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Journal of Computer and System Sciences 67 (2003) 742–756

JOURNAL OF COMPUTER AND SYSTEM SCIENCES

http://www.elsevier.com/locate/jcss

# The parametrized complexity of knot polynomials

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Received 1 October 2001; revised 4 April 2002

**Abstract**

We study the parametrized complexity of the knot (and link) polynomials known as Jones polynomials, Kauffman polynomials and HOMFLY polynomials. It is known that computing these polynomials is ♯**P** hard in general. We look for parameters of the combinatorial presentation of knots and links which make the computation of these polynomials to be fixed parameter tractable, i.e., in the complexity class FPT. If the link is explicitly presented as a closed braid, the number of its strands is known to be such a parameter. In a generalization thereof, if the link is explicitly presented as a combination of compositions and rotations of $k$-tangles the link is called $k$-algebraic, and its algebraicity $k$ is such a parameter. The previously known proofs that, for this parameter, the link polynomials are in FPT uses the so called skein modules, and is algebraic in its nature. Furthermore, it is not clear how to find such an algebraic presentation from a given link diagram. We look at the treewidth of two combinatorial presentation of links: the crossing diagram and its shading diagram, a signed graph. We show that the treewidth of these two presentations and the algebraicity of links are all linearly related to each other. Furthermore, we characterize the $k$-algebraic links using the pathwidth of the crossing diagram. Using this, we can apply algorithms for testing fixed treewidth to find $k$-algebraic presentations in polynomial time. From this we can conclude that also treewidth and pathwidth are parameters of link diagrams for which the knot polynomials are FPT. For the Kauffman and Jones polynomials (but not for the HOMFLY polynomials) we get also a different proof for FPT via the corresponding result for signed Tutte polynomials.


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0022-0000/03/$ - see front matter 
doi:10.1016/S0022-0000(03)00080-1

## 1. Introduction

Let $m \in \mathbb{N}$. A *link* $L$ of $m$ components is a subset of $\mathbb{R}^3$ consisting of $m$ disjoint piecewise linear simple closed curves. A *knot* is a connected link. Two links $L_1$ and $L_2$ are equivalent if there exists an orientation preserving homeomorphism $h : \mathbb{R}^3 \to \mathbb{R}^3$ such that $h(L_1) = L_2$. The *unknot* is a link equivalent to $S^1$ on the plane. A projection of a link $L$ on $\mathbb{R}^2$ where at most two points of the link are mapped on the same point and the set of such points is isolated gives as a picture a plane graph $D(L)$, the crossing diagram. Clearly, many crossing diagrams represent equivalent links. We denote by $Diag(L)$ the set of crossing diagrams for $L$. The fundamental problem of knot theory consists of the following:

LINK-EQUI

*Input*: Two crossing diagrams $D_1$ and $D_2$ with $n_1$ and $n_2$ crossings, respectively.
*Problem*: Do $D_1$ and $D_2$ represent the same link (knot)?

A simpler problem is

UNKNOT

*Input*: A crossing diagram $D$ with $n$ crossings.
*Problem*: Does $D$ represent the unknot?

UNKNOT has recently been shown to be in *NP*, [HLP99], but it seems to be still open whether LINK-EQUI is solvable in finitely iterated exponential time.

A *link invariant* is a link equivalence preserving function. A *polynomial link invariant* is a function mapping links into a Laurent polynomial ring $\mathscr{R} = \mathbf{Z}[X_1^{\pm 1}, \dots, X_m^{\pm 1}]$ such that equivalent links are mapped into the same polynomial. The study of link invariants is an important and beautiful subject of pure but visual mathematics with a rich literature stretching over the last 200 years, cf. among others [BZ85,Lic97,PS97]. For an account of its history, cf. Przytycki's delightful [Prz98].

The first polynomial link invariants were studied by Alexander in 1928 and generalized by Conway in 1970. These Alexander–Conway polynomials do not distinguish even the unknot, but are still useful, especially as they are computable in time polynomial in $n$ from a given crossing diagram with $n$ crossings. More powerful polynomial link invariants were introduced by Jones in 1984, followed immediately by generalizations due to a group of authors known under the acronym HOMFLY–PT,[4] and in 1987 due to Kauffman. These polynomial link invariants are known as *Jones polynomial*, *HOMFLY polynomial*, *Kauffman bracket*, *and Kauffman polynomial*. The Jones polynomial can be obtained from both the Kauffman bracket and the HOMFLY polynomial by simple substitutions. It is known that computing (even evaluating) all these polynomials is $\sharp\mathbf{P}$ hard in general, cf. [Wel93]. These polynomials still do not distinguish equivalent crossing diagrams, and it is open whether they distinguish the unknot. But they have

[4] P. Freyd, J. Hoste, W.B.R. Lickorish, K. Millet, A. Ocneanu, D. Yetter [FYH+85] and J. Przytycki, P. Traczyk [PT87].

proven extremely useful, e.g., in solving Tait's conjecture from around 1880 concerning the minimal crossing number of a link represented by an alternating crossing diagram, [Bol99]. They also have found important applications in physics, chemistry and biology, cf. [Kau91,Kau95a,-Kau95b].

We study the parametrized complexity of these polynomials in the sense of [DF99]. We look for parameters of the combinatorial presentation of knots and links which make the computation of these polynomials to be fixed parameter tractable (FPT), i.e., computable in time $c_k n^d$, where $d$ does not depend on $n$ nor on $k$, $k$ is the parameter and $n$ is the size of the input. Typically, $k$ will be the treewidth of some link diagram of size $n$.

