p) = \Sigma c_k \log p_k \), respectively.

4. Determining All Continuous Ordinately or Equationally Invariant or Interval Invariant Merging Functions

The ordinal and equational invariance and interval invariance are much weaker conditions than those with which we have dealt up to this point. We determine all continuous RMFs satisfying each of these four conditions, individually. We could do, as we will point out, with weaker conditions
than continuity (but not without any conditions at all) to get the same or more general forms for these RMFs, but those conditions are somewhat complicated and probably continuity is not too strong an assumption for practical purposes. (Note also that among the general forms determined above, only the continuous ones seem to be of any practical use. It is enough to mention that the noncontinuous solutions of (8) or (22) have graphs everywhere dense on \( \mathbb{R}^n_+ \times \mathbb{R}_+ \) or on \( \mathbb{R}^n_+ \times \mathbb{R} \), respectively.) We have no results for ordinally or equationally invariant or interval invariant GRMFs, so their determination remains an open problem.

We consolidate also two chains of implications showing the relative strength of the conditions for RMFs and GRMFs. The reader will notice that there is room for improvement there too.

**Theorem 9.** The general continuous equationally interval invariant (eii) RMFs are given by (24a,b), where \( a, b, c_1, \ldots, c_n \) are arbitrary constants.

**Proof.** The implication (eii) means that the value of \( u(rp) - u(rP) \) depends only upon the value of \( u(p) - u(P) \) and \( r \):

\[
u(rp) - u(rP) = f[u(p) - u(P), r].\]

As shown in [10; 2, pp. 22–30; 6], the general nonconstant continuous solutions \( u \) of this equation are given by (24a, b) with \( a \neq 0, c_1^2 + c_2^2 + \cdots + c_n^2 \neq 0 \). If \( u \) is constant, we have one of (24a, b) with \( c_1 = c_2 = \cdots = c_n = 0 \) (or \( a = 0 \)).

**Corollary 10.** The general continuous ordinally interval invariant (oii) RMFs are also given by (24a, b).

**Proof.** Straightforward substitution shows that (24a, b) satisfy (oii) whatever the constants are. The converse follows from (2).

The proofs in [2, pp. 23–30; 6] show that the continuity supposition can be weakened, say to the image \( u(\mathbb{R}^n_+) \) of \( \mathbb{R}^n_+ \) under \( u \) being an interval and \( u(J) = \{ u(p) | p \in J \} \) being bounded for some proper n-dimensional interval \( J \).

In the following theorem we cannot weaken the continuity supposition essentially without changing the result.

**Theorem 11.** The general continuous equationally or ordinally invariant ((e) or (o)) RMFs are of the form

\[
u(p) = u(p_1, \ldots, p_n) = h \left( \prod_{k=1}^{n} p_k^{i_k} \right),\] (25)
where $h: \mathbb{R}_+ \to \mathbb{R}_+$ is arbitrary continuous and strictly monotonic and $c_1, \ldots, c_n$ are arbitrary constants.

**Proof.** The implication (ei) means now that $u(rp)$ depends only upon $u(p)$ and $r$; that is, there exists a function $f: u(\mathbb{R}_+^n) \times \mathbb{R}_+^n \to \mathbb{R}_+$ such that

$$u(rp) = f(u(p), r).$$

Repeated application of this equation gives

$$f[u(p), rs] = u(rsp) = f[u(sp), r] = f(f[u(p), s], r).$$

Since $u$ is continuous, $u(\mathbb{R}_+^n)$ is a (one-dimensional) interval $I$. If $I$ degenerates to a point, then $u$ is constant, which is (25) with $c_1 = \cdots = c_n = 0$. So we can take $I$ to be a proper interval. With $\zeta = u(p)$, $y = \log s = (\log s_1, \ldots, \log s_n)$, $z = \log r = (\log r_1, \ldots, \log r_n)$, and

$$g(\zeta, z) := f(\zeta, e^y) = f(\zeta, e^{z_1}, \ldots, e^{z_n})$$

we have now the *translation equation*

$$g(\zeta, y+z) = g(\zeta, y, z) \quad \text{for all } \zeta \in I, \quad y, z \in \mathbb{R}^n. \quad (28)$$

Moreover, $g$ has the following *transitivity property*:

for any two $\zeta \in I, \eta \in I$ there exists a $z \in \mathbb{R}^n$ such that $g(\zeta, z) = \eta$. \quad (29)

