Topology and its Applications 157 (2010) 760–773
Contents lists available at ScienceDirect
Topology and its Applications
www.elsevier.com/locate/topol

Cascade search principle and its applications to the coincidence problems of n one-valued or multi-valued mappings

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ARTICLE INFO

Article history:
Received 20 November 2008
Received in revised form 11 March 2009

Keywords:
Cascade search
Multi-cascade
Search functional
Coincidence
Common fixed point

ABSTRACT

The problem of the construction of a multi-cascade with a given limit subset A is considered in a metric space X. A multi-cascade is a discrete multi-valued dynamic system with the translation semigroup $(Z_{\geqslant 0}, +)$. The *cascade search principle* using so-called *search* functionals is suggested. It gives a solution of the problem. Also, an estimation is obtained for the distance between any initial point x and every correspondent limit point. Several applications of one-valued and multi-valued versions of the mentioned cascade search principle are given for the cases when the limit subset A is (1) the full (or expanded) preimage of a closed subspace under a mapping from X to another metric space; (2) the coincidence set (or expanded coincidence set) of n mappings from X to another metric space $(n > 1)$; (3) the common preimage (or the expanded one) of a closed subspace under n mappings; and (4) the common fixed point set of n mappings of the space X into itself $(n \geqslant 1)$. Generalizations of the previous authors results are obtained. And, in particular cases, generalizations of some recent results by A.V. Arutyunov on coincidences of two mappings and a generalization of Banach fixed point principle are obtained.

0. Introduction

This article is a development of the author's results shown in [12] and partly in [11]. We consider the problem of the construction of a multi-cascade with a given limit subset A on a metric space X. A cascade[1] on X is defined as a discrete dynamic system with X as the phase space and $(Z_{\geqslant 0}, +)$ as the translation semigroup. Here we use the term *multi-cascade* which means a multi-valued cascade. To construct multi-cascades with a given limit set A, $A \subset X$, we use so-called *search* functionals with their nil-subspace being equal to A and with some special properties. We suggest here one-valued and multi-valued versions of so-called general *cascade search principle* depending on the use of one-valued or multi-valued functionals. Of course, the one-valued version is a particular case of the multi-valued one, but we present them separately for the convenience of references and notations.

We give several applications of both versions of the cascade search principle as well. Let (X, ρ), (Y, d) be metric spaces; $f^{-1}(H)$ be the full preimage of a closed subspace $H \subset Y$ under a mapping $f: X \to Y$; $Coin(f_1, \ldots, f_n) \subseteq X$ be the coincidence set of mappings $f_1, \ldots, f_n: X \to Y$ $(n \geqslant 2)$; and $Comfix(f_1, \ldots, f_n) \subset X$ be the common fixed point set of mappings $f_1, \ldots, f_n: X \to X$ $(n \geqslant 1)$. We apply the one-valued version of the cascade search principle to the cases when A is equal to one of the subsets $f^{-1}(H), Coin(f_1, \ldots, f_n)$, $Comfix(f_1, \ldots, f_n)$, and so-called *common preimage* of H under $f_1, \ldots, f_n$ (see Definition 1.19 below). In this way we obtain generalizations of all our results as exposed in [11,12]. Note that, in those turn, some of the results in [12,11] give a generalization of one recent result on coincidences of two mappings [4, Theorem 1]

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[1] Good term *cascade* was suggested by D.V. Anosov.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2009.08.006

and a (known) generalization of the well-known Banach fixed point principle (see, for example, [2, p. 70]). Also, we apply the cascade search principle to similar cases of the limit set A concerning multi-valued mappings. In particular case, we obtain a generalization of one more recent theorem on two multi-valued mappings [4, Theorem 3].

The results obtained here can be applied to the theory of systems of functional differential equations and inclusions. In some sense, the cascade search method suggested here is a discrete analogue in metric spaces of the well-known gradient search method.

In contrast to other works concerning the common fixed points existence problem (see, for example, [3,5–7]), we do not impose any conditions of commutability or any closed ones to mappings under consideration. Definitions and terminology of the theory of multi-valued mappings can be found in [1]. Nevertheless, for the convenience, in addition to references, we give all necessary definitions and formulations. The article is organized as follows.

In Section 1 we expose the cascade search principle using a *one-valued* search functional (Theorem 1.3). Then we give its applications mentioned above (Theorems 1.4, 1.7, 1.8, 1.11, 1.12, 1.14). Under the more strong conditions, similar results are proved in [12] independently, and here we show how to obtain their generalizations from Theorem 1.3. At the end of the section we give some remarks on modifications of above theorems. Also, we give the general statement (Theorem 1.20) concerning the search for the common preimage of a closed subspace under n mappings. That statement combines the previous results of the section and implies useful Corollary 1.21 concerning the search for the common roots of n mappings.

Section 2 consists of 3 subsections and is devoted to multi-valued mappings.

In Section 2(a) we expose the cascade search principle using a *multi-valued search functional* (Theorem 2.4) and give its application to the cascade search for the full preimage and for the expanded preimage of a closed subspace under a multi-valued mapping (Theorem 2.11).

In Section 2(b) we apply the preceding results to the cascade search for the coincidence set and for the expanded coincidence set of n multi-valued mappings (Theorems 2.13–2.15). Then we show (Statement 2.17) that under $n = 2$ Theorem 2.13 represents a generalization of the recent result [4, Theorem 3]. Much as it was done at the end of Section 1, we give the general statement (Theorem 2.19) concerning the search for the common preimage (or for the expanded one) of a closed subspace under n multi-valued mappings. That statement combines the results of Sections 2(a) and 2(b) and implies useful Corollary 2.21 concerning the search for the common roots of n multi-valued mappings.

In Section 2(c) we apply the preceding results to the cascade search for the common fixed points of n multi-valued mappings (Theorems 2.22, 2.23 and Remark 2.24). At the end of the paper we give a concluding remark concerning generators of considered multi-cascades (Remark 2.25) and an example (Example 2.26) illustrating the comparison of Theorems 2.13 and 2.16.

1. Cascade search principle: one-valued version and its applications

In this section we give a general result (Theorem 1.3) which we call the one-valued *cascade search principle*. Then we show that this principle implies generalizations of all our theorems contained in [12], some of which imply in particular cases a generalization of the recent result [4, Theorem 1], and a generalization of well-known Banach fixed point principle (see, for example [2, p. 70]).

Definition 1.1. Let $f : X \to Y$ be a mapping (one- or multi-valued) between metric spaces, and $Graph(f) \subseteq X \times Y$ be its graph. For a non-empty subset $A \subset Y$, we say the graph $Graph(f)$ is *A-closed* if it contains all its limit points $(x, y) \in X \times Y$, such that $y \in A$. We say $Graph(\varphi)$ is *A-complete* if any fundamental sequence $\{x_n, y_n\}_{n=0,1,\dots} \subseteq Graph(f)$ with $d(y_n, A) \xrightarrow[n\to\infty]{} 0$ converges to some pair $(\xi, \eta) \in Graph(f)$ where $\eta \in A$. So, $\eta \in f(\xi) \cap A$.

Definition 1.2. Let $\varphi : X \to R$ be a non-negative functional on metric space (X, ρ), and $0 < \beta < \alpha$. We say the functional φ is (α, β)*-search* on X (with respect to its nil-subspace $Nil(\varphi) := \{x \in X \mid \varphi(x) = 0\}$) if for any $x \in X$ there is a point $x' \in X$, $\rho(x, x') \leqslant \frac{\varphi(x)}{\alpha}$, such that $\varphi(x') \leqslant \frac{\beta}{\alpha} \cdot \varphi(x)$.

Theorem 1.3 *(Cascade search principle: one-valued version). Let (X, ρ) be a metric space, $\varphi : X \to R$ be a non-negative functional with its non-empty nil-subspace $Nil(\varphi) = A \subset X$. Suppose that either $Graph(\varphi)$ is 0-complete or X is complete and $Graph(\varphi)$ is 0-closed. Let the functional φ be (α, β)-search on X for some numbers α, β, $0 < \beta < \alpha$. Then there exists a multi-cascade on X with the limit set A, and the distance between any initial point $x_0 \in X$ and every correspondent limit point is not greater then $\frac{\varphi(x_0)}{\alpha-\beta}$. In other words, for every point $x_0 \in X$ there exists a convergent sequence $\{x_m\}_{m=0,1,\dots}$ (generally not unique) beginning with x_0 which is an iteration sequence with respect to the generator of the cascade, $\lim_{m\to\infty} x_m = \xi \in A$, and $\rho(x_0, \xi) \leqslant \frac{\varphi(x_0)}{\alpha-\beta}$.*

Proof. Take any point $x_0 \in X$. We shall construct the required sequence by induction. Let $x_1 = x'$ which does exist according to the theorem conditions. And so on. If a point x_m is already chosen and $\varphi(x_m) = 0$, that is $x_m \in A$, then put $x_j = x_m$ for any $j > m$. If $\varphi(x_m) > 0$ then according to the theorem conditions, there exists a point x_{m+1} with $\rho(x_m, x_{m+1}) \leqslant \frac{\varphi(x_m)}{\alpha}$ and $\varphi(x_{m+1}) \leqslant \frac{\beta}{\alpha} \cdot \varphi(x_m)$.

So, we obtain a sequence $\{x_m\}_{m=0,1,\ldots}$. It is fundamental because $\rho(x_m, x_{m+1}) \leqslant \frac{\varphi(x_m)}{\alpha} \leqslant (\frac{\beta}{\alpha})^m \cdot \frac{\varphi(x_0)}{\alpha} \xrightarrow[m\to\infty]{} 0$. One can also see from the last inequalities that $\varphi(x_m) \xrightarrow[m\to\infty]{} 0$. Consequently, the sequence $\{(x_m, \varphi(x_m))\}_{m=0,1,\ldots}$ is fundamental. As *Graph*(φ) is 0-complete or as X is complete and *Graph*(φ) is 0-closed, in both cases it converges to $(\xi, 0)$, and we have $(\xi, 0) \in$ *Graph*(φ). It means that $\varphi(\xi) = 0$, i.e. $\xi \in A$.

Now, let us estimate the distance $\rho(x_0, \xi)$. We have

$$\rho(x_0,\xi) = \lim_{m\to\infty} \rho(x_0, x_m) \leqslant \lim_{m\to\infty} \sum_{k=1}^{m} \rho(x_{k-1}, x_k) \leqslant \lim_{m\to\infty} \sum_{k=1}^{m} \left(\frac{\beta}{\alpha}\right)^{k-1} \cdot \frac{\varphi(x_0)}{\alpha} = \frac{\varphi(x_0)}{\alpha} \cdot \frac{1}{1-\frac{\beta}{\alpha}} = \frac{\varphi(x_0)}{\alpha-\beta}.$$

Theorem 1.3 is proved. □

Below we give a number of applications of Theorem 1.3. We obtain generalizations of our results from [12] where independent proofs were given for all statements. And here we show how their generalizations follow from Theorem 1.3 above.

