ELSEVIER
Available online at www.sciencedirect.com
SCIENCE DIRECT®
J. Math. Anal. Appl. 322 (2006) 403–419
Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS
www.elsevier.com/locate/jmaa

An existence result for a linear–superlinear elliptic system with Neumann boundary conditions

Eugenio Massa [1]

IMECC-UNICAMP, Caixa Postal 6065, 13081-970 Campinas-SP, Brazil

Received 13 December 2004

Available online 12 October 2005

Submitted by S. Heikkilä

Abstract

In this work, we consider an elliptic system of two equations in dimension one (with Neumann boundary conditions) where the nonlinearities are asymptotically linear at $-\infty$ and superlinear at $+\infty$. We obtain that, under suitable hypotheses, a solution exists for any couple of forcing terms in L^2.

We also present a similar result in which the superlinearity is in only one of the two equations, and we discuss the resonant problem too.

Keywords: Elliptic systems; Linear–superlinear problems; Galerkin approximation

1. Introduction

In this work we are mainly concerned with the problem

$$\begin{cases} -u'' = \lambda v + g_1(x, v) + h_1(x) & \text{in } (0, 1), \\ -v'' = \mu u + g_2(x, u) + h_2(x) & \text{in } (0, 1), \\ u'(0) = u'(1) = v'(0) = v'(1) = 0, \end{cases} \tag{1.1}$$

where the principal hypothesis is

(H1) $g_{1,2} \in C^0([0, 1] \times \mathbb{R})$, $\quad \lim_{s \to -\infty} \frac{g_{1,2}(x, s)}{s} = 0, \quad \lim_{s \to +\infty} \frac{g_{1,2}(x, s)}{s} = +\infty$

uniformly with respect to $x \in [0, 1]$, and $h_{1,2} \in L^2(0, 1)$.

E-mail address: eugenio@ime.unicamp.br.
[1] The author was supported by Fapesp/Brazil.

doi:10.1016/j.jmaa.2005.09.037

Some hypotheses on the growth at infinity in the second variable of the nonlinearities $g_{1,2}$ will be needed to obtain the PS condition for the functional associated to problem (1.1): defining $G_{1,2}(x,s) = \int_0^s g_{1,2}(x,\xi)\,d\xi$, we ask

(H2) $\exists \theta \in (0, \frac{1}{2})$, $s_0 > 0$ s.t. $0 < G_{1,2}(x,s) \leqslant \theta s g_{1,2}(x,s)$, $\forall s > s_0$;
(H3) $\exists s_1 > 0$, $C_0 > 0$ s.t. $G_{1,2}(x,s) \leqslant \frac{1}{2} s g_{1,2}(x,s) + C_0$, $\forall s < -s_1$.

Moreover, for certain "resonant" values of λ, μ, also one of the following hypotheses will be assumed:

(HR0) $\lim_{s\to-\infty} g_i(x,s) = 0$, $h_i(x) < -d < 0$ a.e. $x \in [0,1]$, $i = 1$ or 2;
(HR1) $\exists \rho_0 > 0$, $M_0 \in \mathbb{R}$ s.t. $G_1(x,s) + G_2(x,s) + h_1(x)s + h_2(x)s \leqslant M_0$ a.e. $x \in [0,1]$, $\forall s < -\rho_0$.

An example of nonlinearities which satisfy the hypotheses above may be $g_{1,2}(x,s) = e^s$; in this case (HR0) and (HR1) become $h_i(x) < -d < 0$ a.e. and $h_{1,2}(x) \geqslant 0$ a.e., respectively.

We will denote in the following with $0 = \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \cdots \leqslant \lambda_k \leqslant \cdots$ the eigenvalues of $-\Delta$ in $H^1(0,1)$ and with $(\phi_k,\ k = 1,2,\ldots)$ the corresponding eigenfunctions, which will be taken orthogonal and normalized with $\|\phi_k\|_{L^2} = 1$.

The main result of this work is the following theorem.

Theorem 1.1. *For $\lambda, \mu > 0$, $\sqrt{\lambda\mu} \in (0, \lambda_2/4)$, under hypotheses* (H1)–(H3)*, there exists a solution for problem* (1.1) *for any $h_1, h_2 \in L^2(0,1)$.*

We will also consider the two limiting (resonant) cases:

Theorem 1.2. *Under hypotheses* (H1)–(H3) *and with $h_1, h_2 \in L^2(0,1)$ we have*:

(i) *For $\lambda, \mu > 0$, $\sqrt{\lambda\mu} = \lambda_2/4$, if hypothesis* (HR1) *is satisfied too, then there exists a solution for problem* (1.1).
(ii) *If $\lambda = 0$, $\mu > 0$ (or $\lambda > 0$, $\mu = 0$, or $\lambda = \mu = 0$), if hypothesis* (HR0) *is satisfied for $i = 1$ (or $i = 2$, or $i = 1, 2$, respectively), then there exists a solution for problem* (1.1).