Our motivation stems from a result due to Mighton [Mig99]. He considered a different graphical presentation of a link $L$ as a signed graph $S(L)$, and showed that if $S$ is alternating and of treewidth at most 2, then the Jones polynomial can be computed from $S$ in polynomial time. In [Mak01], this was extended to (non-necessarily alternating) signed presentations $S(L)$ of treewidth at most $k$. Consider the problem:

STW-KAUF

*Input*: A signed diagram $S(L)$ of a link $L$ with $n$ crossings.
*Parameter*: $k$, the treewidth of $S$.
*Problem*: Compute the Kauffman bracket of $L$ using $S(L)$.

**Theorem 1** (Makowsky). *STW-KAUF is in FPT.*

The proof in [Mak01] uses a corresponding result for the Tutte polynomial for signed graphs introduced by Kauffman [Kau89] from which the Kauffman bracket and the Jones polynomial can be easily computed. The result for the Tutte polynomials then uses a blend of techniques from logic and dynamic programming as previously developed in [CMR01,MM00,MM02].

Naturally, one would ask, whether the treewidth of $S(L)$ has a natural interpretations in the language of knot theory. Our results show that this is indeed the case.

There is a natural bijection $\sigma : Diag(L) \to Sign(L)$ between crossing diagrams $D$ of $L$ and the signed graph $\sigma(D) = S$ representing $L$.

In Section 2, we show

**Theorem 2.** *The treewidth of $D \in Diag(L)$ and $S = \sigma(D) \in Sign(L)$ are both linearly related to each other.*

It is rather straightforward that for links in braid presentation with at most $k$ strands, i.e., as $k$-braids, the corresponding crossing diagram has treewidth (even pathwidth) at most $k+1$, as they are subgraphs of $(n,k)$-grids. Morton [MS90] has shown how to compute the Jones polynomials for links in $k$-braid presentation in polynomial time using heavy algebraic machinery such as Hecke algebras and the Ocneanu trace.

However, there are crossing diagrams of treewidth 2 which come from braids over arbitrary number of strands. A generalization of $k$-braids are the $k$-algebraic links introduced by Przytycki

[Prz]. We shall give the precise definition in Section 3. Using a modified definition of pathwidth with two parameters, we can characterize the $k$-algebraic links as follows.

**Theorem 3.** *The k-algebraic links are exactly those links which have a crossing diagrams of pathwidth at most* $(2k+1, k)$.

Hence, we get for the following problem:

ALG-KAUF

*Input*: A $k$-algebraic expression $T(L)$ of a link $L$ with $n$ operations.
*Parameter*: $k$.
*Problem*: Compute the Kauffman bracket of $L$ using $T(L)$.

**Corollary 1.** *ALG-KAUF is in FPT.*

For the problem

ALG-HOMFLY

*Input*: A $k$-algebraic expression $T(L)$ of a link $L$ with $n$ operations.
*Parameter*: $k$.
*Problem*: Compute the HOMFLY polynomial of $L$ using $T(L)$.
Przytycki [Prz] has announced the corresponding result, cf. Theorem 10. This seemed open when we first obtained our results.[5]

The paper is now organized as follows. In Section 2, we discuss the treewidth of link diagrams and prove the basic properties. In Section 3, we recall basic facts on braids and discuss $k$-algebraic links and prove Theorem 3. In Section 4, we discuss the advantages and disadvantages of the various parameters for which the computation of the various polynomial link invariants are in FPT.

## 2. Link diagrams and their treewidth

In this section, we give some necessary background from knot theory. But we assume that the reader is familiar with rudimentary basics of knot theory such as Reidemeister moves and crossing diagrams. If not, it is advisable to consult a textbook. In many recent graph theory books, such as

[5] When we showed our results to Przytycki in January 2001, he has informed me [Prz] that this, and also Corollary 1, can be proved with an algebraic approach extending the use of Hecke algebras and Ocneanu traces in a formalism of skein modules. But the exact proof has not yet been written. In any case, our proof of Corollary 1 is combinatorial and more elementary.

Fig. 1. Types of crossings.

[Bol99,GR01], the reader will find more than needed. For modern accounts of knot theory proper, the reader may consult [Lic97,Mur96,PS97,Sos99].

### 2.1. Graphs from links

An excellent discussion of various ways of coding a knot as a graph may be found in the books [Bol99,GR01]. Let $L$ be a link and $Diag(L)$ be the set of crossing diagrams of $L$. $D \in Diag(L)$ is a plane graph of degree 4 with vertex labels $+$ and $-$ according to its crossing. The crossings are classified by starting with an oriented projection of a link and noting that there are only two types of crossings, as indicated in Fig. 1.

As $D$ is a regular plane graph of degree 4, we can color its faces with two colors, black and white, such that no neighboring faces have the same color and such that the outer face is white. This coloring is unique. Given $D$ with such a face-coloring we define a graph $\sigma(D)$ as follows: The vertices of $\sigma(D)$ are the black faces and two faces are connected by an edge signed $+$ $(-)$ if they share a $+$ $(-)$ crossing. We put $Sign(L) = \{\sigma(D) : D \in Diag(L)\}$. $Sign(L)$ is called the set of *signed graphs of* $L$. Note that both graph classes $Diag(L)$ and $Sign(L)$ consist of plane graphs. The underlying graph of $S \in Sign(L)$ is the abstract graph of $S$ without the edges signed. In the other direction, we have a map $\rho : Sign(L) \to Diag(L)$ defined by putting a signed crossing on each edge and by connecting neighboring edges correspondingly. The following is easy to verify, cf. [Sos99].

