ELSEVIER Linear Algebra and its Applications 291 (1999) 235–244

LINEAR ALGEBRA AND ITS APPLICATIONS

Not all GKK τ-matrices are stable

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Received 30 October 1998; accepted 7 December 1998

Submitted by H. Schneider

Abstract

Hermitian positive definite, totally positive, and nonsingular M-matrices enjoy many common properties, in particular:

(A) positivity of all principal minors,
(B) weak sign symmetry,
(C) eigenvalue monotonicity,
(D) positive stability.

The class of GKK matrices is defined by properties (A) and (B), whereas the class of nonsingular τ-matrices by (A) and (C). It was conjectured that:

(A), (B) $\Rightarrow$ (D) [D. Carlson, J. Res. Nat. Bur. Standards Sect. B 78 (1974) 1–2],
(A), (C) $\Rightarrow$ (D) [G.M. Engel and H. Schneider, Linear and Multilinear Algebra 4 (1976) 155–176],
(A), (B) $\Rightarrow$ a property stronger than (D) [R. Varga, Numerical Methods in Linear Algebra, 1978, pp. 5–15],
(A), (B), (C) $\Rightarrow$ (D) [D. Hershkowitz, Linear Algebra Appl. 171 (1992) 161–186].

We describe a class of unstable GKK τ-matrices, thus disproving all four conjectures. © 1999 Elsevier Science Inc. All rights reserved.

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PII: S0024-3795(99)00002-6

1. Definitions and notation

Given a matrix $A \in \mathbb{C}^{n\times n}$, let $A(\alpha,\beta)$ denote the submatrix of A whose rows are indexed by α and columns by β ($\alpha, \beta \in \langle n\rangle := \{1,\dots,n\}$) and let $A[\alpha,\beta]$ denote $\det A(\alpha,\beta)$ if $\#\alpha = \#\beta$ (where # stands for the cardinality of a set) with the convention $A[\emptyset,\emptyset] := 1$.

A matrix A is called a P-matrix if $A[\alpha,\alpha] > 0 \quad \forall \alpha \subseteq \langle n\rangle$. A is *weakly sign-symmetric* if

$$A[\alpha,\beta]A[\beta,\alpha] \geqslant 0 \qquad \forall \alpha,\beta \in \langle n\rangle, \quad \#\alpha = \#\beta = \#\alpha\cup\beta - 1.$$

Weakly sign-symmetric P-matrices are also called GKK after Gantmacher, Krein, and Kotelyansky. It was proven by Gantmacher, Krein [5], and Carlson [2] that a P-matrix is GKK iff it satisfies the generalized Hadamard- -Fisher inequality

$$A[\alpha,\alpha]A[\beta,\beta] \geqslant A[\alpha\cup\beta,\alpha\cup\beta]A[\alpha\cap\beta,\alpha\cap\beta] \quad \forall \alpha,\beta \subseteq \langle n\rangle. \tag{1}$$

Carlson [3] conjectured that the GKK matrices are positive stable, i.e., $\mathrm{Re}\,\lambda > 0 \quad \forall \lambda \in \sigma(A)$ (here $\sigma(A)$ denotes, as usual, the spectrum of A), and showed his conjecture to be true for $n \leqslant 4$.

Let

$$l(A) := \begin{cases} \min\{\lambda \in \sigma(A)\cap\mathbb{R}\} & \text{if } \sigma(A)\cap\mathbb{R} \neq \emptyset, \\ \infty & \text{otherwise.} \end{cases}$$

A is called an ω-matrix if it has eigenvalue monotonicity

$$l(A(\alpha,\alpha)) \leqslant l(A(\beta,\beta)) < \infty \quad \text{whenever } \emptyset \neq \beta \subseteq \alpha \subseteq \langle n\rangle.$$

A is a τ-matrix if, in addition, $l(A) \geqslant 0$.

Engel and Schneider [4] asked if nonsingular τ-matrices, or equivalently, ω-matrices all whose principal minors are positive (see Remark 3.7 in [4]), are positive stable. Varga [9] conjectured even more than stability, viz.

$$|\arg(\lambda - l(A))| \leqslant \frac{\pi}{2} - \frac{\pi}{n} \quad \forall \lambda \in \sigma(A).$$

This inequality was proven for $n \leqslant 3$ by Varga (unpublished) and Hershkowitz and Berman [7] and for $n = 4$ by Mehrmann [8]. In his survey paper [6], Hershkowitz posed the weaker conjecture that τ-matrices that are also GKK are stable.