Let (X, ρ), (Y, d) be metric spaces. Let H be a closed subspace in Y. The following statement is a generalization of our previous result [12, Theorem 1].

Theorem 1.4. *Let $f : X \to Y$ be a mapping which takes any fundamental sequence to a fundamental one, and its graph Graph(f) be H-complete. Let the functional $d(f(x), H)$ is (α, β)-search on X for some numbers α, β, $0 < \beta < \alpha$. Then there exists a multi-cascade on X with its limit set being equal to $F^{-1}(H)$, and the distance between any initial point $x_0 \in X$ and every correspondent limit point is not greater then $\frac{\rho(f(x_0),H)}{\alpha-\beta}$. In other words, for any point $x_0 \in X$ there exists a convergent sequence $\{x_m\}_{m=0,1,\ldots}$ (generally not unique) beginning with x_0 which is an iteration sequence with respect to the generator of the cascade such that $\lim_{m\to\infty} x_m = \xi \in X$, $f(\xi) \in H$, and $\rho(x_0, \xi) \leqslant \frac{\rho(f(x_0),H)}{\alpha-\beta}$.*

Proof. The statement follows from Theorem 1.3. We should put $\varphi(x) := d(f(x), H)$, $x \in X$. It is easy to see that all conditions of Theorem 1.3 are fulfilled for this functional. We comment the condition of 0-completeness of *Graph*(φ). Indeed, let a sequence $\{(x_m, \varphi(x_m))\}_{m=0,1,\ldots}$ be fundamental, and $\varphi(x_m) \xrightarrow[m\to\infty]{} 0$. According to the conditions, f takes fundamental sequences to fundamental ones, consequently the sequence $\{f(x_m)\}_{m=0,1,\ldots}$ is also fundamental. And $\varphi(x_m) = d(f(x_m), H) \xrightarrow[m\to\infty]{} 0$. As *Graph*$(f)$ is H-complete the sequence $\{(x_m, f(x_m))\}_{m=0,1,\ldots}$ converges and its limit $(\xi, z) \in$ *Graph*(f) which means that $z = f(\xi)$. Note that as the metric d is continuous, $d(z, H) = \lim_{m\to\infty} d(f(x_m), H) = \lim_{m\to\infty} \varphi(x_m) = 0$. As H is a closed subspace, $z \in H$. So, $\varphi(\xi) = d(f(\xi), H) = 0$. Consequently, $(\xi, 0) = \lim_{m\to\infty}(x_m, \varphi(x_m)) \in$ *Graph*(φ), hence *Graph*(φ) is 0-complete. □

The problem of the approaching to the coincidence set of n given mappings $f_1, \ldots, f_n : X \to Y$ can be considered as an important particular case of the previous result. Indeed, this problem is equivalent to the problem of the approaching to the diagonal $\Delta_n \subset \underbrace{Y \times \cdots \times Y}_{n} = Y^n$ for the mapping $f = f_1 \times \cdots \times f_n : X \to Y^n$. Let metric D in Y^n be defined as follows $D(x, y) = \sum_{i=1}^{n} d(x_i, y_i)$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in Y^n$.

Definition 1.5. We define the metrical functional $\tilde{D}$ by the following equality. $\tilde{D}(y) := \frac{1}{n-1} \sum_{1\leqslant i<j\leqslant n} d(y_i, y_j)$, $y = (y_1, \ldots, y_n) \in Y^n$.

Now, let us formulate the following technical statement useful for the sequel.

Lemma 1.6. *(Lemma 1 in [12]) For any element $y = (y_1, \ldots, y_n) \in Y^n$, $n \geqslant 2$, the following estimation holds*

$$\tilde{D}(y) \leqslant D(y, \Delta_n) \leqslant \frac{2(n-1)}{n} \tilde{D}(y).$$

Now we appear at the following statement which is a generalization of [12, Theorem 2]. We have replaced the condition of continuity of all mappings $f_1, \ldots, f_n$ in [12] with conditions (j) and (jj) below.

Theorem 1.7. *Let $f_1, \ldots, f_n : X \to Y$ be given mappings between metric spaces (X, ρ), (Y, d), and the following conditions hold*:

(j) *at least one of the mappings $f_1, \ldots, f_n$ takes any fundamental sequence to a fundamental one*;
(jj) *Graph(f) is Δ_n-complete, where $f = f_1 \times \cdots \times f_n : X \to Y^n$; and*
(jjj) *the functional $D(f(x), \Delta_n)$ is (α, β)-search on X with some numbers α, β, $0 < \beta < \alpha$.*

Then there exists a multi-cascade on X with its limit set being equal to $Coin(f_1, \dots, f_n)$, and the distance between any initial point $x_0 \in X$ and every correspondent limit point is not greater then $\frac{D(f(x_0),\Delta_n)}{\alpha-\beta}$. In other words, for any point $x_0 \in X$ there exists a convergent sequence $\{x_m\}_{m=0,1,\dots}$ (generally not unique) which is an iteration sequence with respect to the generator of the cascade beginning with x_0 such that $\lim_{m\to\infty} x_m = \xi = \xi(x_0) \in Coin(f_1, \dots, f_n) \subset X$, and $\rho(x_0, \xi) \leqslant \frac{D(f(x_0),\Delta_n)}{\alpha-\beta}$.

Proof. One can deduce Theorem 1.7 from Theorem 1.3 when taking $\varphi(x) := D(f(x), \Delta_n)$. It is clear that $Nil(\varphi) = Coin(f_1, \dots, f_n)$. The condition (jjj) coincides with the correspondent condition of Theorem 1.3 for φ. Now let us show that the continuity of the metric D and the conditions (j) and (jj) imply the 0-completeness of $Graph(\varphi)$. Indeed, take any fundamental sequence $\{(x_m, \varphi(x_m))\}_{m=0,1,\dots}$ with $\varphi(x_m) \xrightarrow[m\to\infty]{} 0$. According to the condition (j), we can suppose $\{f_1(x_m)\}_{m=0,1,\dots}$ is fundamental. Now, using the inequality of Lemma 1.6, we obtain the following computations $(1 \leqslant i < j \leqslant n)$ $d(f_i(x_m), f_j(x_m)) \leqslant (n-1)\cdot \tilde{D}(f(x_m)) \leqslant (n-1)\cdot D(f(x_m), \Delta_n) = (n-1)\cdot \varphi(x_m) \xrightarrow[m\to\infty]{} 0$. Consequently, all sequences $\{f_i(x_m)\}_{m=0,1,\dots}$ $(i = 1, \dots, n)$ draw together. It follows that all of them are fundamental, that is the sequence $\{(x_m, f(x_m))\}_{m=0,1,\dots}$ is fundamental, and $D(f(x_m), \Delta_n) \xrightarrow[m\to\infty]{} 0$. As $Graph(f)$ is Δ_n-complete, that sequence converges. Let its limit be $(\xi, \tilde{\eta})$, $\tilde{\eta} = (\eta, \dots, \eta) \in \Delta_n \subset Y^n$, and $(\xi, \tilde{\eta}) \in Graph(f)$, that is $\tilde{\eta} = f(\xi)$. It follows that the sequence $\{(x_m, \varphi(x_m))\}_{m=0,1,\dots}$ converges to $(\xi, 0) \in Graph(\varphi)$ because the metric functional $D(\cdot, \Delta_n)$ is continuous. So, $Graph(\varphi)$ is 0-complete. So, all conditions of Theorem 1.3 are fulfilled, and the statement of Theorem 1.7 follows. The proof is over. □

Below we give a similar generalization of one more our theorem [12, Theorem 3]. We again replace the condition of the continuity of all mappings with the conditions (j) and (jj) of Theorem 1.7.

Theorem 1.8. *Let mappings $f_1, \dots, f_n : X \to Y$ be given, $f = f_1 \times \dots \times f_n : X \to Y^n$, and the conditions* (j), (jj) *of Theorem* 1.7 *hold. Besides, suppose for some numbers α, β, $0 < \beta < \alpha$, the functional $\tilde{D}(f(x))$ is $(\alpha, \frac{\beta\cdot n}{2(n-1)})$-search on X. Then there exists a multi-cascade on X with its limit set being equal to $Coin(f_1, \dots, f_n)$, and the distance between any initial point $x_0 \in X$ and every correspondent limit point is not greater then $\frac{\tilde{D}(f(x_0))2(n-1)}{2\alpha(n-1)-\beta n}$. In other words, for any point $x_0 \in X$ there exists a convergent sequence $\{x_m\}_{m=0,1,\dots} \subset X$ (generally not unique) which is an iteration sequence with respect to the generator of the cascade beginning with x_0 such that its limit $\lim_{m\to\infty} x_m = \xi \in Coin(f_1, \dots, f_n) \subset X$, and*

$$\rho(x_0, \xi) \leqslant \frac{\tilde{D}(f(x_0))2(n-1)}{2\alpha(n-1) - \beta n}.$$

Proof. The statement follows from Theorem 1.3. We should put $\varphi(x) := \tilde{D}(f(x))$, and consider in Theorem 1.3 the pair of numbers $(\frac{\beta\cdot n}{2\cdot(n-1)}, \alpha)$ (with (β, α) given in Theorem 1.8) instead of any pair (β, α). The proving is quite similar to the case of Theorem 1.7. □

In case $n = 2$ Theorems 1.7 and 1.8 coincide because in this case we have the equality $D(F(x_0), \Delta_n) = \tilde{D}(F(X_0))$.

In [12], we have shown that in case $n = 2$ Theorems 2 and 3 coincide and give a generalization of one of the theorems by A.V. Arutyunov [4, Theorem 1] (see discussion after Theorem 4 in [12], Remark 3 at the end of the paper, and Example 1, which gives two mappings satisfying the conditions of Theorem 2 in [12], but not satisfying any condition of [4, Theorem 1]). Of course, the same is true for Theorems 1.7 and 1.8 above. For the convenience, below we give the necessary definition and formulate the mentioned Arutyunov's result.

Definition 1.9. A mapping $f : X \to Y$ (one-valued or multi-valued) is called λ-*covering*, if there exists a number $\lambda > 0$, such that for any $r > 0$ and every $x \in X$, the inclusion $B_{\lambda r}(f(x)) \subseteq f(B_r(x))$ is valid.