**Proposition 1.** (i) *$\sigma$ is a bijection between $Diag(L)$ and $Sign(L)$ with $\rho = \sigma^{-1}$.*
(ii) *Given a crossing diagram $D \in Diag(L)$ one can compute $\sigma(D) \in Sign(L)$ in polynomial time.*
(iii) *Given a shading diagram $S \in Sign(L)$ one can compute $\rho(D) \in Diag(L)$ in polynomial time.*

### 2.2. Treewidth and pathwidth of diagrams

There are various equivalent definitions of the treewidth of a graph, as partial $k$-trees, as having $k$-tree decompositions, or as inductively defined classes of $k$-vertex-colored graphs. We need for our proofs the equivalence of two of these definitions. General background on treewidth may be found in [Die96].

#### 2.2.1. Tree decompositions

**Definition 1.** *An* $(m,k)$-tree decomposition of $G$ *is given as follows:*

(i) We have a rooted tree $\mathcal{T} = \langle T, f \rangle$, where $T$ is a set and $f$ is a function mapping nodes onto their father.

(ii) The vertex set $V(G)$ of the graph is covered by sets $V_t$, with $t \in T$ and $|V_t| \leqslant m$.
(iii) For every $t, t' \in T$, the intersection $V_t \cap V_{t'}$ has size at most $k$.
(iv) For every edge $e = (x, y) \in E(G)$, there is a $t \in T$ such that both $x, y \in A_t$.
(v) For each $x \in V$, the set $T(x) = \{t \in T : x \in A_t\}$ is a (connected) subtree of $\mathscr{T}$.

*A k-tree decomposition is an* $(m, k)$*-tree decomposition with* $m = k + 1$. *We denote tree decompositions of* $G = \langle V, E \rangle$ *by* $\langle \mathscr{T}, V_t \rangle$.

**Remark 1.** Under conditions (i)–(iii), (v) is equivalent to: For every connected subgraph $H$ of $G$, the set $\{t \in T : V(H) \cap A_t \neq \emptyset\}$ is a connected subtree of $\mathscr{T}$.

**Definition 2.** $G$ of treewidth at most $(m, k)$, *if there exists an* $(m, k)$*-tree decomposition of* $G$. *If* $m = k + 1$ *we speak of treewidth at most* $k$. *Such graphs are also called* partial $k$-trees. *If the tree does not branch, we speak of* $(m, k)$*-path decompositions and pathwidth.*

For fixed $k$, checking whether $G$ has treewidth at most $k$ (and if yes, finding a witnessing tree decomposition) can be done in polynomial time, cf. [ACP87,BK96,Bod96,Bod97].

**Proposition 2.** *For fixed k and m, checking whether G has treewidth* (*path-width*) *at most* $(m, k)$ *can be done in polynomial* (*linear*) *time.*

**Proof** (Sketch). The class of graphs $TW(m, k)$ which have a $(m, k)$-tree decompositions, is easily seen to be closed under minors, cf. [Die96, Proposition 12.4.2]. From [Die96, Theorem 12.5.2], the Minor Theorem of Robertson and Seymour, it follows that $TW(m, k)$ can be characterized by a finite set of forbidden minors, hence, by the results of [RS95], it is recognizable in polynomial time.

It is likely that one can construct also an $(m, k)$-tree decomposition in linear time, provided one exists. One would have to proceed very much like in [BK96, Section 6]. But we have not verified the details. However, for our applications, this is not needed, as one can always construct an $(m, m-1)$-tree decomposition (which is an $m$-tree decomposition) in liner time using [BK96,Bod96, Section 6]. Getting a proper $(m, k)$-tree decomposition would only improve the constants, but, using current proof techniques, they are bad anyhow, cf. [Bod97].

If we add unary predicates (labels) to $G$, the notion of treewidth does not change. Therefore, the treewidth of a crossing diagram is just the treewidth of its underlying graph. Also the treewidth of an edge-colored graph is, by definition, the same as its treewidth without the coloring.

**Definition 3.** *The class* $TW(m, k)$. A $k$-graph $\hat{G} = (V, E, c)$ is an abstract graph $G = (V, E)$ together with a mapping $c : V \to C = \{1, \ldots, k\}$ such that $c^{-1}(i) \cap c^{-1}(j) = \emptyset$ for $i \neq j$.

For $i \neq j \in C$ and a $k$-graph $\hat{G}$ we define $\rho_{i \to j}(\hat{G}) = (V, E, c')$, where $c'(i) = \emptyset$ and $c'(j) = c(i) \cup c(j)$. $\rho_{i \to j}$ stands for *recolor the vertices colored i by the color j*. $Fuse_i(\hat{G})$ is the $k$-graph obtained from $\hat{G}$ by identifying (fusing) all vertices with color $i$ such that the fused vertex inherits the color and the edges.