Below we describe a class of GKK τ-matrices which are not even nonnegative stable, i.e., have eigenvalues with negative real part. We construct Toeplitz Hessenberg matrices $A_{n,k,t}$ of size n for $k \in \mathbb{N}$ and $t \in \mathbb{R}$. We show that $A_{n,k,t}$ is GKK for any $t \in (0,1)$, a τ-matrix if $n \leqslant 2k+2$ and $t \in (0,1)$ is sufficiently small, and that $A_{2k+2,k,t}$ is unstable for sufficiently large k and sufficiently small positive t. This provides a counterexample to the Hershkowitz conjecture, and therefore, to the Carlson, Engel and Schneider, and Varga conjectures as well.

In what follows, we shall use the following notation:

$$p:q := \begin{cases} \{p, p+1, \ldots, q\} & \text{if } p \leqslant q, \\ \emptyset & \text{otherwise}, \end{cases} \quad \forall p, q \in \mathbb{N},$$

$$x_+ := \begin{cases} x & \text{if } x > 0, \\ 0 & \text{otherwise}, \end{cases} \quad \forall x \in \mathbb{R}.$$

2. Counterexample

Given $k, n \in \mathbb{N}$, and $t \in (0,1)$, let $A_{n,k,t}$ be the following Toeplitz Hessenberg matrix. If $n \leqslant k+1$, set

$$A_{n,k,t} := \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & \cdots & 1 & 1 \end{pmatrix}_{n \times n}.$$

Otherwise let

$$A_{n,k,t} := \begin{pmatrix} 1 & \overbrace{0 \cdots 0\; 0}^{k} & a_1^{k,t} & a_2^{k,t} & \cdots & a_{n-k-2}^{k,t} & a_{n-k-1}^{k,t} \\ 1 & 1 \cdots 0\; 0 & 0 & a_1^{k,t} & \cdots & a_{n-k-3}^{k,t} & a_{n-k-2}^{k,t} \\ \vdots & \vdots \ddots \vdots\; \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 \cdots 1\; 0 & 0 & 0 & \cdots & a_1^{k,t} & a_2^{k,t} \\ 0 & 0 \cdots 1\; 1 & 0 & 0 & \cdots & 0 & a_1^{k,t} \\ 0 & 0 \cdots 0\; 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 \cdots 0\; 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots \ddots \vdots\; \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 \cdots 0\; 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 \cdots 0\; 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}_{n \times n},$$

where $a_j^{k,t}$'s are chosen so that $A_{n,k,t}[\langle k+j+1\rangle, \langle k+j+1\rangle] = t^j$. This definition makes sense for all $j = 1, \ldots, n-k-1$. Indeed, the expansion of $A_{n,k,t}[\langle k+j+1\rangle, \langle k+j+1\rangle]$ by the first row gives

$$
\begin{aligned}
&A_{n,k,t}[\langle k+j+1\rangle, \langle k+j+1\rangle](= \det A_{k+j+1,k,t}) \\
&= A_{n,k,t}[2:k+j+1, 2:k+j+1] \\
&\quad + \sum_{l=1}^{j}(-1)^{k+l} a_l^{k,t} A_{n,k,t}[k+l+2:k+j+1, k+l+2:k+j+1] \\
&= \det A_{k+j,k,t} + \sum_{l=1}^{j}(-1)^{k+l} a_l^{k,t} \det A_{j-l,k,t}
\end{aligned} \tag{2}
$$

(recall that $A_{n,k,t}[\emptyset, \emptyset] = 1$, so the last term in the sum is well defined). As the coefficient of $a_j^{k,t}$ in Eq. (2) is equal to $(-1)^{k+j}$, the equation $A_{n,k,t} = s$ (linear in $a_j^{k,t}$) has a solution for any right-hand side s, in particular, for $s := t^j$. Since $A_{n,k,t}$ is Toeplitz, this implies $A_{n,k,t}[i:i+j-1, i:i+j-1] = t^{(j-k-1)_+}$.