Indeed, for any $\zeta = u(p)$, $\eta = u(q)$ there exists $r$ such that $q = rp$; so, by (26) and (27), we obtain, as asserted,

$$\eta = u(q) = u(rp) = f(\zeta, r) = g(\zeta, \log r).$$

Moszner \cite{8} has proved that every continuous transitive solution of the translation equation (28) is of the form

$$g(\zeta, z) = g(\zeta, z_1, \ldots, z_n) = f(\phi^{-1}(\zeta) + c_1 z_1 + \cdots + c_n z_n),$$

where $\phi: \mathbb{R} \to I$ is an arbitrary continuous and strictly monotonic function, $c_1, \ldots, c_n$ are arbitrary constants, not all 0. (For a weaker form of this theorem, see \cite[pp. 367–370]{1}). So, with (27) and $\psi(t) := \phi(\log t)$, we get

$$f(\zeta, r) = f(\phi^{-1}(\zeta) + c_1 \log r_1 + \cdots + c_n \log r_n) = \psi\left[\psi^{-1}(\xi) \prod_{k=1}^{n} r_k^{c_k}\right],$$

where $\psi: \mathbb{R}_+ \to I$ is again continuous and strictly monotonic. Finally, (26) with $p = 1, a := \psi^{-1}[u(1)]$ yields

$$u(r) = \psi\left(a \prod_{k=1}^{n} r_k^{c_k}\right)$$

(so $a \neq 0$ if $u$ is not constant), which is (25) with $h(t) := \psi(at)$. 

On the other hand, the function \( u \), given by (25) with arbitrary strictly monotonic \( h \) and with arbitrary \( c_1, \ldots, c_n \), satisfies not only (ei) but the ordinal invariance (oi) which (cf. (1)) is stronger. Let \( h \) be, say, strictly decreasing. Then

\[
\begin{align*}
 u(p) > u(q) & \implies h \left( \prod_{k=1}^{n} p_k^{c_k} \right) > h \left( \prod_{k=1}^{n} q_k^{c_k} \right) \\
 & \implies \prod_{k=1}^{n} p_k^{c_k} < \prod_{k=1}^{n} q_k^{c_k} \\
 & \implies \prod_{k=1}^{n} r_k^{c_k} \prod_{k=1}^{n} p_k^{c_k} < \prod_{k=1}^{n} r_k^{c_k} \prod_{k=1}^{n} q_k^{c_k} \\
 & \implies h \left( \prod_{k=1}^{n} (r_k p_k)^{c_k} \right) > h \left( \prod_{k=1}^{n} (r_k q_k^{c_k}) \right) \\
 & \implies u(rp) > u(rq).
\end{align*}
\]

The proof is similar for increasing \( h \). \[\]

Note that if \( n = 1 \), then (25) just means that \( u \) is either constant or a continuous strictly monotonic function of \( p \).

As to replacing the continuity hypothesis, it is implicit in \([1, \text{pp. 367–370}]\), that the solution of (28) is of the form

\[
g(\xi, z) = \phi[\phi^{-1}(\xi) + A(z)],
\]

where \( \phi: \mathbb{R} \to I \) is a bijection and \( A \) an additive function, that is, an arbitrary solution of

\[
A(y + z) = A(y) + A(z)
\]

for all \( y, z \in \mathbb{R}^n \),

if there exists an \( \alpha \in I \) such that, for every \( \xi \in I \), \( y_2, \ldots, y_m \in \mathbb{R} \), there exists a unique \( y_1 \in \mathbb{R} \) such that

\[
g(\alpha, y_1, y_2, \ldots, y_m) = \xi.
\]

This is clearly much stronger than the transitivity property (29). Moszner \([8]\) weakened this condition in the following way. In addition to transitivity, there should exist an \( \alpha \in I \) such that the group \( G_\alpha = \{ y \in \mathbb{R}^n | g(\alpha, y) = \alpha \} \) forms a vector space over the rationals and the cardinality of \( \mathbb{R}/G_\alpha \) is that of the continuum. Clearly neither of these conditions is very attractive for our subject. Let us mention, however, that, in consequence of (30), under these conditions the general equationally invariant (ei) RMFs are

\[
u(p) = h[M(p)].
\]
where \( M: \mathbb{R}_{+}^{n} \rightarrow \mathbb{R}_{+} \) is an arbitrary multiplicative function and \( h: \mathbb{R}_{+} \rightarrow I \) is a bijection.