The class of $k$-graphs $TW(m,k)$ is now defined inductively by

- All $k$-graphs with at most $m$ vertices are in $TW(m,k)$.
- If $G_1$ and $G_2$ are in $TW(m,k)$ so is their disjoint union $G_1 \sqcup G_2$.
- If $G \in TW(m,k)$ so is $\rho_{i \to j}(G)$.
- If $G \in TW(m,k)$ so is $Fuse_i(G)$.

The following is widely used in Courcelle's papers, cf. [Cou90,Cou97,CER93,CM02].

**Lemma 1** (Courcelle). *A graph $G = (V,E)$ is a reduct of a k-graph $\hat{G} = (V,E,c) \in TW(m,k)$ iff G has an $(m,k)$-tree decomposition.*

Hence, the set of graphs of treewidth at most $k$ can be defined inductively as the graphs $G$ obtained from $k$-graphs $\hat{G}$ in $TW(k+1,k)$ by forgetting the coloring.

For a link $L$ a link diagram $D \in Diag(L)$ or $S \in Sign(L)$ has treewidth (pathwidth) at most $(m,k)$ if its underlying graph has treewidth (pathwidth) at most $(m,k)$.

### 2.3. Treewidth of D and $\sigma(D)$

To study the impact of the assumption that a link has a diagram presentation of treewidth at most $k$, we compare the treewidth of $D \in Diag(L)$ and its associated $\sigma(D) \in Sign(L)$.

Now the following observation is easy:

**Proposition 3.** *If $D \in Diag(L)$ has treewidth (pathwidth) at most $k$, then $\sigma(D)$ has treewidth (pathwidth) at most $2k+1$.*

**Proof.** Let $\{V_t^D : t \in T\}$ be a $k$-tree decomposition of $D$. For each crossing $v \in D$ there are two black faces which are connected by an edge $e_v = (u_v, w_v)$ in $\sigma(D)$. We define

$$V_t^C = \{u_v : v \in V_t^D\} \cup \{w_v : v \in V_t^D\}$$

and show that this is indeed a $2k+1$ tree (path) decomposition of $\sigma(D)$. For this we only need that, as each $V_t^D$ has at most $k+1$ vertices, $V_t^C$ has at most $2(k+1)$ vertices. □

Our results in the other direction are:

**Lemma 2.** *Let C be a signed graph of degree at most d and tree (path) width at most k. Then $\sigma^{-1}(C)$ has tree (path) width at most $d(k+1)-1$.*

**Proof.** We start with a $k$-tree decomposition of the underlying graph of $C$. We define a tree decomposition of the underlying graph of $\sigma^{-1}(C)$. The tree of the decomposition is the same, but $V_t$ is replaced by $W_t$ which contains, for each $v \in V_t$ the $d$ medians sitting on the $d$ edges leaving $v$. We have to show that this is a $(d(k+1)-1)$-tree decomposition.

Each $V_t$ has at most $k+1$ vertices, which are replaced by $d$ vertices each. Hence, we have at most $d(k+1)$ vertices in each $W_t$, and hence treewidth at most $d(k+1)-1$. Now assume, for some edge $e=(v_1,v_2)\in E(C)$ both $v_1,v_2\in V_t$. Then all the medians on edges leaving $v_1$ or $v_2$ are in $W_t$ and all the new edges in $E(\sigma^{-1}(C))$ connected to the median on $e$ have end points in $W_t$. Finally, let $w\in V(\sigma^{-1}(C))$ and let $W(w)=\{t: w\in W_t\}$. We have to show that this is a subtree of $T$. $w$ is the median on an edge $(v_1,v_2)$ of $C$. So, $W(w)=V(v_1)\cap V(v_2)$ with $V(v_i)=\{t: v_i\in V_t\}$ and $i=1,2$. □

The next two lemmas are about plane graphs, and collect some useful facts.

**Lemma 3.** *Let G be a plane graph of treewidth at most k. Then G has a plane triangulation Triang(G) of treewidth at most k.*

**Proof.** Let $\langle \mathcal{T}, V_t\rangle$ be a $k$-tree decomposition of $G$. We have to show that every face which is not a triangle, has a chord with end points in some $V_t$.

Let $f$ be such a face. Assume $v$ is a vertex on the boundary of $f$, and $u_1,u_2$ are its immediate neighbors on the boundary of $f$. If $u_1$ and $u_2$ are in the same $V_t$, we are done. Otherwise, one shows by induction over the length of the path between $u_1$ and $u_2$ along the boundary of $f$ not passing through $v$, that there is a $u_4$ on this path, and a $t'$ such that both $v$ and $u_4$ are in $V_{t'}$. For this we use the connectivity condition of $k$-tree decompositions. □

*Triang*$(G)$ is chordal (by definition). For plane chordal graphs $C$, we define *Hex*$(G)$ to be the graph obtained from $G$ by adding on each edge two vertices and in each triangular face three edges, such as to inscribe a hexagon using the new vertices on the sides of the face.