We show that the matrices $A_{n,k,t}$ are GKK for any $t \in (0,1)$. Since $A_{n,k,t}$ is Hessenberg, the submatrix $A_{n,k,t}(\langle n\rangle \setminus i:i+j-1, \langle n\rangle \setminus i:i+j-1)$ is block upper triangular if $1 < i \leqslant i+j-1 < n$, so

$$
\begin{aligned}
&A_{n,k,t}[\alpha \cup \beta, \alpha \cup \beta] = A_{n,k,t}[\alpha, \alpha] A_{n,k,t}[\beta, \beta] \qquad \text{whenever } i < j \\
&- 1 \quad \text{for all} \quad i \in \alpha,\ j \in \beta.
\end{aligned} \tag{3}
$$

This shows that $A_{n,k,t}$ is a P-matrix. Moreover, since $0 < t < 1$ and

$$
\begin{aligned}
&(x+y-k-1)_+ + (x+z-k-1)_+ \\
&\leqslant (x-k-1)_+ + (x+y+z-k-1)_+ \quad \forall x, y, z \geqslant 0,
\end{aligned}
$$

we have

$$
\begin{aligned}
&A_{n,k,t}[i:i+j-1, i:i+j-1] \cdot A_{n,k,t}[l:l+m-1, l:l+m-1] \\
&\quad = t^{(j-k-1)_+ + (m-k-1)_+} \\
&\quad \geqslant t^{(l+m-i-k-1)_+ + (i+j-l-k-1)_+} \\
&\quad = A_{n,k,t}[i:l+m-1, i:l+m-1] \cdot A_{n,k,t}[l:i+j-1, l:i+j-1] \\
&\quad \text{if } l \leqslant i+j-1.
\end{aligned} \tag{4}
$$

Together with Eq. (3), Eq. (4) shows that $A_{n,k,t}$ satisfies Eq. (1) if α, β are sets of consecutive integers.

To prove Eq. (1) in general, first make a definition. Call the subsets $\alpha, \beta \subseteq \langle n\rangle$ *separated* if $|p-q| > 1\ \forall p \in \alpha, q \in \beta$. Suppose $\alpha, \beta_1, \ldots, \beta_j \subseteq \langle n\rangle$ are sets of consecutive integers, β_i $(i = 1, \ldots, j)$ are separated, and

$$
\text{for any } i = 1, \ldots, j, \text{ there exist } p \in \beta_i \text{ and } q \in \alpha \text{ such that } |p-q| \leqslant 1. \tag{5}
$$

Then $A_{n,k,t}$, α, and $\beta := \bigcup_{i=1}^{j} \beta_i$ satisfy Eq. (1). Indeed, Eq. (1) holds for α and β_1. If $1 \leqslant l < j$, then, assuming Eq. (1) for α and $\gamma_l := \bigcup_{i=1}^{l} \beta_i$, we have

$$A_{n,k,t}[\alpha,\alpha]A_{n,k,t}[\gamma_{l+1},\gamma_{l+1}] = A_{n,k,t}[\alpha,\alpha]A_{n,k,t}[\gamma_l,\gamma_l]A_{n,k,t}[\beta_{l+1},\beta_{l+1}]$$
$$\geqslant A_{n,k,t}[\alpha\cup\gamma_l,\alpha\cup\gamma_l]A_{n,k,t}[\alpha\cap\gamma_l,\alpha\cap\gamma_l]A_{n,k,t}[\beta_{l+1},\beta_{l+1}].$$

Due to Eq. (5), $\alpha\cup\gamma_l$ is a set of consecutive integers, so an application of Eq. (1) yields

$$A_{n,k,t}[\alpha\cup\gamma_l,\alpha\cup\gamma_l]A_{n,k,t}[\beta_{l+1},\beta_{l+1}]$$
$$\geqslant A_{n,k,t}[\alpha\cup\gamma_{l+1},\alpha\cup\gamma_{l+1}]A_{n,k,t}[(\alpha\cup\gamma_l)\cap\beta_{l+1},(\alpha\cup\gamma_l)\cap\beta_{l+1}].$$