Theorem 1.10. *([4, Theorem 1]) Let X, Y be metric spaces, and X be complete. Let $f_1, f_2 : X \to Y$ be any two mappings, the first one is continuous and λ-covering, the second one satisfies Lipschitz condition with a constant γ, and $0 < \gamma < \lambda$. Then, for every $x_0 \in X$ there exists a point $\xi = \xi(x_0) \in X$, such that $f_1(\xi) = f_2(\xi)$ and the following estimation holds*

$$\rho(x_0, \xi) \leqslant \frac{\rho(f_1(x_0), f_2(x_0))}{\lambda - \gamma}.$$

Now let us return to the general case of n mappings $(n \geqslant 2)$.

The following statement is a generalization of [12, Theorem 5].

Theorem 1.11. *Let n mappings $f_1, \dots, f_n : X \to Y$ be given; $f = f_1 \times \dots \times f_n : X \to Y^n$ and the conditions* (j), (jj) *of Theorem* 1.7 *are fulfilled. In addition, suppose that for some numbers α, β, $0 < \beta < \alpha$, the functional $T(f(x)) := \sum_{i=1}^{n-1} \rho(f_i(x), f_{i+1}(x))$ is (α, β)-search on X. Then there exists a multi-cascade on X with its limit set being equal to $Coin(f_1, \dots, f_n)$, and the distance between any initial point $x_0 \in X$ and every correspondent limit point is not greater than $\frac{T(f(x_0))}{\alpha-\beta}$. In other words, for any point $x_0 \in X$ there exists* *a convergent sequence* $\{x_m\}_{m=0,1,\ldots} \subset X$ *(generally not unique) which is an iteration sequence with respect to the generator of the cascade beginning with* x_0 *such that its limit* $\lim_{m\to\infty} x_m = \xi \in X$, $f_1(\xi) = \cdots = f_n(\xi)$, *and* $\rho(x_0, \xi) \leqslant \frac{T(f(x_0))}{\alpha-\beta}$.

Proof. The statement follows from Theorem 1.3. The proving is quite similar to ones of Theorems 1.7 and 1.8 above. In this case one should put $\varphi(x) := T(F(x))$, $x \in X$. □

Under $n = 2$, Theorem 1.11 coincides with Theorems 1.7 and 1.8 above.

Now let us consider the problem of the approaching to the common fixed point set of n mappings. It should be noticed that in the case of $X = Y$, Theorems 1.7, 1.8, and 1.11 give solutions of the problem when being applied to $n + 1$ mappings (meaning n given mappings and the identical one). The formulations are not given here. Besides, in [12, Theorem 7] we suggested one more *modus operandi* concerning the solving of this problem. Below we give a generalization of that result.

Theorem 1.12. *Let* $f_1, \ldots, f_n : X \to X$ $(n \geqslant 1)$, *and the condition* (jj) *of Theorem* 1.7 *holds* (*under* $Y = X$) *for the mapping* $f = f_1 \times \cdots \times f_n$. *Put* $\Phi(x) := \sum_{i=1}^{n} \rho(x, f_i(x)) + \frac{2(n-1)}{n} \tilde{D}(f(x))$ *and suppose for some number* α, $0 < \alpha < 1$, *the functional* $\frac{\Phi(x)}{n}$ *is* $(1, \alpha)$*-search on* X. *Then there exists a multi-cascade on* X *with its limit set being equal to Comfix*$(f_1, \ldots, f_n)$, *and the distance between any initial point* $x_0 \in X$ *and every correspondent limit point is not greater than* $\frac{\Phi(x_0)}{n(1-\alpha)}$. *In other words, for any point* $x_0 \in X$ *there exists a convergent sequence* $\{x_m\}_{m=0,1,\ldots} \subset X$ (*generally not unique*) *which is an iteration sequence with respect to the generator of the cascade beginning with* x_0, *with* $\lim_{m\to\infty} x_m = \xi \in$ *Comfix*$(f_1, \ldots, f_n)$, *and*

$$\rho(x_0, \xi) \leqslant \frac{\Phi(x_0)}{n(1-\alpha)}.$$

Proof. Theorem 1.12 also follows from Theorem 1.3. One should put in Theorem 1.3 $X = Y$, $\varphi(x) = \frac{\Phi(x)}{n}$, and take numbers $(1, \alpha)$, $0 < \alpha < 1$, instead of (α, β). □

Note that under the conditions of Theorem 1.12, for any point $x \in X$ there exists a point x', $\rho(x, x') \leqslant \frac{\Phi(x)}{n}$, and $\Phi(x') \leqslant \alpha \cdot \Phi(x)$. And, in [12, Theorem 7] there is the additional condition $x' \in M(x)$, that is $\sum_{j=1}^{n} \rho(f_j(x), x') \leqslant \frac{2(n-1)}{n} \tilde{D}(F(x))$. So, the location set for the point x' in Theorem 1.12 above is essentially wider than in [12, Theorem 7].

Under the conditions of Theorem 1.12, if $n = 1$ (or $n > 1$ and $f_1 = \cdots = f_n$) we obtain the following consequence which is a generalization of our result [12, Theorem 8] and represents a generalization of the well-known *Banach fixed point principle* (see, for example, [2, p. 70]).

Theorem 1.13. *Let* X *be a metric space,* $f : X \to X$ *be any mapping, and Graph*(f) *be complete. Let a number* α, $0 < \alpha < 1$, *be such that the functional* $\rho(x, f(x))$ *is* $(1, \alpha)$*-search on* X. *Then there exists a multi-cascade on* X *with its limit set being equal to Fix*(f), *and the distance between any initial point* $x_0 \in X$ *and every correspondent limit point is not greater than* $\frac{\rho(x_0, f(x_0))}{1-\alpha}$. *In other words, for any point* $x_0 \in X$ *there exists a convergent sequence* $\{x_m\}_{m=0,1,\ldots} \subset X$ (*generally not unique*) *which is an iteration sequence with respect to the generator of the cascade beginning with* x_0, *such that* $\lim_{m\to\infty} x_m = \xi \in$ *Fix*(f), *and*

$$\rho(x_0, \xi) \leqslant \frac{\rho(x_0, f(x_0))}{1-\alpha}.$$

It should be also mentioned here that in [12, Theorem 8] the constructed cascade trajectory beginning with x_0 is just the iteration sequence $\{f^m(x_0)\}_{m=0,1,\ldots}$ because the equality $x' = f(x)$ is necessary. But, in the above Theorem 1.13 the last equality is not necessary. It is just required that $\rho(x, x') \leqslant \rho(x, f(x))$. In this sense, Theorem 1.13 represents a (known) generalization of the Banach fixed point principle.

Theorem 1.13 (like as [12, Theorem 8]) does not guarantee the uniqueness of the fixed point $\xi = \xi(x)$ but is applicable to a much more wide class of mappings than contracting ones.

Now we suggest one more variant of the cascade search for the common fixed point set of n mappings, using the functional $\varphi(x) := \sum_{i=1}^{n} \rho(x, f_i(x))$. In this way, we obtain the following statement which is also a consequence of Theorem 1.3.

Theorem 1.14. *Let* X *be a metric space,* $f_1, \ldots, f_n : X \to X$ $(n \geqslant 1)$, $f = f_1 \times \cdots \times f_n : X \to X^n$. *Let the condition* (jj) *of Theorem* 1.7 *is fulfilled for* f, *under* $Y = X$. *Besides, suppose that the functional* $\theta(x) := \sum_{i=1}^{n} \rho(x, f_i(x))$ *is* (α, β)*-search on* X *for some numbers* β, α, $0 < \beta < \alpha$. *Then there exists a multi-cascade on* X *with its limit set being equal to Comfix*$(f_1, \ldots, f_n)$, *and the distance between any initial point* $x_0 \in X$ *and every correspondent limit point is not greater than* $\frac{\theta(x_0)}{\alpha-\beta}$. *In other words, for any point* $x_0 \in X$ *there exists a convergent sequence* $\{x_m\}_{m=0,1,\ldots} \subset X$ (*generally not unique*) *beginning with* x_0 *which is an iteration sequence with respect to the generator of the cascade with* $\lim_{m\to\infty} x_m = \xi \in$ *Comfix*$(f_1, \ldots, f_n)$, *and* $\rho(x_0, \xi) \leqslant \frac{\theta(x_0)}{\alpha-\beta}$. □

It is clear that there is a lot of other functionals which allow to obtain statements similar to Theorems 1.12, 1.14.

Remark 1.15. Note that Theorem 1.4 remains valid if we replace the condition that the mapping f takes fundamental sequences to fundamental ones with the condition that H is compact subspace. (Compare with Theorem 2.11(b) below.)

Remark 1.16. Theorem 1.7 remains valid if we replace the conditions (j) and (jj) with the following two conditions

(j′) at least one of the mappings $f_1, \ldots, f_n$ takes fundamental sequences to fundamental ones and its graph is complete; and
(jj′) *Graph*(f) is Δ_n-closed.

The same is true for Theorems 1.8 and 1.11.

Remark 1.17. Theorem 1.12 remains valid if we replace the condition (jj) with the following one: (jj″) X is complete, and *Graph*(f) is Δ_n-closed.

Remark 1.18. It was noted in [8–10] that the preimage of a closed subspace under one mapping and the coincidence set of n mappings can be naturally considered as particular cases of the set of common preimages of a closed subspace under n mappings (see Definition 1.19 below).

Definition 1.19. ([9, p. 291]) Let mappings $f_1, \ldots, f_n : X \to Y$ be given, $f := f_1 \times \cdots \times f_n : X \to Y^n$ and H be a closed subspace in Y. We say the set $P(f_1, \ldots, f_n, H) := f^{-1}(\Delta_n(H)) = \{x \in X \mid f_1(x) = \cdots = f_n(x) \in H\}$ is *the common preimage (set) of the subspace H under the mappings* $f_1, \ldots, f_n$. Here $\Delta_n(H) := \{\tilde{h} = (h, \ldots, h) \in \Delta_n \mid h \in H\}$. In case $H = \{c\}$, $c \in Y$, the set $P(f_1, \ldots, f_n, \{c\}) := f^{-1}(\tilde{c}) = \{x \in X \mid f_1(x) = \cdots = f_n(x) = c\}$ is called the *common root set* of the mappings $f_1, \ldots, f_n$ corresponding to the value c.

Now, applying Theorem 1.4 to the mapping $f = f_1 \times \cdots \times f_n : X \to Y^n$ and to the closed subspace $\tilde{H} = \Delta_n(H) \subset Y^n$ and using the sufficient conditions of Theorem 1.7 to provide the 0-completeness of the graph *Graph*(φ) of the functional $\varphi(x) := D(f(x), \Delta_n(H))$, $x \in X$, we obtain the following statement.

