JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS 78, 400–404 (1980)

The Discrepancy of C-Uniformly Distributed Multidimensional Functions

RUDOLF J. TASCHNER

Institut für Mathematische Analysis, Technische Universität, Gusshausstrasse 27–29, A 1040 Vienna, Austria

Submitted by R. P. Boas

We consider a continuous curve ω_l: $\mathbb{R}_0^+ \to \mathbb{R}^L$ in the L-dimensional Euclidean space, assuming that in each interval $[0, T]$, $T > 0$, ω_l has an arc-length $s(T)$ which diverges to infinity as $T \to \infty$. In our notation, l always ranges in the set of natural numbers $1,..., L$. ω_l is called *C-uniformly distributed*, if

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T f(\omega_l(t))\, dt = \int_{[0,1[^L} f(x_l)\, d^L x_l$$

holds for all periodic continuous functions f: $\mathbb{R}^L \to \mathbb{C}$ with period 1 (on each component). Let c_A: $[0, 1[^L \to \mathbb{R}$ denote the characteristic function of $A \subseteq [0, 1[^L$ and extend it on $\mathbb{R}^L$ with period 1. By defining

$$\Delta_T(x_l; \omega_l) = \frac{1}{T} \int_0^T c_{\Pi[0,x_l[}(\omega_l(t))\, dt - \Pi(x_l),$$

where Π stands for the product over $1,..., L$, it is easy to derive the equivalence of the formula

$$\lim_{T\to\infty} \Delta_T(x_l, \omega_l) = 0 \qquad \text{for all} \quad x_l \in [0, 1[^L$$

and the C-uniform distribution of $\omega_l(t)$. Hlawka introduced the notion of the *C-discrepancy* $D_T^*(\omega_l)$ by defining

$$D_T^*(\omega_l) = \sup_{x_l \in [0,1[^L} |\Delta_T(x_l, \omega_l)|.$$

A closely related concept is the *L^2-C-discrepancy*

$$D_T^{(2)}(\omega_l) = \left(\int_{[0,1[^L} \Delta_T(x_l : \omega_l)^2\, d^L x_l \right)^{1/2}.$$

The separability of $[0, 1[^L$ implies that $\lim_{T\to\infty} D_T^*(\omega_l) = 0$ is necessary and sufficient for the C-uniform distribution of ω_l.

0022-247X/80/120400–05$02.00/0

If one interprets the function $\omega_l(t)$ as the motion of a particle in the L-dimensional closed manifold $\mathbb{R}^L/\mathbb{Z}^L$, the C-uniform distribution of $\omega_l(t)$ leads to the observation that the ratio of the length of the particle's stay in any (Riemann-integrable) region of $\mathbb{R}^L/\mathbb{Z}^L$ to the whole time converges to the volume of the region. The C-discrepancy and the L^2-C-discrepancy are thus quantitative measures of the order of this convergence. Weyl has already mentioned this relation between C-uniform distribution and statistical mechanics. Hlawka calculated upper bounds of the C-discrepancy and raised the problem of establishing lower bounds for the C-discrepancy. Using a result of W. M. Schmidt, the author could evaluate a (very small) lower bound of $D_T^*(\omega_l)$ in the case $L = 1$. In this paper we are concerned, however, with the multidimensional case $L > 1$. The main result is as follows:

For any $T > 0$ we have

$$D_T^*(\omega_l) \geqslant D_T^{(2)}(\omega_l) \geqslant c_L \cdot \left(\frac{1}{s(T)}\right)^{L/(L-1)},$$

where c_L is a constant only depending on L.

The proof will show that we can choose

$$c_L = c \cdot 2^{-5L-1},$$

where c is an absolute constant. (We can assume $c \geqslant 2^{-1}$ if $s(T)$ is great enough.)

To prove the theorem, we inductively define numbers $\theta_l(n) \geqslant 0$ by $\theta_l(0) = 0$, and, if we already know $\theta_l(n)$, let $\theta_l(n+1)$ be the least of the numbers $t > \theta_l(n)$ with $N\omega_l(t) = [N\omega_l(\theta_l(n))] \pm 1$. N designates a natural number which will be specified later. From the chain of inequalities

$$s(\theta_l(n)) \geqslant s(\theta_l(n-1)) + \frac{1}{N} \geqslant \cdots \geqslant s(\theta_l(1)) + \frac{n-1}{N} > \frac{n-1}{N}$$

we derive

$$n < Ns(\theta_l(n)) + 1.$$

There are therefore at most

$$2^L(Ns(T) + 1) = 2^L Ns(T) + 2^L \leqslant c^* 2^L Ns(T)$$

intervals $\Pi[(n_l - 1)/N, n_l/N[$, $n_l \in \mathbb{Z}$, which are touched by some $\omega_l(t)$, $0 \leqslant t < T$. c^* designates a suitable constant. (We can assume $c^* \leqslant 1.1$ if $s(T)$ is great enough.)

Let $h(x)$ be a periodic function with period 1 and with $h(x) = -1$ for $0 \leqslant x < \frac{1}{2}$, and $h(x) = 1$ for $\frac{1}{2} \leqslant x < 1$. We define $g(x_l) = 0$, if there exists a $t \leqslant T$ with $[x_l] = [N(\omega_l(t) - [\omega_l(t)])]$, otherwise $g(x_l) = \Pi(h(x_l))$. Instead of $\omega_l(t) - [\omega_l(t)]$ we write $\omega_l^r(t)$ (the modulo 1 "reduced" function), and we finally define $f(x_l) = g(Nx_l)$.