But $(\alpha\cup\gamma_l)\cap\beta_{l+1} = \alpha\cap\beta_{l+1}$ since the sets β_i are pairwise disjoint. So,

$$A_{n,k,t}[\alpha,\alpha]A_{n,k,t}[\gamma_{l+1},\gamma_{l+1}]$$
$$\geqslant A_{n,k,t}[\alpha\cup\gamma_{l+1},\alpha\cup\gamma_{l+1}]A_{n,k,t}[\alpha\cap\gamma_l,\alpha\cap\gamma_l]A_{n,k,t}[\alpha\cap\beta_{l+1},\alpha\cap\beta_{l+1}]$$
$$= A_{n,k,t}[\alpha\cup\gamma_{l+1},\alpha\cup\gamma_{l+1}]A_{n,k,t}[\alpha\cap\gamma_{l+1},\alpha\cap\gamma_{l+1}]. \qquad (6)$$

Now, given a set of consecutive integers $\alpha\subseteq\langle n\rangle$ and any index set $\beta\subseteq\langle n\rangle$, write $\beta=\gamma_1\bigcup\gamma_2$ where $\gamma_1 := \bigcup_{i=1}^{l}\beta_i$, $\gamma_2 := \bigcup_{i=l+1}^{l+m}\beta_i$, all β_i ($i=1,\ldots,l+m$) are separated, and β_i satisfies Eq. (5) if and only if $i\leqslant l$. Then

$$A_{n,k,t}[\alpha,\alpha]A_{n,k,t}[\beta,\beta] = A_{n,k,t}[\alpha,\alpha]A_{n,k,t}[\gamma_1,\gamma_1]A_{n,k,t}[\gamma_2,\gamma_2]$$
$$\geqslant A_{n,k,t}[\alpha\cup\gamma_1,\alpha\cup\gamma_1]A_{n,k,t}[\alpha\cap\gamma_1,\alpha\cap\gamma_1]A_{n,k,t}[\gamma_2,\gamma_2]$$
$$= A_{n,k,t}[\alpha\cup\gamma_1\cup\gamma_2,\alpha\cup\gamma_1\cup\gamma_2]A_{n,k,t}[\alpha\cap\gamma_1,\alpha\cap\gamma_1]$$
$$= A_{n,k,t}[\alpha\cup\beta,\alpha\cup\beta]A_{n,k,t}[\alpha\cap\beta,\alpha\cap\beta].$$

In other words, $A_{n,k,t}$ satisfies Eq. (1) if $\alpha\subseteq\langle n\rangle$ is a set of consecutive integers and $\beta\subseteq\langle n\rangle$ is arbitrary.

Finally, if $\alpha_1,\alpha_2,\beta\subseteq\langle n\rangle$, the sets α_i ($i=1,2$) are separated, Eq. (1) holds for α_1 and β, and α_2 is a set of consecutive integers, then Eq. (1) holds for $\alpha := \alpha_1\cup\alpha_2$ and β:

$$A_{n,k,t}[\alpha,\alpha]A_{n,k,t}[\beta,\beta] = A_{n,k,t}[\alpha_1,\alpha_1]A_{n,k,t}[\alpha_2,\alpha_2]A_{n,k,t}[\beta,\beta]$$
$$\geqslant A_{n,k,t}[\alpha_1\cup\beta,\alpha_1\cup\beta]A_{n,k,t}[\alpha_1\cap\beta,\alpha_1\cap\beta]A_{n,k,t}[\alpha_2,\alpha_2]$$
$$\geqslant A_{n,k,t}[(\alpha_1\cup\beta)\cup\alpha_2,(\alpha_1\cup\beta)\cup\alpha_2]A_{n,k,t}[(\alpha_1\cup\beta)\cap\alpha_2,(\alpha_1\cup\beta)\cap\alpha_2]$$
$$A_{n,k,t}[\alpha_1\cap\beta,\alpha_1\cap\beta]$$
$$= A_{n,k,t}[\alpha\cup\beta,\alpha\cup\beta]A_{n,k,t}[\alpha_1\cap\beta,\alpha_1\cap\beta]A_{n,k,t}[\alpha_2\cap\beta,\alpha_2\cap\beta]$$
$$= A_{n,k,t}[\alpha\cup\beta,\alpha\cup\beta]A_{n,k,t}[\alpha\cap\beta,\alpha\cap\beta].$$

So, by induction on the number of 'components' of α, Eq. (1) holds for any α, $\beta\subseteq\langle n\rangle$. Thus, by the Gantmacher–Krein–Carlson theorem, $A_{n,k,t}$ is GKK for any $t\in(0,1)$ and any $k,n\in\mathbb{N}$.