**Lemma 4.** *If G is a plane chordal graph of treewidth at most k, the treewidth of Hex(G) is at most* $7k-6$.

**Proof.** Let $\langle \mathcal{T}, V_t\rangle$ be a $k$-tree decomposition of $G$. Each $V_t$ contains at most $3(k+1)-6=3(k-1)$ edges, as $G$ is planar, cf. [Die96, Proposition 4.4.1]. We add for each edge in $V_t$ the two new vertices to $V_t$. Hence, $V_t$ contains at most $7k-5$ vertices. The vertices of each triangle occur together in at least one $V_t$. So we can add the new edges in the corresponding $V_t$. This gives us a $(7k-6)$-tree decomposition of *Hex*$(G)$ with the same underlying tree $\mathcal{T}$. □

**Lemma 5.** *Let* $D\in Diag(L)$ *with n crossings and the treewidth of* $\sigma(D)=k\geqslant 2$. *Then there is* $D^\star\in Diag(L)$ *such that*

(i) $D^\star$ *has at most* $3n$ *crossings, and*
(ii) $\sigma(D^\star)$ *is of degree at most* 3.
(iii) *The treewidths of* $\sigma(D^\star)$ *is at most* $7k-6$.

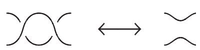

Fig. 2. Reidemeister II move.

**Proof.** Look at the coloring of the faces of $D$. If a black face $f$ has $m \geqslant 4$ black neighboring faces, we can perform a Reidemeister move of type II, cf. Fig. 2 i.e., pulling the rope overneath (or underneath) another part of the rope, such that $f$ is cut into three faces $f_1, f_2$ and $f_3$, all colored black, and such that $f_1$ and $f_3$ have both $<m$ neighboring black faces, and $f_2$ is has only $f_1$ and $f_3$ as black neighboring faces (see Fig. 2). Let $D'$ be the resulting crossing diagram. Repeating this trick we get the result $D^{\star}$.

To see that $D^{\star}$ has treewidth at most $7k$ we use Lemma 4. We first note that $D^{\star}$ is a minor of $Hex(Triang(D))$, hence, cf. [Die96, Proposition 12.4.2], has at most the treewidth of $Hex(Triang(D))$, which is bounded by $7k - 6$. □

**Theorem 4.** *Let $D \in Diag(L)$ with n crossings and such that $\sigma(D)$ has treewidth at most $k$. Then there is $D' \in Diag(L)$ such that*

(i) *$D'$ has at most $3n$ crossings, and*
(ii) *the treewidth of $D'$ is at most $21k - 16$.*

**Proof.** We apply Lemma 2 with $d = 3$ and Lemma 5. □

We suspect the constant in Theorem 4 is not optimal.

**Problem 1.** Let $C$ be a signed graph of treewidth at most $k$. What is the treewidth of $\sigma^{-1}(C)$?

### 2.4. *Link polynomials on diagrams of fixed treewidth*

In the introduction we have defined the parametrized problem STW-KAUF and seen, Theorem 1, that it is in FPT. We now look at

DTW-KAUF

*Input*: A crossing diagram $D(L)$ of a link $L$ with $n$ crossings.
*Parameter*: $k$, the treewidth of $D$.
*Problem*: Compute the Kauffman bracket of $L$ using $D(L)$.
and see, using Theorems 1 and 4:

**Theorem 5.** *DTW-KAUF is in FPT.*

For the corresponding problems STW-HOMFLY and DTW-HOMFLY

DTW-HOMFLY (STW-HOMFLY)

*Input:* A crossing (signed) diagram $D(L)$ $(S(L))$ of a link $L$ with $n$ crossings.
*Parameter*: $k$, the treewidth of $D$ $(S)$.
*Problem*: Compute the HOMFLY polynomial of $L$ using $D(L)$ $(S(L))$.

the techniques of the proof of Theorem 1 do not apply. We shall see in the next section, that their analogues for pathwidth are also in FPT. For the treewidth case this remains open.

### 2.5. Treewidth and pathwidth of a link

We have used the treewidth as a parameter of diagrams representing knots and links. We can define the treewidth (pathwidth) of a link $L$ as the minimal treewidth (pathwidth) among all crossing diagrams $C \in Diag(L)$. Analoguously, there is also a corresponding notion of s-treewidth and s-pathwidth via shading diagrams.

But the pathwidth of a knot or link has its own interest, as it asks what is the *narrowest carpet* [6] which can be made out of the given link. Very little is known about how to find crossing or shading diagrams of a given link which are of minimal treewidth or pathwidth. It is not even clear whether there are links of arbitrarily large treewidth or pathwidth.

Recall that the *crossing number* of a link $L$ as the the minimal number of vertices among all crossing diagrams $C \in Diag(L)$. It is not even clear whether the treewidth of a link is realized in a diagram of minimal crossing number.

In short, studying the treewidth of links may still prove to be an interesting challenge.

## 3. Braids and $k$-algebraic links

### 3.1. Braid presentation of links

Here we follow liberally [PS97]. A *braid in n strands, or an n-braid*, in $\mathbb{R}^3$ is defined as a set of pairwise non-intersecting ascending polygonal lines (the strands) joining the points $A_1, \ldots, A_n$ to the points $B_1, \ldots, B_n$ (in any order). The *closure* of a braid $b$ is defined as the link $\beta(b)$ obtained by joining $A_i$ to $B_i$ for each $i \leqslant n$. The question arises which links are closures of braids. The answer is the classical theorem of Alexander (1923):

**Theorem 6** (Alexander's Braiding Theorem). *The closure map $\beta$ is surjective, i.e., any link (in particular every knot) is the closure of some braid.*

Vogel [Vog90] turned this into a feasible algorithm. With each oriented crossing diagram $D$ one can associate a new diagram $D'$ by "undoing the crossings" while preserving the orientation. $D'$ consists of circles in the plane, the *Seifert circles*, and their number is easily computed from the diagram $D'$ and also $D$. Hence, the number of Seifert circles of a crossing diagram can be computed in polynomial time.