From (24a, b), from Theorems 8, 9, 11, and Corollary 10 we see that for continuous RMVs

\[(\text{rii}) \Leftrightarrow (\text{eii}) \Leftrightarrow (\text{oi}) \Leftrightarrow (\text{ei}).\]

Roberts [9] proved for all (not only continuous) RMFs (however, under the inclusion of (oi) into (rii))

\[(\text{rii}) \Rightarrow (\text{oi}) 
\Rightarrow (\text{ei}).\]

We have also (1), (2), and (23) for all RMVs. Roberts proved also a significant part of our (23), namely

\[(p) \Rightarrow (gm) \Rightarrow (ri) \Rightarrow (rii).\]

In fact, he proved the chain of implications

\[(p) \Rightarrow (gm) \Rightarrow (ri) \Rightarrow (rii) \Rightarrow (oi) \Rightarrow (ei) \Rightarrow (ei). \]

For GRMFs we have (1), (2), (14), and (19). In particular

\[(\text{C}) \Rightarrow (\text{RI}) \Rightarrow (\text{RII}), \ (\text{GM}) \Rightarrow (\text{RI}) \Rightarrow (\text{RII}).\]

Roberts proved the second chain of implications \((\text{GM}) \Rightarrow (\text{RI}) \Rightarrow (\text{RII}).\) Moreover, he proved also \((\text{RII}) \Rightarrow (\text{OII}) \Rightarrow (\text{OI}) \Rightarrow (\text{EI}),\) (as mentioned above, he included (OI) in (RII)). So we have now

\[(\text{C}) \Rightarrow (\text{RI}) \Rightarrow (\text{RII}) \Rightarrow (\text{OII}) \Rightarrow (\text{OI}) \Rightarrow (\text{EI}).\]

5. **Merger Functions Under Homogeneity, Proportionality, Identity, or Agreement Conditions**

In what follows, the geometric mean

\[\gamma(p) = \gamma(p_1, p_2, ..., p_n) = (p_1p_2 \cdots p_n)^{1/n}\]

will play an essential role, mainly because it is linearly homogeneous (lh), multiplicative (m), agreeing (a), and symmetric (s). Actually we have the following (cf. [7]).

**Proposition 12.** The geometric mean \( \gamma \) is the only agreeing, multiplicative, and symmetric function \( u: \mathbb{R}_{+}^{n} \rightarrow \mathbb{R}_{+}. \)
Proof.

\[ u(p) = \prod_{k=1}^{n} M_k(p_k) = \prod_{k=1}^{n} M_1(p_k) = M_1(p_1 p_2 \cdots p_n), \]

by (11), (s) and the multiplicativity of \( M_1 \), while by (a)

\[ \rho = u(\rho) = M_1(\rho)^n \quad \text{and} \quad u(p) = (p_1 p_2 \cdots p_n)^{1/n}, \]

as asserted. The converse was mentioned above and is obvious. \( \Box \)

However, occasionally we need only a part of these properties and then the geometric mean can be replaced by an arbitrary but fixed \( \gamma \) satisfying those conditions. For instance, if only (a), (lh), and (s) are required then any root-mean-power \( \left( \frac{p_1^b + \cdots + p_n^b}{n} \right)^{1/b} \) will do; if only (a) and (lh) then the weighted geometric mean \( \prod p_k^{q_k} \) or any weighted root-mean-power \( \left( \sum q_k p_k^b \right)^{1/b} \), where \( \sum q_k = 1 \), will be appropriate, the former satisfying also multiplicativity (m).

**Proposition 13.** The general linearly homogeneous (lh) \( u: \mathbb{R}_+^n \to \mathbb{R}_+ \) is given by

\[ u(p) = \gamma(p) f \left( \frac{1}{\gamma(p)} p \right), \tag{34} \]

where \( f: \mathbb{R}_+^n \to \mathbb{R}_+ \) is arbitrary while \( \gamma \) is the geometric mean (33) or any arbitrary but fixed linearly homogeneous function.

**Proof.** Choosing \( \rho = 1/\gamma(p) \) in (lh), we get

\[ u \left( \frac{1}{\gamma(p)} p \right) = \frac{1}{\gamma(p)} u(p) \]

that is, (34) with \( f = u \). On the other hand (34) satisfies (lh) with any \( f \). \( \Box \)

**Proposition 14.** The general ratio invariant (RI) GRMFs which satisfy the identity condition (I) are given by

\[ U(x, y) = F(x)/F(y), \tag{6} \]

where \( F: \mathbb{R}_+^n \to \mathbb{R} \) is arbitrary. Thus [(RI) and (I)] \( \Rightarrow \) (C).