Now check that $A_{n,k,t}$ have eigenvalue monotonicity if $n\leqslant 2k+2$ and $t\in(0,1)$ is sufficiently small. Let $\varphi_j^{k,t}(\lambda) := \det(A_{k+j+1,k,t}-\lambda I)$ for $j=1,\ldots,k+1$. We show by induction that

$$\varphi_j^{k,t}(\lambda) = \begin{cases} (1-\lambda)^{k+2} - (1-t) & \text{if } j = 1, \\ (1-\lambda)^{j+k+1} - j(1-t)(1-\lambda)^{j-1} + (j-1)(1-t)^2(1-\lambda)^{j-2} & \\ + \frac{t(1-t)^2}{((1-\lambda)-t)^2}\left[t^{j-1} - (j-1)t(1-\lambda)^{j-2} + (j-2)(1-\lambda)^{j-1}\right] & \text{if } j > 1, \end{cases} \tag{7}$$

$$a_j^{k,t} = \begin{cases} (-1)^k(1-t) & \text{if } j = 1, \\ (-1)^{k+j}t^{j-2}(1-t)^2 & \text{if } j > 1, \end{cases} \tag{8}$$

$$g_j^{k,t}(\lambda) := (-1)^{k+1}\det\begin{pmatrix} (1-\lambda) & 0 & \dots & 0 & a_j^{k,t} \\ 1 & (1-\lambda) & \dots & 0 & a_{j-1}^{k,t} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & (1-\lambda) & a_2^{k,t} \\ 0 & 0 & \dots & 1 & a_1^{k,t} \end{pmatrix}$$

$$= -(1-t)(1-\lambda)^{j-1} + (1-t)^2\frac{(1-\lambda)^{j-1} - t^{j-1}}{(1-\lambda)-t} \qquad \forall j \in \mathbb{N}. \tag{9}$$

By direct calculation, $\varphi_1^{k,t}(\lambda) = (1-\lambda)^{k+2} - (-1)^k a_1^{k,t}$, so, since $\varphi_1^{k,t}(0) = t$, we have $a_1^{k,t} = (-1)^k(1-t)$. Thus Eqs. (7)–(9) hold for $j = 1$. Now suppose that $j \geqslant 2$ and our formulas are true for $j-1$. Expansion of $\varphi_j(\lambda)$ by its last row gives

$$\varphi_j^{k,t}(\lambda) = (1-\lambda)\varphi_{j-1}^{k,t}(\lambda) + g_j^{k,t}(\lambda). \tag{10}$$

Since $\varphi_j^{k,t}(0) = t^j \ \forall j \in \mathbb{N}$, this implies $g_j^{k,t}(0) = t^j - t^{j-1}$. On the other hand, expanding $g_j^{k,t}(\lambda)$ by its first row, we get

$$g_j^{k,t}(\lambda) = (1-\lambda)g_{j-1}^{k,t}(\lambda) + (-1)^{j+k}a_j^{k,t}, \tag{11}$$

so $a_j^{k,t} = (-1)^{k+j}[g_j^{k,t}(\lambda) - (1-\lambda)g_{j-1}^{k,t}(\lambda)]_{\lambda=0} = (-1)^{j+k}t^{j-2}(1-t)^2$, which gives Eq. (8). Now, using Eq. (11) again together with the inductive hypothesis on $g_{j-1}^{k,t}$, we get Eq. (9):

$$g_j^{k,t}(\lambda) = -(1-t)(1-\lambda)^{j-1} + (1-t)^2\frac{(1-\lambda)^{j-1} - (1-\lambda)t^{j-2}}{(1-\lambda)-t} + t^{j-2}(1-t)^2$$

$$= -(1-t)(1-\lambda)^{j-1} + (1-t)^2\frac{(1-\lambda)^{j-1} - t^{j-1}}{(1-\lambda)-t}.$$

Finally, substituting the expression for $\varphi_{j-1}^{k,t}(\lambda)$ and the just verified expression for $g_j^{k,t}(\lambda)$ into Eq. (10) yields Eq. (7).