We remark that problem (1.1) with $\lambda, \mu > 0$, $\sqrt{\lambda\mu} > \lambda_2/4$ seems much more difficult to work with, due to the more complicated interaction of the nonlinearity with the spectrum.

In the case $\lambda < 0$ or $\mu < 0$ instead, it is simple to show that no result similar to Theorem 1.1 may be achieved, actually we will show in Proposition 7.1 that one may always find functions $h_1, h_2 \in L^2$ for which no solution exists.

Observe that in problem (1.1), we are assuming a linear–superlinear nonlinearity in both equations; however, we will show that few modifications in the proofs allow to treat also the problem with the linear–superlinear term in one equation and a jumping nonlinearity in the other: namely

$$\begin{cases} -u'' = \lambda v + g_1(x,v) + h_1(x) & \text{in } (0,1), \\ -v'' = \mu^+ u^+ - \mu^- u^- + g_2(x,u) + h_2(x) & \text{in } (0,1), \\ u'(0) = u'(1) = v'(0) = v'(1) = 0, \end{cases} \tag{1.2}$$

where $u^{\pm}(x) = \max\{0, \pm u(x)\}$ and now

(H1*) $g_{1,2} \in \mathcal{C}^0([0,1] \times \mathbb{R})$, $\lim_{s \to -\infty} \frac{g_1(x,s)}{s} = 0$, $\lim_{s \to +\infty} \frac{g_1(x,s)}{s} = +\infty$,
$\lim_{s \to -\infty} \frac{g_2(x,s)}{s} = 0$, $\lim_{s \to +\infty} \frac{g_2(x,s)}{s} = 0$

uniformly with respect to $x \in [0,1]$, and still $h_{1,2} \in L^2(0,1)$.

In this case we will assume hypothesis (H2) only for g_1, while for g_2 we will assume the equivalent of (H3) also at $+\infty$ too, namely

(H3*) $G_2(x,s) \leqslant \frac{1}{2} s g_2(x,s) + C_0$, $\forall s > s_1$.

The result is the following

Theorem 1.3. *For $\lambda > 0$, $\mu^+ > \mu^- > 0$ and $\sqrt{\mu^- \lambda} \in (0, \lambda_2/4)$, under hypotheses* (H1*), (H2) *only for g_1,* (H3) *and* (H3*)*, there exists a solution for problem* (1.2) *for any $h_1, h_2 \in L^2(0,1)$.*

1.1. Some comments about the techniques used and some related results

The main theorems will be proved by finding a critical point of the functional associated to problem (1.1):

$$F : E = H^1 \times H^1 \to \mathbb{R} : \mathbf{u} = (u,v) \mapsto F(\mathbf{u})$$
$$= \int_0^1 u'v' - \int_0^1 \left(\frac{\lambda}{2} v^2 + \frac{\mu}{2} u^2 \right) - \int_0^1 \big(G_1(x,v) + G_2(x,u) \big) - \int_0^1 (h_1 v + h_2 u), \tag{1.3}$$

or to problem (1.2), which is analogous to this except for the second integral being replaced by

$$\int_0^1 \left(\frac{\lambda}{2} v^2 + \frac{\mu^+}{2} (u^+)^2 + \frac{\mu^-}{2} (u^-)^2 \right).$$

We observe that one important characteristic of this kind of system is that, in order to treat it variationally, we are led to work with this functional, which is strongly indefinite, in the sense that there exist two infinite dimensional subspaces of E such that F is unbounded from above in one and from below in the other (see Lemma 2.1). This implies that the standard linking theorems are no more available to find critical points. Some of the techniques used in approaching this kind of problems may be seen in [1–4]; in particular, we will use an approximation technique (Galerkin procedure), namely we will solve finite dimensional problems, then take limit on the dimension of such problems and prove that the result is actually the critical point we were looking for (see, for example, [4]).

The scalar counterpart of problem (1.1) is

$$\begin{cases} -u'' = \lambda u + g(x,u) + h(x) & \text{in } (0,1), \\ u'(0) = u'(1) = 0 \end{cases} \tag{1.4}$$

and it has been considered in many works.

