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## The Discrepancy of $C$ -Uniformly Distributed Multidimensional Functions

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We consider a continuous curve  $\omega_l: \mathbb{R}_0^+ \rightarrow \mathbb{R}^L$  in the  $L$ -dimensional Euclidean space, assuming that in each interval  $[0, T]$ ,  $T > 0$ ,  $\omega_l$  has an arc-length  $s(T)$  which diverges to infinity as  $T \rightarrow \infty$ . In our notation,  $l$  always ranges in the set of natural numbers  $1, \dots, L$ .  $\omega_l$  is called  $C$ -uniformly distributed, if

$$\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T f(\omega_l(t)) dt = \int_{[0,1]^L} f(x_l) d^L x_l$$

holds for all periodic continuous functions  $f: \mathbb{R}^L \rightarrow \mathbb{C}$  with period 1 (on each component). Let  $c_A: [0, 1]^L \rightarrow \mathbb{R}$  denote the characteristic function of  $A \subseteq [0, 1]^L$  and extend it on  $\mathbb{R}^L$  with period 1. By defining

$$\Delta_T(x_l; \omega_l) = \frac{1}{T} \int_0^T c_{\Pi[0, x_l]}(\omega_l(t)) dt - \Pi(x_l),$$

where  $\Pi$  stands for the product over  $1, \dots, L$ , it is easy to derive the equivalence of the formula

$$\lim_{T \rightarrow \infty} \Delta_T(x_l; \omega_l) = 0 \quad \text{for all } x_l \in [0, 1]^L$$

and the  $C$ -uniform distribution of  $\omega_l(t)$ . Hlawka introduced the notion of the  $C$ -discrepancy  $D_T^*(\omega_l)$  by defining

$$D_T^*(\omega_l) = \sup_{x_l \in [0, 1]^L} |\Delta_T(x_l; \omega_l)|.$$

A closely related concept is the  $L^2$ - $C$ -discrepancy

$$D_T^{(2)}(\omega_l) = \left( \int_{[0, 1]^L} \Delta_T(x_l; \omega_l)^2 d^L x_l \right)^{1/2}.$$

The separability of  $[0, 1]^L$  implies that  $\lim_{T \rightarrow \infty} D_T^*(\omega_l) = 0$  is necessary and sufficient for the  $C$ -uniform distribution of  $\omega_l$ .

If one interprets the function  $\omega_l(t)$  as the motion of a particle in the  $L$ -dimensional closed manifold  $\mathbb{R}^L/\mathbb{Z}^L$ , the  $C$ -uniform distribution of  $\omega_l(t)$  leads to the observation that the ratio of the length of the particle's stay in any (Riemann-integrable) region of  $\mathbb{R}^L/\mathbb{Z}^L$  to the whole time converges to the volume of the region. The  $C$ -discrepancy and the  $L^2$ - $C$ -discrepancy are thus quantitative measures of the order of this convergence. Weyl has already mentioned this relation between  $C$ -uniform distribution and statistical mechanics. Hlawka calculated upper bounds of the  $C$ -discrepancy and raised the problem of establishing lower bounds for the  $C$ -discrepancy. Using a result of W. M. Schmidt, the author could evaluate a (very small) lower bound of  $D_T^*(\omega_l)$  in the case  $L = 1$ . In this paper we are concerned, however, with the multidimensional case  $L > 1$ . The main result is as follows:

For any  $T > 0$  we have

$$D_T^*(\omega_l) \geq D_T^{(2)}(\omega_l) \geq c_L \cdot \left(\frac{1}{s(T)}\right)^{L/(L-1)},$$

where  $c_L$  is a constant only depending on  $L$ .

The proof will show that we can choose

$$c_L = c \cdot 2^{-5L-1},$$

where  $c$  is an absolute constant. (We can assume  $c \geq 2^{-1}$  if  $s(T)$  is great enough.)

To prove the theorem, we inductively define numbers  $\theta_l(n) \geq 0$  by  $\theta_l(0) = 0$ , and, if we already know  $\theta_l(n)$ , let  $\theta_l(n+1)$  be the least of the numbers  $t > \theta_l(n)$  with  $N\omega_l(t) = [N\omega_l(\theta_l(n))] \pm 1$ .  $N$  designates a natural number which will be specified later. From the chain of inequalities

$$s(\theta_l(n)) \geq s(\theta_l(n-1)) + \frac{1}{N} \geq \dots \geq s(\theta_l(1)) + \frac{n-1}{N} > \frac{n-1}{N}$$

we derive

$$n < Ns(\theta_l(n)) + 1.$$

There are therefore at most

$$2^L(Ns(T) + 1) = 2^L Ns(T) + 2^L \leq c^* 2^L Ns(T)$$

intervals  $\Pi[(n_l - 1)/N, n_l/N]$ ,  $n_l \in \mathbb{Z}$ , which are touched by some  $\omega_l(t)$ ,  $0 \leq t < T$ .  $c^*$  designates a suitable constant. (We can assume  $c^* \leq 1.1$  if  $s(T)$  is great enough.)