If $\langle n\rangle \supseteq \alpha = \bigcup_{i=1}^{l}\alpha_i$ is the union of separated sets of consecutive integers, then $\det(A_{n,k,t}(\alpha,\alpha) - \lambda I) = \prod_{i=1}^{l}\det(A(\alpha_i,\alpha_i) - \lambda I)$ since $A_{n,k,t} - \lambda I$ is Hessenberg (the same observation earlier led to Eq. (3)). Since $A_{n,k,t} - \lambda I$ is Toeplitz, the product in the right-hand side equals $\prod_{i=1}^{l}\det(A_{n,k,t}(\langle\#\alpha_i\rangle,\langle\#\alpha_i\rangle) - \lambda I)$. Hence, to prove eigenvalue monotonicity of $A_{n,k,t}$ for $n \leqslant 2k+2$ it is enough to prove it for leading principal submatrices of $A_{n,k,t}$ only, i.e., to show

$$l(A_{k+j+1,k,t}) \leqslant l(A_{k+j,k,t}) \quad \forall j \in \mathbb{N},$$

i.e., that $\varphi_j^{k,t}$ has a root in $(0,1]$ for any $j \leqslant k+1$, and

$$\min\{\lambda \in (0,1] : \varphi_j(\lambda) = 0\} \leqslant \min\{\lambda \in (0,1] : \varphi_{j-1}(\lambda) = 0\},$$
$$j = 2,\ldots,k+1$$

(since $A_{k+j,k,t}$ is a P-matrix, the coefficients of its characteristic polynomial are strictly alternating, so $A_{k+j,k,t}$ has no nonpositive eigenvalues). Observe that $\varphi_j^{k,t}(\lambda) = t^j - \lambda\tilde{\varphi}_j^{k,t}(\lambda)$ where

$$\tilde{\varphi}_j^{k,t}(0) = -\left.\frac{\mathrm{d}\varphi_j^{k,t}(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=0} \underset{t\to 0+}{\to} = -\left.\frac{\mathrm{d}v_j^k(\lambda)}{\mathrm{d}\lambda}\right|_{\lambda=0},$$

$$v_j^k(\lambda) := \lim_{t\to 0+}\varphi_j^{k,t}(\lambda) = (1-\lambda)^{j+k+1} - j(1-\lambda)^{j-1} + (j-1)(1-\lambda)^{j-2}.$$

So, $\lim_{t\to 0+}\tilde{\varphi}_j^{k,t}(0) = k+3-j \geqslant 2 \ \forall j = 1,\ldots,k+1$.

Since 0, the minimal real root of v_j^k, is simple, the minimal real root λ_j of $\varphi_j^{k,t}$ is positive and simple for all $j = 1,\ldots,k+1$ whenever t is sufficiently small. But then $\tilde{\varphi}_j^{k,t}(\lambda_j)$ is bounded below by a positive constant for any $j = 1,\ldots,k+1$, hence

$$\lambda_j = \frac{t^j}{\tilde{\varphi}_j^{k,t}(\lambda_j)} < \frac{t^{j-1}}{\tilde{\varphi}_{j-1}^{k,t}(\lambda_{j-1})} = \lambda_{j-1} \quad \forall j = 1,\ldots,k+1,$$

if t is small. So, for any $k \in \mathbb{N}$ and $n \leqslant 2k+2$, there exists $t(k) \in (0,1)$ such that $A_{n,k,t}$ is a τ-matrix for all $t \in (0, t(k))$.

Now let $B_k := \lim_{t\to 0+} A_{2k+2,k,t}$. The matrix B_k is Toeplitz with first column

$$(1,1,\underbrace{0,\ldots,0}_{(2k\ \text{times})})^{\mathrm{T}}$$

and first row

$$(1,\underbrace{0,\ldots,0}_{k\ \text{times}},(-1)^k,(-1)^k,\underbrace{0,\ldots,0}_{k-1\ \text{times}}).$$

We show that there exists $K \in \mathbb{N}$ such that, for all $k > K$, B_k has an eigenvalue λ with $\mathrm{Re}\,\lambda < 0$. As the eigenvalues depend continuously on the entries of the matrix, this will demonstrate that, for any $k > K$, there exists $t \in (0,1)$ such that the GKK τ-matrix $A_{2k+2,k,t}$ has an eigenvalue with negative real part.