Theorem 1.20. *Let $f_1, \ldots, f_n : X \to Y$ be given mappings, $f = f_1 \times \cdots \times f_n : X \to Y^n$ and the following conditions hold*:

(j) *at least one of the mappings $f_1, \ldots, f_n$ takes any fundamental sequence to a fundamental one*;
(jj) *Graph(f) is $\Delta_n(H)$-complete*; *and*
(jjj) *the functional $D(f(x), \Delta_n(H))$ is (α, β)-search on X with some numbers α, β, $0 < \beta < \alpha$.*

Then there exists a multi-cascade on X with its limit set being equal to $P(f_1, \ldots, f_n, H) \subseteq Coin(f_1, \ldots, f_n) \subset X$, and the distance between any initial point $x_0 \in X$ and every correspondent limit point is not greater then $\frac{D(f(x_0), \Delta_n(H))}{\alpha - \beta}$. In other words, for any point $x_0 \in X$ there exists a convergent sequence $\{x_m\}_{m=0,1,\ldots}$ (generally not unique) which is an iteration sequence with respect to the generator of the cascade beginning with x_0, such that $\lim_{m\to\infty} x_m = \xi = \xi(x_0) \in P(f_1, \ldots, f_n, \tilde{H})$, and $\rho(x_0, \xi) \leqslant \frac{D(f(x_0), \Delta_n(H))}{\alpha - \beta}$.

It is not difficult to see that in case $H = Y$ Theorem 1.20 implies Theorem 1.7, and in case $n = 1$ it implies Theorem 1.4. Similarly to Remark 1.16 above, it should be noted that Theorem 1.20 remains valid if we replace the conditions (j) and (jj) with the following conditions: ($\tilde{\text{j}}$) at least one of the mappings $f_1, \ldots, f_n$ takes fundamental sequences to fundamental ones and its graph is H-complete and ($\tilde{\text{jj}}$) *Graph*(f) is $\Delta_n(H)$-closed. Besides, when applying Theorem 1.20 to the case $H = \{c\}$, $c \in Y$, we obtain the following useful statement, which solves the problem of the searching for common roots of the mappings $f_1, \ldots, f_n$.

Corollary 1.21. *Let $f_1, \ldots, f_n : X \to Y$ be given mappings between metric spaces (X, ρ), (Y, d), $f = f_1 \times \cdots \times f_n : X \to Y^n$, and $c \in Y$ be a given point. Suppose that the following conditions hold*:

(j) *at least one of the mappings $f_1, \ldots, f_n$ takes any fundamental sequence to a fundamental one*;
(jj) *Graph(f) is $\{\tilde{c}\}$-complete where $\tilde{c} = (c, \ldots, c) \in Y^n$*; *and*
(jjj) *the functional $D(f(x), \tilde{c}) = \sum_{i=1}^n d(f_i(x), c)$ is (α, β)-search on X for some numbers α, β, $0 < \beta < \alpha$.*

Then there exists a multi-cascade on X with its limit set being equal to $P(f_1, \ldots, f_n, \tilde{c})$, and the distance between any initial point $x_0 \in X$ and every correspondent limit point is not greater then $\frac{D(f(x_0), \tilde{c})}{\alpha - \beta}$. In other words, for any point $x_0 \in X$ there exists a convergent sequence $\{x_m\}_{m=0,1,\ldots}$ (generally not unique) which is an iteration sequence with respect to the generator of the cascade beginning with x_0, such that $\lim_{m\to\infty} x_m = \xi = \xi(x_0) \in P(f_1, \ldots, f_n, \tilde{c})$, and $\rho(x_0, \xi) \leqslant \frac{D(f(x_0), \tilde{c})}{\alpha - \beta}$.

In the next section we shall consider similar problems for the case of multi-valued functionals and multi-valued mappings.

2. Cascade search in case of multi-valued mappings

2(a). Cascade search for the preimage of a closed subspace

Definition 2.1. Let $\varphi : X \to P(R)$ be a non-negative multi-valued functional, $P(R)$ be the totality of non-empty subsets of R. We define *nil-subspace* of φ as $Nil(\varphi) = \{x \in X \mid 0 \in \varphi(x)\}$, and *expanded nil-subspace* as $Nil_+(\varphi) = \{x \in X \mid \varphi_*(x) = 0\}$. Here and everywhere below $\varphi_*(x)$ stands for $\inf_{\gamma \in \varphi(x)}\{\gamma\}$.

Definition 2.2. Let (X, ρ) be a metric space, and $0 < \beta < \alpha$. We say a non-negative multi-valued functional $\varphi : X \to P(R)$ is (α, β)-*search* on X, if the one-valued functional $\varphi_*(x)$ is (α, β)-search is the sense of Definition 1.2.

Definition 2.3. We say that the graph *Graph*(φ) of a multi-valued functional φ is 0-*closed* (*weakly* 0-*closed*) if for any its limit element of the form $(\xi, 0)$, $\xi \in Nil(\varphi)$ ($\xi \in Nil_+(\varphi)$). And we say *Graph*(φ) is 0-*complete* (weakly 0-*complete*) if any fundamental sequence $\{(x_m, \varphi_m)\}_{m=9,1,\dots} \subset Graph(\varphi)$ with $\varphi_m \xrightarrow[m\to\infty]{} 0$ converges to a pair $(\xi, 0)$ where $\xi \in Nil(\varphi)$ ($\xi \in Nil_+(\varphi)$).

Theorem 2.4 *(Cascade search principle: multi-valued version). Let (X, ρ) be a metric space, a multi-valued functional $\varphi : X \to P(R)$ is non-negative and (α, β)-search on X for some α, β, $0 < \beta < \alpha$, and one of the following conditions is fulfilled*:

(I) *Graph(φ) is* 0*-complete, or X is complete and Graph(φ) is* 0*-closed*;
(II) *Graph(φ) is weakly* 0*-complete, or X is complete and Graph(φ) is weakly* 0*-closed.*

Then there exists a multi-cascade on X with the limit set being equal either to $Nil(\varphi)$ in the case of (I)*, or to $Nil_+(\varphi)$ in the case of* (II)*, and the distance between any initial point $x_0 \in X$ and every correspondent limit point is not greater than $\frac{\varphi_*(x_0)}{\alpha-\beta}$. In other words, for every point $x_0 \in X$, there exists a convergent sequence $\{x_m\}_{m=0,1,\dots}$ (generally not unique) beginning with x_0 which is an iteration sequence with respect to the generator of the cascade, with its limit $\lim_{m\to\infty} x_m = \xi \in Nil(\varphi)$ in the case of* (I)*, or $\xi \in Nil_+(\varphi)$ in the case of* (II)*. And, in both cases, $\rho(x_0, \xi) \leqslant \frac{\varphi_*(x_0)}{\alpha-\beta}$.*

Proof. We shall proceed similarly to the proving of Theorem 1.3 above. Take any point $x_0 \in X$. We can construct the required convergent sequence by induction. If a point x_m is already chosen, and $x_m \in Nil(\varphi)$ ($x_m \in Nil_+(\varphi)$), then one should put $x_j = x_m$ for any $j > m$. If not, then according to the condition that φ is (α, β)-search, there exists a point $x_{m+1} = x_{m+1}(x_m)$ with $\rho(x_m, x_{m+1}) \leqslant \frac{\varphi_*(x_m)}{\alpha}$ and $\varphi_*(x_{m+1}) \leqslant \frac{\beta}{\alpha} \cdot \varphi_*(x_m)$. In such a way, we obtain the sequence $\{x_m\}_{m=0,1,\dots}$ which is fundamental because

$$\rho(x_m, x_{m+1}) \leqslant \frac{\varphi_*(x_m)}{\alpha} \leqslant \left(\frac{\beta}{\alpha}\right)^m \cdot \frac{\varphi_*(x_0)}{\alpha} \xrightarrow[m\to\infty]{} 0.$$

Now, for any $\delta > 0$ consider a sequence $\{\gamma_m\}_{m=0,1,\dots}$, where $\gamma_m \in \varphi(x_m)$, and $\gamma_m \leqslant \varphi_*(x_m) \cdot (1+\delta)$. Then

$$\gamma_m \leqslant \varphi_*(x_m) \cdot (1+\delta) \leqslant \left(\frac{\beta}{\alpha}\right)^m \cdot \varphi_*(x_0) \cdot (1+\delta) \xrightarrow[m\to\infty]{} 0.$$

As $\gamma_m \geqslant 0$, it follows that there exists $\lim_{m\to\infty} \gamma_m = 0$. Then each of the conditions sets (I) and (II) implies that the sequence of pairs $\{(x_m, \gamma_m)\}_{m=0,1,\dots}$ converges to some pair $(\xi, 0)$, where $\xi \in Nil(\varphi)$ in the case of (I), or $\xi \in Nil_+(\varphi)$ in the case of (II), respectively.

Let us estimate the distance $\rho(x_0, \xi)$. In view of the continuity of the metric, we have

$$\rho(x_0, \xi) = \lim_{m\to\infty} \rho(x_0, x_m) \leqslant \lim_{m\to\infty} \sum_{k=1}^{m} \rho(x_{k-1}, x_k) \leqslant \lim_{m\to\infty} \sum_{k=1}^{m} \left(\frac{\beta}{\alpha}\right)^{k-1} \cdot \frac{\varphi_*(x_0)}{\alpha} = \frac{\varphi_*(x_0)}{\alpha} \cdot \frac{1}{1-\frac{\beta}{\alpha}} = \frac{\varphi_*(x_0)}{\alpha-\beta}.$$

Theorem 2.4 is proved. □

Below we give a number of consequences of Theorem 2.4 concerning multi-valued mappings. Therefore, now we turn to multi-valued mappings and give definitions we shall need hereinafter.

Let $F : X \to C(Y)$ be a multi-valued mapping between metric spaces (X, ρ), (Y, d), and $C(Y)$ be the totality of all non-empty closed subsets of Y. As above, we shall consider Y^n as a metric space with the metric D where $D(y, z) := \sum_{i=1}^{n} d(y_i, z_i)$, $y = (y_1, \dots, y_n)$, $z = (z_1, \dots, z_n) \in Y^n$.

Definition 2.5. The mapping F is called *k-Lipschitz* ($k > 0$) if for any two points x, x' of the space X the following inequality holds $h(F(x), F(x')) \leqslant k \cdot \rho(x, x')$ where $h : C(Y) \times C(Y) \to R \cup \{\infty\}$ is the (*expanded*) *Hausdorff metric* defined for any $A, B \in C(Y)$ by the following formula

$$h(A, B) := \max\Big\{\sup_{a\in A}\big\{d(a, B)\big\}, \sup_{b\in B}\big\{d(b, A)\big\}\Big\} \quad \text{where } d(a, B) := \inf_{b\in B}\big\{d(a, b)\big\}.$$

Recall that the graph *Graph*(F) of a mapping F is called *complete* if any fundamental sequence of its elements converges to some element belonging to *Graph*(F). We say *Graph*(F) is *closed* if limits of all convergent sequences of its elements are contained in it.