**Proof.** This follows immediately by substituting (13) into (I) (and comparison to Proposition 1). \( \Box \)

We use Propositions 13 and 14 to determine homogeneous (linearly in the first vector or of degree -1 in the second vector if there is any), proportional or agreeing GRMFs and RMFs.
**Theorem 15.** The general ratio invariant (RI) GRMFs with identity (I) and also the general circular (C) GRMFs, satisfying either proportionality (P), linear homogeneity in the first vector (LH), or homogeneity of degree \(-1\) in the second vector (H\(^{-1}\)) are given by

\[
U(x, z) = \frac{\gamma(x)}{\gamma(z)} f \left( \frac{1}{\gamma(x)} x \right) / f \left( \frac{1}{\gamma(z)} z \right),
\]

where \(f: \mathbb{R}^n_+ \to \mathbb{R}_+\) is arbitrary and \(\gamma: \mathbb{R}^n_+ \to \mathbb{R}_+\) is an arbitrary but fixed linearly homogeneous function, for instance the geometric mean. In the latter case, or for any other multiplicative (m) and agreeing (a) function \(\gamma\), (35) holds also with the factor \(\gamma(x)/\gamma(z)\) replaced by \(\gamma(x/z)\).

**Proof.** By (5) and Proposition 14 it is enough to consider (P) and (6):

\[
F(\rho x)/F(x) = \rho, \quad \text{that is} \quad F(\rho x) = \rho F(x),
\]

\(F\) is linearly homogeneous. So Proposition 13 gives \(F(x) = \gamma(x)f((1/\gamma(x))x)\) and (35) follows from (6). The rest is obvious.

The proof of the next theorem is very similar but we give the details because it is a new and in our opinion simpler proof of Theorem 9 in [9].

**Theorem 16.** Under (LH) linear homogeneity with respect to the first and (H\(^{-1}\)) homogeneity of degree \(-1\) with respect to the second vector, the equationally, ordinally, or rationally invariant ((EI), (OI), or (RI)) GRMFs are exactly of the form

\[
U(x, z) = \frac{\gamma(x)}{\gamma(z)} f \left( \frac{1}{\gamma(x)} x \right) g \left( \frac{1}{\gamma(z)} z \right),
\]

where \(f, g: \mathbb{R}^n_+ \to \mathbb{R}_+\) are arbitrary and \(\gamma: \mathbb{R}^n_+ \to \mathbb{R}_+\) is an arbitrary but fixed linearly homogeneous (lh) function, for instance the geometric mean (33). In the latter case, or for any other multiplicative (m) and agreeing (a) function \(\gamma\), (36) holds also with the factor \(\gamma(x)/\gamma(z)\) replaced by \(\gamma(x/z)\).

**Proof.** First we show that

\[
[(\text{EI}) \text{ and } (\text{LH})] \Rightarrow (\text{RI}).
\]

This has been proved in [9, Theorem 8 and the remark following it]) but for completeness's sake and since it is short we do it here too. If

\[
U(x, z) = cU(y, z) \quad \text{for a triple } x, y, z \in \mathbb{R}_+^n
\]

then, by (LH),

\[
U(x, z) = U(cy, z)
\]
and, by (EI) and (LH),
\[ U(x, w) = U(\gamma y, w) = cU(y, w) \quad \text{for every } w \in \mathbb{R}_+^n, \]
that is, we have indeed (RI). By (1), also \[(01) \text{ and } (LI) \Rightarrow (RI). \]
So, in view of Theorem 5,
\[ U(x, z) = F(x) G(z). \tag{37} \]

By (LH) and (H⁻¹),
\[ F(\rho x) G(z) = \rho F(x) G(z), \quad F(x) G(\rho z) = (1/\rho) F(x) G(z). \tag{38} \]

Using Proposition 13, the first equality gives
\[ F(x) = \gamma(x) f \left( \frac{1}{\gamma(x)} x \right). \tag{39} \]

Similarly, the second equation of (38), that is \[ G(\rho z) = (1/\rho) G(z), \] gives, with \( \rho = 1/\gamma(z), \)
\[ G(z) = \gamma(z)^{-1} g \left( \frac{1}{\gamma(z)} z \right). \tag{40} \]

The combination of (37), (39), and (40) shows that \( U \) is indeed of the form (36). Conversely, (36) with any \( f \) satisfies (EI), even (01) and (RI). The rest is again obvious. \( \square \)

In the case (LH), by the first part of the proof of Theorem 16, (RI) may be replaced also in Theorem 15 by (OI) or (EI). For linear homogeneous (LH) ratio interval invariant (RII) and ordinal interval inariant (OII) GRMFs and for further results see [9].