The polynomial ν^k_{k+1} has a root with negative real part iff the polynomial ψ_k where

$$\psi_k(\lambda) := \frac{\nu^k_{k+1}(-\lambda)}{(1+\lambda)^{k-1}} = (1+\lambda)^{k+3} - (k+1)(1+\lambda) + k$$

has a root with positive real part. Since

$$\psi_k(\lambda) = \lambda\left[\sum_{j=0}^{k+1}\binom{k+3}{j}\lambda^{k+3-j-1} + 2\right],$$

it is, in turn, enough to show that η_k where

$$\eta_k(\lambda) := \lambda^{k+3}\psi_k\left(\frac{1}{\lambda}\right) = 2\lambda^{k+2} + \sum_{j=2}^{k+3}\binom{k+3}{j}\lambda^{k+3-j}$$

has a root with positive real part. The Hurwitz matrix for the polynomial η_k is

$$H_k := \begin{pmatrix} \binom{k+3}{2} & \binom{k+3}{4} & \binom{k+3}{6} & \binom{k+3}{8} & \binom{k+3}{10} & \cdots \\ 2 & \binom{k+3}{3} & \binom{k+3}{5} & \binom{k+3}{7} & \binom{k+3}{9} & \cdots \\ 0 & \binom{k+3}{2} & \binom{k+3}{4} & \binom{k+3}{6} & \binom{k+3}{8} & \cdots \\ 0 & 2 & \binom{k+3}{3} & \binom{k+3}{5} & \binom{k+3}{7} & \cdots \\ 0 & 0 & \binom{k+3}{2} & \binom{k+3}{4} & \binom{k+3}{6} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}_{(k+2)\times(k+2)}$$

Compute the minor $H_k[2:5,2:5]$, taking out the factors $\binom{k+3}{2}$, $\binom{k+3}{4}$, $\binom{k+3}{6}$ from its second, third, and fourth columns respectively. We obtain

$$H_k[2:5,2:5] = -\frac{1}{132\,300}(3k^3 - 49k^2 - 210k - 318)(k+4)^2(k+5)\binom{k+3}{2}\binom{k+3}{4}\binom{k+3}{6}.$$

It follows that $H_k[2:5,2:5] < 0$ for k large enough, precisely, for all $k > 20$. But the Hurwitz matrix of a nonpositive stable polynomial is totally nonnegative (see [1]). So, for $k > 20$, η_k has a zero with positive real part, therefore,

γ_{k+1}^k has a zero with negative real part. This completes the proof that the GKK τ-matrices $A_{2k+2,k,t}$ are unstable for sufficiently large k and small t.

Remark. *To illustrate the result, consider the matrix* $A_{44,21,1/2}$, *i.e., the Toeplitz matrix whose first column is*

$$(1, 1, \underbrace{0, \ldots, 0}_{(42 \text{ times})})^{\mathrm{T}}$$

and first row is

$$(1, \underbrace{0, \ldots, 0}_{21 \text{ times}}, -1/2, -1/2^2, 1/2^3, -1/2^4, \ldots, -1/2^{22})$$

and the limit matrix B_{21}, *with the same first column as* $A_{44,21,1/2}$ *and first row equal to*

$$(1, \underbrace{0, \ldots, 0}_{21 \text{ times}}, -1, -1, \underbrace{0, \ldots, 0}_{20 \text{ times}}).$$

According to MATLAB, the two eigenvalues with minimal real part of the first matrix are $-2.809929189497896 \cdot 10^{-2} \pm 3.275076252367531 \cdot 10^{-1}\mathrm{i}$; *those of the second are* $-3.420708309454068 \cdot 10^{-2} \pm 3.400425852703498 \cdot 10^{-1}\mathrm{i}$.

Acknowledgements

I am grateful to Professor Hans Schneider for many valuable remarks and suggestions and to the referee for careful reading.

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