**Theorem 7** (Vogel). *Let $D(L)$ be a crossing diagram of a link L with n Seifert circles. Then one can construct in polynomial time a braid b such that $\beta(b) = D(L)$.*

[6] We think here of a *carpet* as an embedding of the crossing diagram in a rectangular grid. In contrast to a braid, in a carpet the threads may go back and forth.

For a link $L$ the smallest $k$ such that $L$ is the closure of a $k$-braid is called the *braid index of* $L$. It is open how difficult it is to determine the braid index of $L$.

If the link is explicitly presented as a closed $k$-braid, where $k$ is the number of its strands, all the above polynomials can be computed from the braid presentation in time polynomial in the number of crossings in the braid. In other words, define

BRAID-HOMFLY (BRAID-KAUF)

*Input*: A crossing diagram $D(L)$ of a link $L$ with $n$ crossings.
*Parameter*: $k$, the number of Seifert circles of $D(L)$.
*Problem*: Compute the HOMFLY polynomial (Kauffman bracket) of $L$ using $D(L)$.

For the HOMFLY polynomial the following is due to Morton and Short [MS90].

**Theorem 8** (Morton and Short). *BRAID-HOMFLY is in FPT.*

Their method is algebraic and involved, and the constants depend exponentially on the number of strands. They conjecture in [MS90], and seem to confirm it in [MT90,MW], that the same method also works for Kauffman brackets.

**Theorem 9** (Morton et al.). *BRAID-KAUF is in FPT.*

But here is a proof for BRAID-KAUF using treewidth. From the definition of tree decompositions one sees easily the following lemma.

**Lemma 6.** *A crossing diagram $D$ which is the closure of $k$-braid has a $2k$-tree decomposition, and therefore treewidth at most $2k$.*

Hence, we get Theorem 9 using Theorem 1.

### 3.2. *Algebraic presentation of links*

A generalization of braids are tangles. We follow [MT90,PT01] or [Mur96]. On the sphere $S^2$ place $2n$ points. An $(n,n)$-*tangle* ($n$-tangle for short) $T$ is formed by attaching $n$ curves to these points, none of which should intersect each other. We think of $n$-tangles also in their projection onto $\mathbb{R}^2$, hence as a crossing diagram. A *simple $k$-tangle* is a $k$-tangle with at most one crossing and the end points of the strands numbered cyclically $\{1,\dots,k,k+1,\dots,2k\}$. $k$-tangles are obtained from simple $k$-tangles by juxtaposition of two $k$-tangles or rotation of the numbering of the end points. This gives an *algebraic presentation* of a link $L$ as a term $T(L)$ over simple $k$-tangles with juxtaposition and rotation as operators. $T(L)$ is called a $k$-algebraic expression for $L$.

We denote by $Alg_k(L)$ the set of $k$-algebraic presentations of $L$. If such a $T \in Alg_k(L)$ exists, $L$ is called $k$-algebraic. Clearly, $k$-braids are $k$-algebraic. Conway showed how to prove that not every link is 2-algebraic, cf. [Con69]. But it is still open whether every link is 3-algebraic. The experts conjecture that the answer is negative.

Next we define:

ALG-POL

*Input*: A $k$-algebraic expression $T(L)$ of a link $L$ with $n$ operations.

*Parameter*: $k$.

*Problem*: Compute the Jones polynomial, Kauffman bracket, Kauffman polynomial or HOMFLY polynomial of $L$ using $T(L)$.

Przytycki [Prz] has announced that

**Theorem 10.** *ALG-POL is in FPT.*

The proof for this uses the so-called skein modules, and is algebraic in its nature. It generalizes the proof in [MS90]. Similar ideas may be found in [MW,MT90] based on an invariant introduced by Kauffman [Kau90].

For a link $L$, the smallest $k$ such that $L$ has a $k$-algebraic presentation is called the *algebraic index* (or its *algebraicity*) of $L$. It is open how difficult it is to determine the algebraic index of $L$. It is even open whether there are links of algebraic index strictly greater than 3. Note that it is not at all obvious how to find an algebraic presentation $T \in Alg_k(L)$ from a given crossing diagram $D \in Diag(L)$ which is better, in as much as $k$ is smaller than the number of strands in a braid presentation derived from $D$. We shall see below how the pathwidth of $D$ will give rise to a $k$-algebraic presentation.

### *3.3. The pathwidth of k-algebraic links*

Our first results is the following theorem.

**Theorem 11.** *Let $T \in Alg_k(L)$ be a $k$-algebraic expression of a link $L$. Then one can find in polynomial time in the size of $T$ a crossing diagram $C \in Diag(L)$ of pathwidth (and hence treewidth) $(2k+1, k)$.*

**Proof.** We show how to build a crossing diagram $C \in Diag(L)$ which is in $TW(k+1)$. The simple $k$-tangles are made into a graph consisting of $k+1$ edges on $2k+1$ vertices, including the only crossing vertex. We color the $2k$ vertices from the end points with $2k$ colors, leaving one color as a spare color for recoloring. For juxtaposition of two $k$-tangles we use a disjoint union and appropriate $Fuse_i$ operations. For rotation of the labels of the end points we use recolorings making use of the spare color. This shows that we have treewidth at most $2k+1$.