For $\lambda < \lambda_1$ (no matter whether the boundary conditions are Neumann or Dirichlet) it is the so-called Ambrosetti–Prodi problem (first considered in [5]) and it has zero, at least one or at least two solutions, depending on the forcing term $h \in L^2$. The result in Propositions 7.1 and 7.2 suggests that a similar phenomenon may happen for our system too.

For $\lambda > \lambda_1$, the behavior is quite different for Neumann and Dirichlet conditions: in [6] it is shown that, in the Dirichlet case, for any $\lambda > \lambda_1$, there exist examples in which no solution exists, while for the Neumann case (in dimension one), it was obtained in [7] and later in [8] that for $\lambda \in (0, \lambda_2/4)$, a solution exists for any $h \in L^2$; this result was then extended to $\lambda \in (\lambda_k/4, \lambda_{k+1}/4)$, $k \geqslant 2$ in [9].

Our Theorems 1.1 and 1.3 look to be the equivalent of the results in [7,8] for the problems (1.1) and (1.2), while the result in [9] appears much more difficult to be extended to these systems.

In [7], also the resonant case $\lambda = 0$ is considered, with a nonresonance condition similar, but weaker, to our (HR0); the resonance for $\lambda = \lambda_2/4$ was considered in [10] and in [9]; in this last one, the nonresonance condition is quite similar to our (HR1), although it is interesting to remark that in (HR1) we could assume a joint condition on the nonlinearities in the two equations, which is much weaker than asking the condition in [9] for both, separately.

Finally, we remark that problem (1.2) with $\mu^+ = \mu^- = 1$ and $g_2 \equiv 0$, $h_2 \equiv 0$, becomes a fourth order scalar problem, which was considered in [11] and (for higher values of λ) in [6]: the result here may be seen as a generalization of that in [11]; however, since here we are considering a more general nonlinearity, the result in [11] is stronger: it was obtained up to dimension three and, for dimension one, the existence was proved for $\lambda \in (0, \gamma)$, where γ was approximatively $0.32\pi^4$: a value much larger than $\lambda_2^2/16 = \pi^4/16 \simeq 0.0625\pi^4$, which results from Theorem 1.3. This is due to the fact that, since here we are considering a more general nonlinearity, the sets chosen to estimate the functional may not be adapted to the problem as well as there.

The techniques we will use in order to prove the main theorems will be inspired by those in [7,8] (which we will briefly describe in Section 3), but will need to be adapted to the more complex characteristics of the functional (1.3) and of its variational setting, which forces us to use the Galerkin approximation technique described above.

2. Definitions and notations

Consider the eigenvalue problem

$$\begin{cases} -u'' = \lambda v & \text{in } (0,1), \\ -v'' = \lambda u & \text{in } (0,1), \\ u'(0) = u'(1) = v'(0) = v'(1) = 0, \end{cases} \tag{2.1}$$

it is known that the eigenvalues of problem (2.1) are:

- λ_k, $k = 1, 2, \ldots$ (with corresponding eigenfunctions the couples (ϕ_k, ϕ_k)),
- $-\lambda_k$, $k = 1, 2, \ldots$ (with corresponding eigenfunctions the couples $(\phi_k, -\phi_k)$).

In view of the above structure, let $H = H^1(0,1)$, $E = H \times H$ (with norm $\|(u,v)\|_E^2 = \|u\|_H^2 + \|v\|_H^2$) and define

$$E^+ = \{(u,v) \in E\colon\ u = v\}, \qquad E^- = \{(u,v) \in E\colon\ u = -v\}, \tag{2.2}$$

$$E_n^+ = \{(u,v) \in E\colon\ u = v \in \operatorname{span}\{\phi_1, \ldots, \phi_n\}\}, \tag{2.3}$$

$$E_n^- = \{(u,v) \in E\colon\ u = -v \in \operatorname{span}\{\phi_1, \ldots, \phi_n\}\} \tag{2.4}$$

and finally,

$$E_n = E_n^+ \oplus E_n^-, \tag{2.5}$$

so that $\overline{\bigcup_{h\in\mathbb{N}} E_h} = E$.