In order to determine RMFs under linear homogeneity (lh) or agreement (a) conditions we state the following.

**Proposition 17.** A multiplicative \( (m) \) function \( u: \mathbb{R}_+^n \rightarrow \mathbb{R}_+ \) satisfies either the linear homogeneity (lh) or the agreement (a) property if, and only if
\[ u(p) = \gamma(p) M \left( \frac{1}{\gamma(p)} p \right), \tag{41} \]
where \( M \) is an arbitrary multiplicative function and \( \gamma \) is the geometric mean (33) or an arbitrary but fixed agreeing multiplicative function.

**Proof.** We have seen in (5) that \( (m) \) and (a) imply (lh), so it is enough to consider \( (m) \) and (lh). Then Proposition 13 gives (41) while (41) satisfies
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(m), (lh), and (a) with arbitrary multiplicative \( M \) whenever \( \gamma \) is multiplicative and agreeing (thus also linearly homogeneous).

**Corollary 18.** The general linearly homogeneous (lh) generalized multiplicative \((gm)\) or ratio invariant \((ri)\) or ratio interval invariant \((rii)\) RMFs are given by

\[
u(p) = a\gamma(p) M \left( \frac{1}{\gamma(p)} p \right), \tag{42}\]

where \( a > 0 \) is an arbitrary constant, \( M \) an arbitrary multiplicative function, and \( \gamma \) is the geometric mean or an arbitrary but fixed agreeing multiplicative function. If (lh) is replaced by the agreement (a) condition, then \( a = 1 \) in (42).

**Proof.** The statement on (lh) and (gm) or (ri) follows from Proposition 17 and from Corollaries 3 and 6. If instead of (lh) we have (a), then by Corollaries 3 and 6, we must substitute (15) into (a). So \( \rho = aM(\rho) \) for all \( \rho \in \mathbb{R}_+ \). But then, by (15) and (8),

\[
u(p) = aM(p) = aM[\gamma(p)] M \left( \frac{1}{\gamma(p)} p \right) = \gamma(p) M \left( \frac{1}{\gamma(p)} p \right),
\]

as asserted. As to (rii), it was shown in [3, Theorem 3, Case 3] that, among our solutions (20) and (21) (with (8) and (22)) of (rii), only (42) is linearly homogeneous (lh) and only (42) with \( a = 1 \) is agreeing.

**Corollary 19.** The general linearly homogeneous (lh) locally bounded or locally continuous \((gm)\) or (ri) or (rii) RMFs are given by

\[
u(p) = a \prod_{k=1}^{n} p_k^{b_k}, \quad \left(a > 0, \sum_{k=1}^{n} b_k = 1 \right).
\]

If (lh) is replaced by agreement (a) then \( a = 1 \), that is, we get weighted geometric means.

**Corollary 20.** The general \((gm)\) or (ri) or (rii) symmetric \((s)\) RMFs, which are linearly homogeneous \((lh)\) are constant multiples of the geometric mean. If (lh) is replaced by the agreement (a) condition, then the geometric mean (33) is the only solution.

The proofs, based upon Corollary 18, are similar to those of Corollary 4 and Proposition 12. The agreement (a) statements of Corollaries 19 and 20 are contained also in [9, Theorems 4 and 5, although with the definition of (rii) which includes (oi)], also for (oi) and (oii) RMFs. There are also similar results for GRMFs in [9]; for instance (Theorems 13 and 14...
there), the locally bounded or locally continuous generalized multiplicative (GM) and proportional (P) GRMFs are exactly the weighted geometric means of the relative scores

$$U(x, z) = \prod_{k=1}^{n} \left( x_k/z_k \right)^{b_k} \left( \sum_{k=1}^{n} b_k = 1 \right)$$

and the generalized multiplicative (GM), proportional (P), and symmetric (S) GRMFs are given exactly by the geometric mean of the relative scores

$$U(x, z) = \left[ \prod_{k=1}^{n} \left( x_k/z_k \right) \right]^{1/n}.$$

All this suggests that the geometric mean may be the appropriate merging function in many situations.

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