To see that we can also get pathwidth $2k+1$ we observe that we use always disjoint unions of two graphs coming from $k$-tangles which act like concatenation.

The complexity is easily checked. □

**Theorem 12.** *Let $C \in Diag(L)$ be a crossing diagram of $L$ of pathwidth $(2k+1, k)$. Then one can find in polynomial time in the size of $C$ a $k$-algebraic expression $T \in Alg_k(L)$ which represents $L$.*

**Proof.** Using the path decomposition of $C$, we construct $T \in Alg_k(L)$ inductively. For a path decomposition of length 1 we have $2k+1$ vertices which form a crossing diagram. Without loss of generality we can assume that it has no loops, so it can be presented, using Theorem 7, as a $k$-braid, which is $k$-algebraic.

Assume we have a path decomposition of length $n \geqslant 2$. Then we can cut it into two pieces $P_1$ and $P_2$ where an even number $m \leqslant k$ of vertices are identified between the two parts. We obtain from $P_1$ and $P_2$ two crossing diagrams $C_1 \in Diag(L_1)$ and $C_2 \in Diag(L_2)$ by choosing in each of them a closure along those $m$ points. By the induction hypothesis there are $k$-tangles $T_1$ and $T_2$ such that their closures (at the given $m$ end points, and whatever remains) are in $Alg_k(L_1)$ (respectively $Alg_k(L_2)$). By joining $T_1$ and $T_2$ at those $m$ end points appropriately, we get $T$, such that its closure is in $Alg_k(L)$.

Again, the complexity is easily checked. □

Now consider the parametrized problems

PW-POL

*Input*: A crossing diagram $C(L)$ of a link $L$ of pathwidth $(2k+1, k)$.
*Parameter*: $k$.
*Problem*: Compute the Jones, Kauffman bracket or HOMFLY polynomial of $L$ using $T(L)$.
The last two theorems give us an interesting new corollary.

**Corollary 2.** *All the problems, PW-POL, DTW-KAUF and STW-KAUF are in FPT.*

For the Kauffman brackets this follows also from Theorems 1 and 2. For the problems *DTW-HOMFLY* and *STW-HOMFLY* this remains open.

## 4. Conclusion

We have discussed the parametrized complexity of the link polynomials known as HOMFLY polynomials and Kauffman brackets (and their specialization, the Jones polynomial).

We considered the computation of these polynomials from three presentations of links: as crossing diagrams, as shading diagrams of crossing diagrams, and as algebraic terms representing compositions of $k$-tangles. We have shown that the pathwidth of both presentations and the algebraicity are linearly related, and conversion can be computed in polynomial time.

The following remain open:

- Given a link $L$, what is the complexity of deciding whether it has crossing (shading) diagram $C \in Diag(L)$ ($C \in Sign(L)$) of treewidth (pathwidth) at most $k$?

- How is the treewidth (pathwidth) of a link related to its crossing number? Is the minimal width achieved with the smallest crossing number?
- Are there arbitrarily large $k$ such that there are links of treewidth at most $k$ which are not of treewidth at most $k-1$?
- Are there arbitrarily large $k$ such that there are links of treewidth at most $k$ which are not $k$-algebraic?
- Given a link $L$, what is the complexity of deciding whether it is $k$-algebraic?
- Are there arbitrarily large $k$ such that there are $k$-algebraic links which are not $(k-1)$-algebraic?

## Acknowledgments

We are indebted to A. Glikson, J. Mighton, J. Przytycki, E. Ravve and B. Wajnryb for useful comments. We are also indebted to two anonymous referees for their careful reading and comments, and pointing out a gap, and to B. Courcelle for suggesting Lemmas 3 and 4.

## References

[ACP87] S. Arnborg, D.G. Corneil, A. Proskurowski, Complexity of finding embedding in a $k$-tree, SIAM J. Algebraic Discrete Methods 8 (1987) 277–284.

[Bod96] H.L. Bodlaender, A linear time algorithm for finding tree-decompositions of small treewidth, SIAM J. Comput. 25 (1996) 1305–1317.

[Bod97] H. Bodlaender, Treewidth: algorithmic techniques and results, in: I. Privara, P. Ruzicka (Eds.), Proceedings of the 22th International Symposium on the Mathematical Foundation of Computer Science, MFCS'97, Lecture Notes in Computer Science, Vol. 1295, Springer, Berlin, 1997, pp. 29–36.

[BK96] H. Bodlaender, T. Kloks, Efficient and constructive algorithms for the pathwidth and treewidth of graphs, J. Algorithms 21 (1996) 358–402.

[Bol99] B. Bollobás, Modern Graph Theory, Springer, Berlin, 1999.

[BZ85] G. Burde, H. Zieschang, Knots, in: Studies in Mathematics, Vol. 5, de Gruyter, Berlin, 1985.

[Con69] J.H. Conway, An enumeration of knots and links, in: J. Leech (Ed.), Computational Problems in Abstract Algebra, Pergamon Press, Oxford, 1969, pp. 329–358.

[Cou90] B. Courcelle, Graph rewriting: an algebraic approach, in: J. van Leeuwen (Ed.), Handbook of Theoretical Computer Science, Vol. 2, Elsevier, Amsterdam, 1990 (Chapter 5).

[Cou97] B. Courcelle, The expression of graph properties and graph transformations in monadic second-order logic, in: G. Rozenberg (Ed.), Handbook of Graph Grammars and Computing by Graph Transformations, Vol. 1: Foundations, World Scientific, Singapore, 1997, pp. 313–400 (Chapter 5).