Let  $h(x)$  be a periodic function with period 1 and with  $h(x) = -1$  for  $0 \leq x < \frac{1}{2}$ , and  $h(x) = 1$  for  $\frac{1}{2} \leq x < 1$ . We define  $g(x_i) = 0$ , if there exists a  $t \leq T$  with  $[x_i] = [N(\omega_i(t) - [\omega_i(t)])]$ , otherwise  $g(x_i) = \Pi(h(x_i))$ . Instead of  $\omega_i(t) - [\omega_i(t)]$  we write  $\omega_i^r(t)$  (the modulo 1 "reduced" function), and we finally define  $f(x_i) = g(Nx_i)$ .

From  $|f(x_i)| \leq 1$  it obviously follows that

$$\int_{[0,1]^L} f(x_i)^2 d^L x_i \leq 1;$$

on the other hand we have

$$\int_A^B f(x_j) dx_j = 0 \tag{1}$$

for each  $j \leq L$  and all numbers  $A = a/N, B = b/N$  with integers  $a, b, a < b$ . This follows from

$$\int_A^B f(x_j) dx_j = \frac{1}{N} \int_a^b g(\dots, u, \dots) du = \frac{1}{N} \sum_{c=a+1}^b \int_{c-1}^c g(\dots, u, \dots) du,$$

$u$  appearing at the place of the  $j$ th component. As  $[x_i] = [y_i]$  implies the equivalence of  $g(x_i) = 0$  and  $g(y_i) = 0$ , resp. the equivalence of  $g(x_i) = \Pi(h(x_i))$  and  $g(y_i) = \Pi(h(y_i))$ , each of the integrals on the right-hand side vanishes. Equation (1) leads to

$$\int_{\Pi[\omega_i^r(t), 1]} f(x_i) d^L x_i = 0 \tag{2}$$

for all  $t \leq T$ . To show this, we define  $A_t$  to be the least integral multiple of  $1/N$  that is  $\geq \omega_i^r(t)$  and from

$$\int_{\Pi[\omega_i^r(t), 1]} f(x_i) d^L x_i = \int_{\Pi[\omega_i^r(t), A_t]} f(x_i) d^L x_i$$

we reach (2), because  $x_i \in \Pi[\omega_i^r(t), A_t]$  leads to  $[Nx_i] = [N\omega_i^r(t)]$ , i.e.,  $f(x_i) = 0$ .

We note further that

$$\begin{aligned} \int_{[0,1]^L} f(x_i) \Pi(x_i) d^L x_i &= \frac{1}{N^L} \int_{[0, N]^L} g(u_i) \Pi\left(\frac{u_i}{N}\right) d^L u_i \\ &= N^{-2L} \sum_{\substack{n_l = N, l = L \\ n_l = 1, l = 1; n_l \notin E}} \int_{\Pi[n_l - 1, n_l]} \Pi(h(u_i) u_i) d^L u_i. \end{aligned}$$

Here  $E$  denotes the set of all  $n_t$  with  $1 \leq n_t \leq N$  and  $n_t - 1 = [N\omega'_t(t)]$  for a  $t \leq T$ . The number of elements in  $E$  is at most  $c^*2^L Ns(T)$ . From

$$\int_0^1 h(x)x \, dx = \frac{1}{4}$$

we conclude that

$$\int_{[0,1]^L} f(x_t) \Pi(x_t) \, d^L x_t \geq N^{-2L} \cdot \frac{1}{4^L} (N^L - 2^L N c^* s(T)).$$

Using the Cauchy–Schwarz inequality, we now derive from

$$\begin{aligned} & \left| \int_{[0,1]^L} \Delta_T(x_t; \omega_t) f(x_t) \, d^L x_t \right| \\ &= \left| \frac{1}{T} \int_0^T \int_{\Pi|\omega'_t(t),1} f(x_t) \, d^L x_t - \int_{[0,1]^L} f(x_t) \Pi(x_t) \, d^L x_t \right| \\ &\geq \frac{1}{4^L} \cdot N^{1-2L} (N^{L-1} - 2^L c^* s(T)) \end{aligned}$$

the formula

$$\begin{aligned} & \frac{1}{4^{2L}} \cdot N^{2-4L} (N^{L-1} - 2^L c^* s(T))^2 \\ &\leq \left( \int_{[0,1]^L} \Delta_T(x_t, \omega_t)^2 \, d^L x_t \right) \left( \int_{[0,1]^L} f(x_t)^2 \, d^L x_t \right) \\ &\leq D_T^{(2)}(\omega_t)^2 \leq D_T^*(\omega_t)^2. \end{aligned}$$

Thus we have

$$D_T^*(\omega_t) \geq D_T^{(2)}(\omega_t) \geq \frac{1}{4^L} \cdot N^{1-2L} (N^{L-1} - 2^L c^* s(T)).$$

The theorem is established if we define  $N$  to be the nearest integer to  $((2L - 1) c^* s(T) 2^L / L)^{1/(L-1)}$ .

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