Definition 2.6. Let a multi-valued mapping $F : X \to Y$ be given, and H be a closed subspace of Y. We say the set $F^{-1}(H) = \{x \in X \mid F(x) \cap H \neq \emptyset\}$ is the *full preimage of* H under the mapping F, and the set $F_+^{-1}(H) = \{x \in X \mid d(F(x), H) = 0\}$ is the *expanded preimage* of H under F.

Definition 2.7. We say the graph $Graph(F) \subseteq X \times Y$ of a multi-valued mapping $F : X \to C(Y)$ is *H-complete*, if any fundamental sequence $\{(x_m, y_m)\}_{m=0,1,\dots} \subseteq Graph(F)$ with $d(y_m, H) \xrightarrow[m\to\infty]{} 0$, converges to some pair $(x, y) \in Graph(F)$ with $y \in F(x) \cap H$.

Definition 2.8. A multi-valued mapping $F : X \to Y$ is called *upper semi-continuous* at a point $x \in X$, if for any open set $V \subset Y$ with $F(x) \subset V$, there exists a neighbourhood $U(x)$ of the point x such that $F(U(x)) \subset V$. A mapping F is upper semi-continuous on X if it is upper semi-continuous at any point x, $x \in X$.

Definition 2.9. We say that a multi-valued mapping $F : X \to Y$ is *sequentially upper semi-continuous* at a point ξ, if for any convergent sequence $\{x_k\}_{k=0,1,\dots}$ with $\lim_{k\to\infty} x_k = \xi$, any sequence $\{y_k\}_{k=0,1,\dots}$ with $y_k \in F(x_k)$ has the property that $\lim_{k\to\infty} d(y_k, F(\xi)) = 0$. And, a multi-valued mapping F is called sequentially upper semi-continuous on X, if it has that property at any point of X.

Note that if a mapping is upper semi-continuous, then it is sequentially upper semi-continuous. But, generally speaking, the converse is not true. The following simple example shows that sequentially upper semi-continuous mappings are not, in general, upper semi-continuous.

Example 2.10. Let X be the axes OX, and $Y = \{(x, y) \mid y > 0\}$ be the upper semi-plane of the Cartesian plane with usual metric. Let $F : X \to Y$ be a mapping taking any point $x \in X$ to the set $F(x) = \{(x, y) \mid y > 0\} \subset Y$. It is easy to see that the mapping F is 1-Lipschitz mapping and consequently it is sequentially upper semi-continuous. But one can easily check that it is not an upper semi-continuous.

The following Theorem 2.11 solves the problems of the cascade search for the expanded preimage and for the full preimage of a closed subspace under a multi-valued mapping. Note that everywhere here we interpret *compact metric spaces* as spaces in which any sequence has a convergent subsequence.

Theorem 2.11. *Let (X, ρ), (Y, d) be metric spaces, $F : X \to C(Y)$ be sequentially upper semi-continuous multi-valued mapping, and $H \subset Y$ be a closed subspace of the space Y. Suppose the multi-valued functional $d_{(F,H)}(x) := \{d = d(y, H) \mid y \in F(x)\}$, $x \in X$, is (α, β)-search on X for some α, β, $0 < \beta < \alpha$, and one of the following conditions is fulfilled*:

(a) *X is complete*;
(b) *H is compact, and Graph(F) is H-complete.*

Then there exists a multi-cascade on X with its non-empty limit set A, where either $A = F_+^{-1}(H)$ in the case of (a), *or $A = F^{-1}(H)$ in the case of* (b), *and in both cases the distance between any initial point $x_0 \in X$ and every correspondent limit point is not greater than $\frac{d(F(x_0),H)}{\alpha-\beta}$. In other words, for any point $x_0 \in X$ there exists a sequence $\{x_m\}_{m=0,1,\dots}$ (generally not unique) beginning with x_0 which is an iteration sequence with respect to the generator of the cascade $x_m \xrightarrow[m\to\infty]{} \xi$ either with $\xi \in F_+^{-1}(H)$ in the case of* (a), *or with $\xi \in F^{-1}(H)$ in the case of* (b). *And in both cases $\rho(x_0, \xi) \leqslant \frac{d^*_{(F,H)}(x_0)}{\alpha-\beta} = \frac{d(F(x_0),H)}{\alpha-\beta}$.*

Proof. Put $\varphi(x) := d_{(F,H)}(x)$, $x \in X$. Then $Nil_+(\varphi) = F_+^{-1}(H)$, $Nil(\varphi) = F^{-1}(H)$, and φ is (α, β)-search on X.

Let the condition (a) is fulfilled, that is X is complete. Show that in this case *Graph*(φ) is weakly 0-closed. Indeed, take any sequence $\{(x_m, \gamma_m)\}_{m=0,1,\dots} \subseteq Graph(\varphi)$, that is $\gamma_m = d(y_m, H)$, $y_m \in F(x_m)$. Let it converge to $(\xi, 0) \in X \times R$. The sequence $\{x_m\}_{m=0,1,\dots}$ converges to ξ. Then, according to the condition that F is sequentially upper semi-continuous, $\{y_m\}_{m=0,1,\dots}$ converges to the set $F(\xi)$, that is $d(y_m, F(\xi)) \xrightarrow[m\to\infty]{} 0$. And, at the same time, $\gamma_m = d(y_m, H)$ goes to 0. It follows that $\varphi_*(\xi)=d(F(\xi),H)=0$, that is $Graph(\varphi)$ is weakly 0-closed. Then according to Theorem 2.4(II), there exists a multi-cascade on X with the limit set being equal to $Nil_+(\varphi)=F_+^{-1}(H)$.

Let now the condition (b) is fulfilled, that is H is compact and $Graph(F)$ is H-complete. Show that in this case $Graph(\varphi)$ is 0-complete. Take any fundamental sequence $\{(x_m,\gamma_m)\}_{m=0,1,\ldots}\subseteq Graph(\varphi)$, where $\gamma_m=d(y_m,H)$, $y_m\in F(x_m)$, and γ_m goes to 0. Then it follows that there exists a sequence $\{z_m\}_{m=0,1,\ldots}\subset H$ such that $d(y_m,z_m)\xrightarrow[m\to\infty]{}0$. As H is compact, there is a convergent subsequence $\{z_{m_k}\}_{m=0,1,\ldots}$, and we have $d(y_{m_k},z_{m_k})\xrightarrow[m\to\infty]{}0$. It follows that the subsequence $\{y_{m_k}\}_{m=0,1,\ldots}$ is also convergent, and hence the same is true for the subsequence $\{(x_{m_k},y_{m_k})\}_{k=1,2,\ldots}$. As $Graph(F)$ is H-complete, the last subsequence converges to some pair $(\xi,\eta)\in Graph(F)$ that is $\xi\in X$, $\eta\in F(\xi)$, and $d(\eta,H)=0$. Moreover, as the initial sequence $\{x_m\}_{m=0,1,\ldots}$ is fundamental, it also converges to ξ. So, we obtain that the initial sequence of pairs $\{(x_m,\gamma_m)\}_{m=0,1,\ldots}$ converges to the pair $(\xi,0)\in Graph(\varphi)$. It means that $Graph(\varphi)$ is 0-complete. Then according to Theorem 2.4(I), there exists a multi-cascade on X with its limit set A, where $A=F^{-1}(H)$.

So, in this case the statement follows from Theorem 2.4(I). The proof is over. □

2(b). Cascade search for the coincidences of n multi-valued mappings

Now, let us consider the problem of the cascade search for the coincidence set and for the expanded coincidence set of n multi-valued mappings ($n>1$).

Definition 2.12. A point $\xi\in X$ is called *a coincidence point* of multi-valued mappings $F_1,\ldots,F_n:X\to C(Y)$, if $\bigcap_{i=1}^n F_i(\xi)\neq\emptyset$. The set of all coincidence points is called the *coincidence set* and denoted by $Coin(F_1,\ldots,F_n)$. We call the set $Coin_+(F_1,\ldots,F_n)=\{x\in X\mid D((F_1\times\cdots\times F_n)(x),\Delta_n)=0\}$ the *expanded coincidence set* of the family $F_1,\ldots,F_n$.

Theorem 2.13. *Let X be a complete metric space, $F_1,\ldots,F_n:X\to C(Y)$ be multi-valued sequentially upper semi-continuous mappings, and $F=F_1\times\cdots\times F_n:X\to C(Y^n)$. Let there exist numbers $0<\beta<\alpha$, such that the multi-valued functional $\Psi(x):=\{\psi=D(y,\Delta_n)\mid y\in F(x)\}$ is (α,β)-search on X. Then there exists a multi-cascade on X with the non-empty limit set being equal to $Coin_+(F_1,\ldots,F_n)$, and the distance between any initial point $x_0\in X$ and every correspondent limit point is not greater than $\frac{D(F(x_0),\Delta_n)}{\alpha-\beta}$. In other words, for any $x_0\in X$ there exists a sequence $\{x_m\}_{m=0,1,\ldots}$ (generally not unique) beginning with x_0 which is an iteration sequence with respect to the generator of the cascade $x_m\xrightarrow[m\to\infty]{}\xi\in X$, such that $\xi\in Coin_+(F_1,\ldots,F_n)$ and $\rho(x_0,\xi)\leqslant\frac{D(F(x_0),\Delta_n)}{\alpha-\beta}$.*

Proof. The statement follows from Theorem 2.11(a). Indeed, as all multi-valued mappings $F_1,\ldots,F_n:X\to C(Y)$ are sequentially upper semi-continuous, the same is true for the mapping $F=F_1\times\cdots\times F_n:X\to C(Y^n)$. Taking $H=\Delta_n$ we see that all conditions of Theorem 2.11(a) are fulfilled. The proof is over. □

Theorem 2.14. *Let all conditions of Theorem 2.13 be fulfilled, and additionally at least one of the mappings F_i, $1\leqslant i\leqslant n$, takes convergent sequences to compact subsets. Then there exists a multi-cascade on X with the non-empty limit set being equal to $Coin(F_1,\ldots,F_n)$, and the distance between any initial point $x_0\in X$ and every correspondent limit point is not greater than $\frac{D(F(x_0),\Delta_n)}{\alpha-\beta}$. In other words, for any point $x_0\in X$ there exists a sequence $\{x_m\}_{m=0,1,\ldots}$ (generally not unique) beginning with x_0 which is an iteration sequence with respect to the generator of the cascade with $x_m\xrightarrow[m\to\infty]{}\xi\in X$, such that $\xi\in Coin(F_1,\ldots,F_n)$ and $\rho(x_0,\xi)\leqslant\frac{D(F(x_0),\Delta_n)}{\alpha-\beta}$.*