From $|f(x_l)| \leqslant 1$ it obviously follows that

$$\int_{[0,1[^L} f(x_l)^2 \, d^L x_l \leqslant 1;$$

on the other hand we have

$$\int_A^B f(x_l) \, dx_j = 0 \tag{1}$$

for each $j \leqslant L$ and all numbers $A = a/N$, $B = b/N$ with integers $a, b, a < b$. This follows from

$$\int_A^B f(x_l) \, dx_j = \frac{1}{N} \int_a^b g(..., u, ...) \, du = \frac{1}{N} \sum_{c=a+1}^{b} \int_{c-1}^{c} g(..., u, ...) \, du,$$

u appearing at the place of the jth component. As $[x_l] = [y_l]$ implies the equivalence of $g(x_l) = 0$ and $g(y_l) = 0$, resp. the equivalence of $g(x_l) = \Pi(h(x_l))$ and $g(y_l) = \Pi(h(y_l))$, each of the integrals on the right-hand side vanishes. Equation (1) leads to

$$\int_{\Pi]\omega_l^r(t), 1]} f(x_l) \, d^L x_l = 0 \tag{2}$$

for all $t \leqslant T$. To show this, we define A_l to be the least integral multiple of $1/N$ that is $\geqslant \omega_l^r(t)$ and from

$$\int_{\Pi]\omega_l^r(t), 1]} f(x_l) \, d^L x_l = \int_{\Pi[\omega_l^r(t), A_l[} f(x_l) \, d^L x_l$$

we reach (2), because $x_l \in \Pi[\omega_l^r(t), A_l[$ leads to $[Nx_l] = [N\omega_l^r(t)]$, i.e., $f(x_l) = 0$.

We note further that

$$\int_{[0,1[^L} f(x_l) \, \Pi(x_l) \, d^L x_l = \frac{1}{N^L} \int_{[0,N[^L} g(u_l) \, \Pi\left(\frac{u_l}{N}\right) d^L u_l$$
$$= N^{-2L} \sum_{n_l=1, l=1; n_l \notin E}^{n_l=N, l=L} \int_{\Pi[n_l-1, n_l[} \Pi(h(u_l) u_l) \, d^L u_l.$$

Here E denotes the set of all n_l with $1 \leqslant n_l \leqslant N$ and $n_l - 1 = [N\omega_l^r(t)]$ for a $t \leqslant T$. The number of elements in E is at most $c^* 2^L N s(T)$. From

$$\int_0^1 h(x) x \, dx = \frac{1}{4}$$

we conclude that

$$\int_{[0,1[^L} f(x_l) \, \Pi(x_l) \, d^L x_l \geqslant N^{-2L} \cdot \frac{1}{4^L} (N^L - 2^L N c^* s(T)).$$

Using the Cauchy–Schwarz inequality, we now derive from

$$\begin{aligned} &\left| \int_{[0,1[^L} \Delta_T(x_l; \omega_l) \, f(x_l) \, d^L x_l \right| \\ &\quad = \left| \frac{1}{T} \int_0^T \int_{\Pi]\omega_l^r(t), 1]} f(x_l) \, d^L x_l - \int_{[0,1[^L} f(x_l) \, \Pi(x_l) \, d^L x_l \right| \\ &\quad \geqslant \frac{1}{4^L} \cdot N^{1-2L} (N^{L-1} - 2^L c^* s(T)) \end{aligned}$$

the formula

$$\begin{aligned} &\frac{1}{4^{2L}} \cdot N^{2-4L} (N^{L-1} - 2^L c^* s(T))^2 \\ &\quad \leqslant \left(\int_{[0,1[^L} \Delta_T(x_l, \omega_l)^2 \, d^L x_l \right) \left(\int_{[0,1[^L} f(x_l)^2 \, d^L x_l \right) \\ &\quad \leqslant D_T^{(2)}(\omega_l)^2 \leqslant D_T^*(\omega_l)^2. \end{aligned}$$

Thus we have

$$D_T^*(\omega_l) \geqslant D_T^{(2)}(\omega_l) \geqslant \frac{1}{4^L} \cdot N^{1-2L} (N^{L-1} - 2^L c^* s(T)).$$

The theorem is established if we define N to be the nearest integer to $((2L-1) \, c^* s(T) 2^L / L)^{1/(L-1)}$.

ACKNOWLEDGMENT

This paper was written during the author's visit at Stanford University. He wishes to thank the Department of Mathematics at Stanford for its hospitality.

REFERENCES

1. E. HLAWKA, Ueber C-Gleichverteilung, *Ann. Mat. Pura Appl. (IV)* **49** (1960), 311–326.
2. E. HLAWKA, "Theorie der Gleichverteilung," Bibliographisches Institut, Mannheim, 1979.
3. L. KUIPERS AND H. NIEDERREITER, "Uniform Distribution of Sequences," Wiley, New York, 1974.
4. K. F. ROTH, On irregularities of distribution, *Mathematika* **1** (1954), 73–79.
5. R. J. TASCHNER, Gleichverteilte Doppelfolgen und eine Abschaetzung der C-Diskrepanz, *Monatsh. Math.*, in press.
6. H. WEYL, Sur une application de la théorie des nombres à la mécanique statistique et la theorie des perturbations, *Enseignement Math.* **16** (1914), 455–467.
7. H. WEYL, Ueber die Gleichverteilung von Zahlen mod. Eins, *Math. Ann.* **77** (1916), 313–352.