[CER93] B. Courcelle, J. Engelfriet, G. Rozenberg, Handle-rewriting hypergraph grammars, J. Comput. System Sci. 46 (1993) 218–270.

[CM02] B. Courcelle, J.A. Makowsky, Fusion on relational structures and the verification of monadic second order properties, Mathematical Struct. Comput. Science 12 (2002) 203–235.

[CMR01] B. Courcelle, J.A. Makowsky, U. Rotics, On the fixed parameter complexity of graph enumeration problems definable in monadic second order logic, Discrete Appl. Math. 108 (1–2) (2001) 23–52.

[Die96] R. Diestel, Graph Theory, Graduate Texts in Mathematics, Springer, Berlin, 1996.

[DF99] R.G. Downey, M.F. Fellows, Parametrized Complexity, Springer, Berlin, 1999.

[FYH+85] P. Freyd, D. Yetter, J. Hoste, W.B.R. Lickorish, K.C. Millet, A. Ocneanu, A new polynomial invariant of knots and links, Bull. Amer. Math. Soc. 12 (1985) 239–246.

[GR01] C. Godsil, G. Royle, Algebraic Graph Theory, Graduate Texts in Mathematics, Springer, Berlin, 2001.

[HLP99] J. Hass, J.C. Lagarias, N. Pippenger, The computational complexity of knot and link problems, J. ACM 46.2 (1999) 185–211.

[Kau89] L.H. Kauffman, A Tutte polynomial for signed graphs, Discrete Appl. Math. 25 (1989) 105–127.

[Kau90] L.H. Kauffman, An invariant for regular isotopy, Trans. Amer. Math. Soc. 318 (1990) 417–471.

[Kau91] L.H. Kauffman, Knots and Physics, World Scientific, Singapore, 1991.

[Kau95a] L.H. Kauffman (Ed.), Knots and Applications, World Scientific, Singapore, 1995.

[Kau95b] L.H. Kauffman, Preface, in: L.H. Kauffman (Ed.), Knots and Applications, World Scientific, Singapore, 1995, pp. 9–36.

[Lic97] W.B.R. Lickorish, Introduction to Knot Theory, Graduate Texts in Mathematics, Springer, Berlin, 1997.

[Mak01] J.A. Makowsky, Colored Tutte polynomials and Kauffman brackets on graphs of bounded tree width, in: Discrete Applied Mathematics, to appear in 2003. Extended abstract published in Proceedings of the 12th Annual ACM-SIAM Symposium on Discrete Algorithms SODA'01, Washington D.C., ACM and SIAM, 2001, pp. 487–495.

[MM00] J.A. Makowsky, J.P. Mariño, Farrell polynomials on graphs of bounded tree width, Presented at FPSAC'01, 2000.

[MM02] J.A. Makowsky, J.P. Mariño, Farrell polynomials on graphs of bounded treewidth, Adv. Appl. Math. 30 (2003) 160–176.

[Mig99] J. Mighton, Knot theory on bipartite graphs, Ph.D. Thesis, Department of Mathematics, University of Toronto, Toronto, Canada, 1999.

[MS90] H.R. Morton, H.B. Short, Calculating the 2-variable polynomial for knots presented as closed braids, J. Algorithms 11 (1990) 117–131.

[MT90] H.R. Morton, P. Traczyk, Knots and algebras, in: E. Martin-Peinador, A. Rodez Usan (Eds.), Contribuciones Metematicas en homenaje al profesor D. Antonio Plans Sanz de Bremond, University of Zaragoza, Zaragoza, 1990, pp. 201–220.

[MW] H.R. Morton, A.J. Wassermann, A basis for the Birman–Wenzl algebra, Preprint 1989, updated January 2000, http://www.liv.ac.uk/~su14/knotprints.html.

[Mur96] K. Murasugi, Knot Theory and its Applications, Birkhäuser, Basel, 1996.

[PS97] V.V. Prasolov, A.B. Sossinsky, Knots, Links, Braids and 3-Manifolds, Translations of Mathematical Monographs, Vol. 154, Amer. Math. Soc., Providence, RI, 1997.

[Prz98] J. Przytycki, Classical roots of knot theory, Chaos, Solitons Fractals 9 (4–5) (1998) 531–545.

[Prz] J. Przytycki, private communication, Washington, January 2001.

[PT87] J. Przytycki, P. Traczyk, Invariants of Conway type, Kobe J. Math. 4 (1987) 115–139.

[PT01] J. Przytycki, T. Tsukamoto, The fourth skein module and the Montesinos–Nakanishi conjecture for 3-algebraic links, J. Knot Theory Ramifications 10 (7) (2001) 959–982.

[RS95] N. Robertson, P.D. Seymour, Graph minors. XIII. The disjoint paths problem, J. Combin. Theory Ser. B 63 (1995) 65–110.

[Sos99] A. Sossinsky, Noeuds: Genèse d'une théorie mathématique, Seuil, Paris, 1999.

[Vog90] P. Vogel, Representations of links by braids: a new algorithm, Comment. Math. Helv. 65 (1990) 104–113.

[Wel93] D.J.A. Welsh, Complexity: Knots, Colourings and Counting, in: London Mathematical Society Lecture Notes Series, Vol. 186, Cambridge University Press, Cambridge, 1993.