Proof. Show that under the conditions of the theorem the graph $Graph(\Psi)$ is 0-closed, and the statement follows from Theorem 2.4(I). Take any convergent sequence $\{(x_m,\psi_m)\}_{m=0,1,\ldots}\subset Graph(\Psi)$ such that $(x_m,\psi_m)\xrightarrow[m\to\infty]{}(\xi,0)$. Let $\psi_m=d(y_m,\Delta_n)\in\Psi(x_m)$, $y_m=(y_{1m},\ldots,y_{nm})\in F(x_m)$. Suppose that the mapping F_1 takes any convergent sequence to a compact subset. Then, the whole image of the convergent sequence $\{x_m\}_{m=0,1,\ldots}\subset X$ under F_1 is compact in Y. In particular, it follows that one can choose a convergent subsequence $\{y_{1m_k}\}_{k=0,1,\ldots}$ from the sequence $\{y_{1m}\}_{m=0,1,\ldots}$, $y_{1m}\in F_1(x)$. So, the subsequence $\{(x_{m_k},y_{1m_k})\}_{k=0,1,\ldots}\subset Graph(F_1)$ converges to some pair $(\xi,\eta)\in X\times Y$. As $D(y_m,\Delta_n)$ goes to 0, all sequences $\{y_{im_k}\}_{k=0,1,\ldots}$ $(i=1,\ldots,n)$ draw together. Consequently, all of them converge and have the same limit $\eta\in Y$. In other words, there exists $\lim_{k\to\infty}y_{m_k}=\tilde\eta=(\eta,\ldots,\eta)\in\Delta_n$. On the other hand, as all mappings are sequentially upper semi-continuous, we have also $d(\eta,F_i(\xi))=0$, $i=1,\ldots,n$. It means that $\tilde\eta\in F(\xi)$, that is $(\xi,0)\in Graph(\Psi)$. So, $Graph(\Psi)$ is 0-closed, and we can see that the conditions of Theorem 2.4(I) are fulfilled for the functional Ψ. Note that $Nil(\Psi)=Coin(F_1,\ldots,F_n)$. So, the statement follows from Theorem 2.4(I). □

Theorem 2.15. *Let $F_1,\ldots,F_n:X\to C(Y)$ be multi-valued sequentially upper semi-continuous mappings, and $F=F_1\times\cdots\times F_n:X\to C(Y^n)$. Let the multi-valued functional $\Psi(x):=\{\psi=D(y,\Delta_n)\mid y\in F(x)\}$ be (α,β)-search on X where $0<\beta<\alpha$. Let also Y be compact, and at least one of the graphs $Graph(F_1),\ldots,Graph(F_n)$ be complete. Then there exists a multi-cascade on X with the non-empty limit set $Coin(F_1,\ldots,F_n)$, and the distance between any initial point $x_0\in X$ and every correspondent limit point is not greater than $\frac{D(F(x_0),\Delta_n)}{\alpha-\beta}$. In other words, for any $x_0\in X$ there exists a sequence $\{x_m\}_{m=0,1,\ldots}$ (generally not unique) which is an iteration sequence with respect to the generator of the cascade beginning with x_0, $x_m\xrightarrow[m\to\infty]{}\xi\in X$, such that $\xi\in Coin(F_1,\ldots,F_n)$ and $\rho(x_0,\xi)\leqslant\frac{D(F(x_0),\Delta_n)}{\alpha-\beta}$.*

Proof. Show that the statement follows from Theorem 2.4(I). It is sufficient to prove that under the theorem conditions the functional $\Psi(x)$ has 0-complete graph. Take any fundamental sequence $\{(x_m, \psi_m)\}_{m=0,1,\dots}$, $\psi_m \in \Psi(x_m)$, that is $\psi_m = D(y_m, \Delta_n)$ where $y_m \in F(x_m)$. Let $\psi_m \xrightarrow[m\to\infty]{} 0$. Let $Graph(F_1)$ be complete. As Y is compact, there is a convergent subsequence $\{y_{1m_k}\}_{k=1,2,\dots}$ of the sequence $\{y_{1m}\}_{m=0,1,\dots}$. So, we have the fundamental subsequence $\{(x_{m_k}, y_{1m_k})\}_{k=1,2,\dots} \subseteq Graph(F_1)$. As $Graph(F_1)$ is complete, there is the limit $\lim_{k\to\infty}(x_{m_k}, y_{1m_k}) = (\xi, \eta) \in Graph(F_1)$, that is $\eta \in F_1(\xi)$, $\xi = \lim_{k\to\infty} x_m$. Furthermore, as $\psi_m = D(y_m, \Delta_n)$ goes to 0, all subsequences $\{y_{im_k}\}_{k=1,2,\dots}$, $i = 1, \dots, n$, draw together. It follows that all of them converge to the same limit η. It means that the subsequence $\{(x_{m_k}, y_{m_k})\}_{k=1,2,\dots}$ goes to $(\xi, \tilde{\eta})$, $\tilde{\eta} = (\eta, \dots, \eta) \in \Delta_n$. On the other hand, as all mappings F_i are sequentially upper semi-continuous, $d(\eta, F_i(\xi)) = 0$, $i = 1, \dots, n$. It follows that $(\xi, 0) \in Graph(\Psi)$. So, $Graph(\Psi)$ is 0-complete, and the statement follows from Theorem 2.4(I). □

Now, let us compare Theorem 2.13 with the following recent result.

Theorem 2.16. *([4, Theorem 3 and footnote 1)]) Let (X, ρ), (Y, d) be metric spaces, X be complete. Suppose a multi-valued mapping $Q : X \to C(Y)$ is $\tilde{\alpha}$-covering and sequentially upper semi-continuous, and a multi-valued mapping $P : X \to C(Y)$ is $\tilde{\beta}$-Lipschitz with $0 < \tilde{\beta} < \tilde{\alpha}$. Then for any $x_0 \in X$ and for any $\varepsilon > 0$ there exists $\xi = \xi(x_0) \in X$ such that $d(Q(\xi), P(\xi)) = 0$ and $\rho(x_0, \xi) \leqslant \frac{d(Q(x_0), P(x_0))}{\tilde{\alpha}-\tilde{\beta}} + \varepsilon$.*

Note that the author of [4] considers multi-valued sequentially upper semi-continuous mappings as upper semi-continuous ones (see [4, footnote 1)]) which does not correspond to the commonly used definition of an upper semi-continuous mapping (see, for example, [1, Definition 1.2.13] and Example 2.10 above).

Statement 2.17. Theorem 2.16 follows from Theorem 2.13, under $n = 2$.

Proof. Indeed, it is easily seen that any Lipschitz multi-valued mapping is sequentially upper semi-continuous. Now, let us show that under the conditions of Theorem 2.16 the one-valued functional $\varphi(x) := D((Q \times P)(x), \Delta_2)$ is (α, β)-search on X for some α, β, $0 < \beta < \alpha$. At first, note that $\varphi(x) = d(Q(x), P(x))$. Indeed,

$$\varphi(x) = D\big((Q \times P)(x), \Delta_2\big) := \inf_{y_1, y_2, z}\big\{D\big((y_1, y_2), (z, z)\big)\big\} = \inf_{y_1, y_2, z}\big\{d(y_1, z) + d(y_2, z)\big\}.$$

And we have the following inequalities

$$\begin{aligned} d\big(Q(x), P(x)\big) &:= \inf_{y_1, y_2}\big\{d(y_1, y_2)\big\} \leqslant \inf_{y_1, y_2, z}\big\{d(y_1, z) + d(z, y_2)\big\} = \varphi(x) \\ &\leqslant \inf_{y_1, y_2}\big\{d(y_1, y_2) + d(y_2, y_2)\big\} = \inf_{y_1, y_2}\big\{d(y_1, y_2)\big\} = d\big(Q(x), P(x)\big). \end{aligned}$$

Above we took *inf* along $y_1 \in Q(x)$, $y_2 \in P(x)$, $z \in Y$. So, $\varphi(x) = d(Q(x), P(x))$. Now, let x be any point in X. Fix any μ, $0 < \mu < \frac{\tilde{\alpha}-\tilde{\beta}}{\tilde{\beta}}$, and take any $y_2 \in P(x)$, such that $d(y_2, Q(x)) \leqslant (1+\mu)\varphi(x)$. As the mapping Q is $\tilde{\alpha}$-covering, there exists a point $x' \in X$, $\rho(x, x') \leqslant \frac{d(y_2, Q(x))}{\alpha}$, such that $y_2 \in Q(x')$. So, $y_2 \in Q(x') \cap P(x)$. Then, as the mapping P is β-Lipschitz, we have the following computation

$$\begin{aligned} \varphi(x') &= d\big(Q(x'), P(x')\big) \leqslant d\big(y_2, P(x')\big) \leqslant h\big(P(x), P(x')\big) \\ &\leqslant \tilde{\beta} \cdot \rho(x, x') \leqslant \frac{\tilde{\beta}}{\tilde{\alpha}} \cdot d\big(y_2, Q(x)\big) \leqslant \frac{\tilde{\beta}}{\tilde{\alpha}} \cdot (1+\mu)\varphi(x). \end{aligned}$$

Finally, we have: $\varphi(x') \leqslant \frac{\beta}{\alpha} \cdot \varphi(x)$ where $\alpha = \tilde{\alpha}$, $\beta = (1+\mu) \cdot \tilde{\beta}$. And $0 < \beta < \alpha$ because $\tilde{\beta} \cdot (1+\mu) < \tilde{\alpha}$. So, the functional $\varphi(x) := D((Q \times P)(x), \Delta_2)$ is (α, β)-search on X. All conditions of Theorem 2.13 (under $n = 2$) are fulfilled. So, we obtain that for any $x_0 \in X$ there exists a sequence $\{x_m\}_{m=0,1,\dots}$ (generally not unique) beginning with x_0, $x_m \xrightarrow[m\to\infty]{} \xi \in X$, such that $\varphi(\xi) = d(Q(\xi), P(\xi)) = 0$, and

$$\rho(x_0, \xi) \leqslant \frac{d(Q(x_0), P(x_0))}{\alpha - \beta} = \frac{d(Q(x_0), P(x_0))}{\tilde{\alpha} - \tilde{\beta} \cdot (1+\mu)} = \frac{d(Q(x_0), P(x_0))}{\tilde{\alpha} - \tilde{\beta}} + \varepsilon$$

where

$$\varepsilon = \frac{d(Q(x_0), P(x_0))}{\tilde{\alpha} - \tilde{\beta} \cdot (1+\mu)} - \frac{d(Q(x_0), P(x_0))}{\tilde{\alpha} - \tilde{\beta}} = \frac{d(Q(x_0), P(x_0)) \cdot \tilde{\beta}\mu}{(\tilde{\alpha} - \tilde{\beta})(\tilde{\alpha} - \tilde{\beta} \cdot (1+\mu))} \xrightarrow[\mu\to 0]{} 0.$$

So, we can see that ε can be indefinitely small. The proof is over. □

It is easy to see that if all *infimums* are reached in the definitions of distances between sets and points in Statement 2.17, then one can take $\mu = 0$, and in this case the statement comes to its one-valued version.

As shown above, under $n=2$ the conditions $D((F_1\times F_2)(x),\Delta_2)=0$ and $d(F_1(x),F_2(x))=0$ are equivalent. But, generally speaking, it is not true under $n>2$. The condition $\sum_{1\leqslant i<j\leqslant n} d(F_i(x),F_j(x))=0$ does not imply the condition $D((F_1\times\cdots\times F_n)(x),\Delta_n)=0$. The simplest example is represented by three segments forming a triangle in the Cartesian plane.

In conclusion of the subsection, we give some integrating reasoning similar to Remark 1.18, Theorem 1.20 and Corollary 1.21 in Section 1. We shall need the following definition.

Definition 2.18. (Compare with Definition 1.19 above.) Let $F_1,\dots,F_n: X\to C(Y)$ be given multi-valued mappings, $F=F_1\times\cdots\times F_n: X\to C(Y^n)$ and H be a closed subspace in Y. We call *the full common preimage* (*or expanded common preimage*) *of the subspace H under the mappings $F_1,\dots,F_n$* the following set $P(F_1,\dots,F_n,H):=F^{-1}(\Delta_n(H))=\{x\in X \mid (\bigcap_{i=1}^n F_i(x))\cap H\neq\emptyset\}$ (or the set $P_+(F_1,\dots,F_n,H):=F_+^{-1}(\Delta_n(H))=\{x\in X\mid D(F(x),\Delta_n(H))=\inf_{y_i\in F_i(x),\,h\in H}\{\sum_{i=1}^n d(y_i,h)\}=0\}$).

Now, much as it was done at the end of Section 1, we can formulate the following general statement combining the statements of Theorems 2.11, 2.13–2.15.

Theorem 2.19. *Let $F_1,\dots,F_n: X\to C(Y)$ be multi-valued sequentially upper semi-continuous mappings, $F=F_1\times\cdots\times F_n: X\to C(Y^n)$, $H\subset Y$ be a closed subspace in Y. Let there exist numbers $0<\beta<\alpha$, such that the multi-valued functional $\Psi(x):=\{\psi=D(y,\Delta_n(H))\mid y\in F(x)\}$ is (α,β)-search on X. Let also one of the following conditions holds*:

(J) *X is complete*;
(JJ) *X is complete, and at least one of the mappings F_i, $1\leqslant i\leqslant n$, takes convergent sequences to compact subsets*;
(JJJ) *H be compact, and at least one of the graphs $Graph(F_1),\dots,Graph(F_n)$ is H-complete.*

Then there exists a multi-cascade on X with the non-empty limit set $A\subset X$, where $A=P_+(F_1,\dots,F_n,H)$ in case (J), *and $A=P(F_1,\dots,F_n,H)$ in cases* (JJ) *and* (JJJ). *The distance between any initial point $x_0\in X$ and every correspondent limit point of the multi-cascade is not greater than $\frac{D(F(x_0),\Delta_n(H))}{\alpha-\beta}$. In other words, for any $x_0\in X$ there exists a sequence $\{x_m\}_{m=0,1,\dots}$* (*generally not unique*) *which is an iteration sequence with respect to the generator of the multi-cascade beginning with x_0, $x_m\xrightarrow[m\to\infty]{}\xi$, such that $\xi\in A$ and $\rho(x_0,\xi)\leqslant\frac{D(F(x_0),\Delta_n(H))}{\alpha-\beta}$.*

The proof of Theorem 2.19 is quite similar to ones of the mentioned theorems, so we do not give it here.

It is easy to see that under $H=Y$, Theorem 2.19 implies Theorems 2.13, 2.14 and 2.15 in cases (J), (JJ), and (JJJ) respectively. Under $n=1$, Theorem 2.19 implies Theorem 2.11(a) in case (J) and Theorem 2.11(b) in case (JJJ). Moreover, in case $H=\{c\}$, $c\in Y$, Theorem 2.19 implies a useful statement (see below Corollary 2.21) which solves the problem of the search for the common roots of n multi-valued mappings $F_1,\dots,F_n$ (compare Corollary 1.21 above). At first, let us define the set of common roots of n multi-valued mappings.

Definition 2.20. Let $F_1,\dots,F_n: X\to C(Y)$ be given multi-valued mappings, $F=F_1\times\cdots\times F_n$, $c\in Y$ be a given point, $\tilde c=(c,\dots,c)\in Y^n$. We define *the set of common roots* of the mappings $F_1,\dots,F_n$ corresponding to the value $c\in Y$ as the set $P(F_1,\dots,F_n,c):=\{x\in X\mid c\in\bigcap_{i=1}^n F_i(x)\}$.

Corollary 2.21. *Let $F_1,\dots,F_n: X\to C(Y)$ be given multi-valued sequentially upper semi-continuous mappings between metric spaces (X,ρ), (Y,d), $F=F_1\times\cdots\times F_n: X\to Y^n$, $c\in Y$ be a given point, $\tilde c=(c,\dots,c)\in Y^n$, and the functional Ψ, $\Psi(x):=\{\psi\mid\psi=D(y,\tilde c),\ y\in F(x)\}$, $x\in X$, is (α,β)-search on X for some numbers α,β, $0<\beta<\alpha$. Let also one of the following conditions holds*:

(V) *X is complete*;
(VV) *at least one of the graphs $Graph(F_1),\dots,Graph(F_n)$ is $\{c\}$-complete.*

Then there exists a multi-cascade on X with the non-empty limit set $A\subset X$, where $A=P(F_1,\dots,F_n,c)$. The distance between any initial point $x_0\in X$ and every correspondent limit point of the multi-cascade is not greater than $\frac{D(F(x_0),\tilde c)}{\alpha-\beta}$. In other words, for any $x_0\in X$ there exists a sequence $\{x_m\}_{m=0,1,\dots}$ which is an iteration sequence with respect to the generator of the multi-cascade (*and in general not a single one*) *beginning with x_0, $x_m\xrightarrow[m\to\infty]{}\xi$, such that $\xi\in A$ and $\rho(x_0,\xi)\leqslant\frac{D(F(x_0),\tilde c)}{\alpha-\beta}$.*

Proof. Let the condition (V) be fulfilled, that is X is complete. Take any fundamental sequence $\{(x_m,\psi_m)\}_{m=0,1,\dots}$, $\psi_m\in\Psi(x_m)$, that is $\psi_m=D(y_m,\tilde c)$ for some $y_m\in F(x_m)$. Let $\psi_m\xrightarrow[m\to\infty]{}0$. It means that $y_m\xrightarrow[m\to\infty]{}\tilde c$. As X is complete, the sequence $\{x_m\}_{m=0,1,\dots}$ is also convergent. Let $\xi\in X$ be its limit. So, the sequence $\{(x_m,y_m)\}_{m=0,1,\dots}$ converges to the pair $(\xi,\tilde c)$. On the other hand, as the mappings $F_1,\dots,F_n$ are sequentially upper semi-continuous, we have $D(y_m,F(\xi))\xrightarrow[m\to\infty]{}0$, hence $D(\tilde c,F(\xi))=0$ that is $\tilde c\in F(\xi)$. It means that $(\xi,0)\in Graph(\Psi)$. So, $Graph(\Psi)$ is 0-complete, and the statement follows from Theorem 2.4(I).

Let now the condition (VV) be fulfilled. Suppose that $Graph(F_1)$ is c-complete. Show that in this case $Graph(\Psi)$ is also 0-complete. Take again any fundamental sequence $\{(x_m, \psi_m)\}_{m=0,1,\dots}$, $\psi_m \in \Psi(x_m)$, that is $\psi_m = D(y_m, \tilde{c})$ for some $y_m \in F(x_m)$. Let $\psi_m \xrightarrow[m\to\infty]{} 0$. As above, it means that $y_m \xrightarrow[m\to\infty]{} \tilde{c}$. In particular, we have the fundamental sequence $\{(x_m, y_{1m})\}_{m=0,1,\dots}$ with $y_{1m} \xrightarrow[m\to\infty]{} c$. As $Graph(F_1)$ is c-complete, it follows that $\{(x_m, y_{1m})\}_{m=0,1,\dots} \xrightarrow[m\to\infty]{} (\xi, c) \in Graph(F_1)$ with some $\xi \in X$. Then $(\xi, \tilde{c}) = \lim_{m\to\infty}(x_m, y_m)$. On the other hand, as the mappings $F_1, \dots, F_n$ are sequentially upper semi-continuous, we have $D(y_m, F(\xi)) \xrightarrow[m\to\infty]{} 0$. It means that $\tilde{c} \in F(\xi)$ that is $(\xi, \tilde{c}) \in Graph(F)$. Consequently, $(\xi, 0) \in Graph(\Psi)$. So, $Graph(\Psi)$ is 0-complete, and the statement follows from Theorem 2.4(I). □

2(c). Cascade search for the common fixed points of n multi-valued mappings

Now, let us consider the cascade search problem for the common fixed point set of n given multi-valued mappings. First of all, it should be mentioned that Theorem 2.13 can be applied to the case when $X = Y$ is a complete space, and one of $n+1$ given mappings is equal to Id_X. As the identical mapping has compact images of compact sets, the conditions of Theorem 2.14 are automatically fulfilled, and we obtain the following statement.

Theorem 2.22. *Let X be a complete metric space, $F_1, \dots, F_n : X \to C(X)$ be sequentially upper semi-continuous multi-valued mappings, and $F = Id_X \times F_1 \times \dots \times F_n : X \to C(X^{n+1})$. Suppose the one-valued functional $\varphi : X \to R$, $\varphi(x) = D(F(x), \Delta_{n+1})$ is (α, β)-search for some α, β, $0 < \beta < \alpha$. Then there exists a multi-cascade on X with the non-empty limit set $Comfix(F_1, \dots, F_n)$, and the distance between any initial point $x_0 \in X$ and every correspondent limit point is not greater than $\frac{D(F(x_0), \Delta_{n+1})}{(\alpha-\beta)}$. In other words, for any point $x_0 \in X$, there is a convergent sequence $\{x_m\}_{m=0,1,\dots}$ (generally not unique) beginning with x_0 which is an iteration sequence with respect to the generator of the cascade, with $\lim_{m\to\infty} x_m = \xi$, $\xi \in Comfix(F_1, \dots, F_n)$, and $\rho(x_0, \xi) \leqslant \frac{D(F(x_0), \Delta_{n+1})}{(\alpha-\beta)}$.*

Now let us consider one more way to solve the problem which gives a generalization of Theorem 1.14 to the multi-valued case.

Let X be a metric space, $F_1, \dots, F_n : X \to C(X)$ be multi-valued mappings, $F = F_1 \times \dots \times F_n : X \to C(X^n)$. Define a multi-valued functional $\Theta(x) := \{\gamma \in R \mid \gamma = \gamma(x, y) = \sum_{i=1}^n \rho(x, y_i),\ \ y = (y_1, \dots, y_n) \in F(x)\}$, and $\Theta^*(x) := \inf_{\gamma\in\Theta(x)}\{\gamma\}$.

Theorem 2.23. *Let X be a metric space, $F_1, \dots, F_n : X \to C(X)$ be multi-valued mappings ($n \geqslant 2$), and $F = F_1 \times \dots \times F_n : X \to C(X^n)$. Let $Graph(F)$ be Δ_n-complete, and the multi-valued functional $\Theta(x)$ be (α, β)-search on X for some numbers α, β, $0 < \beta < \alpha$. Then there exists a multi-cascade on X with the non-empty limit set $Comfix(F_1, \dots, F_n)$, and the distance between any initial point $x_0 \in X$ and every correspondent limit point is not greater than $\frac{\Theta^*(x_0)}{(\alpha-\beta)}$. In other words, for any point $x_0 \in X$, there is a convergent sequence $\{x_m\}_{m=0,1,\dots}$ (generally not unique) beginning with x_0 which is an iteration sequence with respect to the generator of the cascade, with $\lim_{m\to\infty} x_m = \xi$, $\xi \in Comfix(F_1, \dots, F_n)$, and $\rho(x_0, \xi) \leqslant \frac{\Theta^*(x_0)}{(\alpha-\beta)}$.*

Proof. Show that Theorem 2.23 follows from Theorem 2.4(I). It is clear that $Nil(\Theta) = Comfix(F_1, \dots, F_n)$. Let us show that the conditions of Theorem 2.4(I) are fulfilled. We should check that $Graph(\Theta)$ is 0-complete. Take any fundamental sequence $\{(x_m, \gamma_m)\}_{m=0,1,\dots} \subseteq Graph(\Theta)$ with $\gamma_m \xrightarrow[m\to\infty]{} 0$. As $\gamma_m = \gamma(x_m, y_m) \in \Theta(x_m)$, it follows that the sequences $\{x_m\}_{m=0,1,\dots}$, $\{y_{im}\}_{m=0,1,\dots}$ $(i = 1, \dots, n)$ draw together. As $\{x_m\}_{m=0,1,\dots}$ is fundamental, it follows that all of them are fundamental, consequently the sequence $\{(x_m, y_m)\} \subseteq Graph(F)$ is fundamental. And, in view of the inequalities of Lemma 1.6, the condition $\gamma_m \xrightarrow[m\to\infty]{} 0$ implies that $D(y_m, \Delta_n) \xrightarrow[m\to\infty]{} 0$. Then, as $Graph(F)$ is Δ_n-complete, $\{x_m, y_m\}_{m=0,1,\dots} \xrightarrow[m\to\infty]{} (\xi, \tilde{\xi})$ where $\tilde{\xi} = (\xi, \dots, \xi) \in \Delta_n \subset X^n$, and $(\xi, \tilde{\xi}) \in Graph(F)$, that is we have $0 \in \Theta(\xi)$. It follows that $Graph(\Theta)$ is 0-complete. So, all conditions of Theorem 2.4(I) are fulfilled, and we conclude that Theorem 2.23 follows from Theorem 2.4(I). □

Remark 2.24. One can modify Theorems 2.23 and 2.22, replacing the functional Θ (or φ respectively) with some other metric functional which nil-subspace is equal to $Comfix(F_1, \dots, F_n)$. For example, one can use one of the following multi-valued (or respectively, one-valued) functionals

$$\Phi(x) = \left\{\gamma \;\middle|\; \gamma = \sum_{i=1}^n \rho(x, y_i) + \sum_{1\leqslant k<m\leqslant n} \rho(y_k, y_m),\ y_i \in F_i(x),\ i = 1, \dots, n\right\}, \tag{1}$$

and its one-valued version $\Phi^*(x) = \inf_{\gamma\in\Phi(x)}\{\gamma\}$;

$$W(x) = \left\{\lambda \in R \;\middle|\; \lambda = \rho(x, y_1) + \sum_{i=1}^{n-1} \rho(y_i, y_{i+1}),\ y_i \in F_i(x),\ i = 1, \dots, n\right\}, \tag{2}$$

and its one-valued version $W^*(x) = \inf_{\lambda\in W(x)}\{\lambda\}$.

Now, we would like to make the following general remark concerning generators of multi-cascades considered here (Remark 2.25) and to give an example (see Example 2.26 below) confirming that the conditions of Theorem 2.13, under $n=2$, are essentially weaker than the conditions of Theorem 2.16 (= [4, Theorem 3]).

Remark 2.25. Let us note (like as we have noted in [12]) that the *generator* of the multi-cascade in our cascade search principle is defined as the mapping (in general, multi-valued one) which associates with any point $x \in X$ the set of all points $x' \in X$ indicated in Definition 1.2 (for one-valued version) and in Definition 2.2 (for multi-valued version). So, all required sequences we constructed (the cascade trajectories) are of course iteration ones with respect to that generator.

Now we give an example illustrating the comparison of Theorems 2.13 and 2.16.

Example 2.26. Define the metric space X as the set on the plane, equal to the union of a countable family of closed unit segments $\{I_n = [A; B_n]\}_{n=0,1,\dots}$, intersecting in one common end-point A. Parametrize each segment with a parameter x, $x \in [0; 1]$ (put $x = 0$ at the point A for all segments). Define a metric on X as follows. In each segment I_n it is induced by usual Euclidean metric, and the distance between points belonging to different segments is defined as the sum of the distances from the chosen points to the common point A. It is easy to see that the space X is complete in the described metric. Define the space Y as a ray $Y = (-\infty; 1]$ with the ordinary metric. And define two multi-valued mappings as follows

$$f_1(x) = \begin{cases} (-\infty; 0], & x = 0, \\ \frac{x}{2}, & 0 < x \leqslant \frac{1}{2},\ x \in I_0, \\ \frac{1-x}{2}, & \frac{1}{2} < x \leqslant 1,\ x \in I_0, \\ -nx, & x \in I_n,\ n = 1, 2, \dots, \end{cases} \qquad f_2(x) = \begin{cases} (-\infty; 0], & x = 0, \\ x, & 0 < x \leqslant \frac{1}{2},\ x \in I_0, \\ 1-x, & \frac{1}{2} < x \leqslant 1,\ x \in I_0, \\ -2nx, & x \in I_n,\ n = 1, 2, \dots. \end{cases}$$

So, under the mapping f_1 the segment $I_0 = [A; B_0]$ is folded in two, contracted in two, and then is put isometrically onto the segment $[0; 0{,}25] \in Y$. Each of the segments $I_n = [A; B_n]$, $n \geqslant 1$, is stretched out in n times and is put onto the segment $[0; -n] \subset Y$. Under the mapping f_2, the segment $I_0 = [A; B_0]$ is also folded in two and put isometrically onto the segment $[0; 0{,}5] \in Y$. Each of the segments $I_n = [A; B_n]$, $n \geqslant 1$, under the mapping f_2 is stretched in $2n$ times and is put isometrically onto the segment $[0; -2n] \subset Y$.

We can easily see that both mappings are sequentially upper semi-continuous. Indeed, the mappings are clearly continuous at every point $x \in X \setminus \{A\}$. And, for any sequence $\{x_m\}_{m=0,1,\dots}$ converging to the point A, the correspondent sequences $f_i(x_m)$ converge to $f_i(A) = [-\infty; 0]$ which means that $d(f_i(x_m), f_i(A))$ goes to zero $(i = 1, 2)$.

It is easy to see as well, that no one of the described mappings f_1, f_2 is Lipschitz. And, no one of them is a covering mapping because the image of any neighbourhood of the point $x = 0{,}5$ (the middle of the segment I_0) does not cover any neighbourhood of its image under the mapping f_1, or under the mapping f_2. Consequently, no one of the mentioned conditions of Arutyunov Theorem [4, Theorem 3] is fulfilled for the mappings f_1, f_2.

Now, let us show that all conditions of our Theorem 2.13 above (under $n = 2$) are fully fulfilled with $\alpha = 1$, $\beta = \frac{1}{2}$. Indeed,

$$d(x) = D\big((f_1 \times f_2)(x), \Delta_2\big) = d\big(f_1(x), f_2(x)\big) = \begin{cases} 0, & x = 0 \text{ (i.e. } d(A) = 0), \\ \frac{x}{2}, & 0 < x \leqslant \frac{1}{2},\ x \in I_0, \\ \frac{1-x}{2}, & \frac{1}{2} < x \leqslant 1,\ x \in I_0, \\ nx, & x \in I_n,\ n = 1, 2, \dots. \end{cases}$$

Taking $\alpha = 1$ one should consider $r(x) = d(x)$ as the radius of the ball (in this case it is the interval with x as the center) in which there should exist a point x', satisfying the following inequality $d(x') = d(f_1(x'), f_2(x')) \leqslant \frac{1}{2} d(f_1(x), f_2(x)) = \frac{1}{2} d(x)$.

One can easily check that points $x' = x'(x)$ defined by the following formula meet the conditions $\rho(x, x') \leqslant d(x)$, $d(x') \leqslant \frac{1}{2} d(x)$.

$$x'(x) \in \begin{cases} 0, & x = 0 \text{ (i.e. } x'(A) = A), \\ \frac{x}{2}, & 0 \leqslant x < \frac{1}{2},\ x \in I_0, \\ \frac{1+x}{2}, & \frac{1}{2} \leqslant x \leqslant 1,\ x \in I_0, \\ [0; \frac{x}{2}], & x \in I_n,\ n = 1, 2, \dots. \end{cases}$$

Note that in the last formula we have presented only parts of the full admissible sets of such points $x' = x'(x)$ for $x \in I_n$, $n \geqslant 2$.

So, all conditions of Theorem 2.13 are fulfilled for the mappings f_1, f_2.

Acknowledgement

I am grateful to Olga D. Frolkina who has drawn my attention to the convenient concept of the common preimage of n one-valued mappings and suggested to insert Theorem 1.20 and Corollary 1.21 into Section 1. As a result of that, I have inserted similar statements into Section 2(c) (Theorem 2.19 and Corollary 2.21).

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