Since the functional (1.3) has the term $\int_0^1 u'v'$ as its principal part, the following estimates will be useful:

Lemma 2.1.

$$\int_0^1 2u'v' \geqslant \lambda_{k+1} \int_0^1 \left(u^2 + v^2\right) \quad \textit{for } \mathbf{u} = (u, v) \in \left(E^- \oplus E_k^+\right)^\perp, \tag{2.6}$$

$$\int_0^1 2u'v' \leqslant -\lambda_{k+1} \int_0^1 \left(u^2 + v^2\right) \quad \textit{for } \mathbf{u} = (u, v) \in \left(E_k^- \oplus E^+\right)^\perp, \tag{2.7}$$

$$\int_0^1 2u'v' \leqslant \lambda_k \int_0^1 \left(u^2 + v^2\right) \quad \textit{for } \mathbf{u} = (u, v) \in E^- \oplus E_k^+, \tag{2.8}$$

$$\int_0^1 2u'v' \geqslant -\lambda_k \int_0^1 \left(u^2 + v^2\right) \quad \textit{for } \mathbf{u} = (u, v) \in E_k^- \oplus E^+. \tag{2.9}$$

Proof. In $(E^- \oplus E_k^+)^\perp$ one has $u = v$ and then

$$\int_0^1 2u'v' = 2\int_0^1 |u'|^2 \geqslant 2\lambda_{k+1} \int_0^1 u^2 = \lambda_{k+1} \int_0^1 u^2 + v^2, \tag{2.10}$$

proving (2.6).

Then observe that

$$\int_0^1 2u'v' = \frac{1}{2} \int_0^1 \left|(u+v)'\right|^2 - \left|(u-v)'\right|^2$$

and that for $\mathbf{u} \in E^- \oplus E_k^+$ one has $(u+v, u+v) \in E_k^+$, then

$$\int_0^1 2u'v' \leqslant \frac{1}{2} \int_0^1 \left|(u+v)'\right|^2 \leqslant \lambda_k \frac{1}{2} \int_0^1 \left(u^2 + v^2 + 2uv\right) \leqslant \lambda_k \int_0^1 u^2 + v^2, \tag{2.11}$$

proving (2.8).

The same argument gives the other two estimates. □

3. Proof of Theorem 1.1

In [7,8], the solution of problem (1.4) is found as a mountain pass critical point: the functional J associated to the problem is such that:

- J is bounded from below in the set

$$S = \Big\{ u \in H^1(0,1) \text{ such that } \sup_{x\in[0,1]} u(x) = 0 \Big\}, \tag{3.1}$$

  provided $\lambda < \pi^2/4$,
- $\lim_{t\to\pm\infty} J(t\phi_1) = -\infty$, provided $\lambda > 0$;

since $H^1(0,1) \subseteq \mathcal{C}([0,1])$, the set S splits $H^1(0,1)$ into two components and $\pm\phi_1$ lie on the opposite sides of it, so one gets the linking structure which provides (through the PS condition) a critical point. Moreover, the value $\pi^2/4 = \lambda_2/4$ was obtained through the variational characterization

$$\frac{\pi^2}{4} = \inf\left\{ \frac{\int_0^1 (u')^2}{\int_0^1 u^2} \text{ with } u \in S\setminus\{0\} \right\} \tag{3.2}$$

(this characterization is the one used in [8], the one used in [7] it is slightly different).

We will try to adapt this idea to our problem.

First of all, the following lemma will allow us to work with simpler hypotheses.

Lemma 3.1. *In the hypotheses of Theorem* 1.1*, problem* (1.1) *admits a solution with the parameters λ, μ if and only if it admits a solution with parameters $\hat\lambda = \hat\mu = \sqrt{\lambda\mu}$.*

Proof. If we change the unknown functions u, v with the new ones $U = u$ and $V = \delta v$, being $\delta = \sqrt{\lambda/\mu}$, then we obtain a new system with parameters $\hat\lambda = \hat\mu = \sqrt{\lambda\mu}$, and in which the given hypotheses are still satisfied; then the two problems are equivalent. □

Then, we make the following definitions: given $\mathbf{u} = (u,u) \in E^+$, we define

$$\sigma(\mathbf{u}) = \sup_{x\in[0,1]} u(x); \tag{3.3}$$

then we define (for $n > 1$) the following sets and quantities:

$$T_n = \left\{ \mathbf{u} = (u,u) \in E_n^+ \colon \int_0^1 u\phi_1 = 0 \right\}, \tag{3.4}$$

$$S_n = \big\{ \mathbf{u} = (u,u) \in E_n^+ \colon \sigma(\mathbf{u}) = 0 \big\}, \tag{3.5}$$

$$\gamma_n = \inf\left\{ \frac{\int_0^1 (u')^2}{\int_0^1 u^2} \text{ with } \mathbf{u} = (u,u) \in S_n\setminus\{0\} \right\}, \tag{3.6}$$

$$L_n = \left\{ \mathbf{u} = (u,v) \in \big(E_n^- \oplus E_1^+\big) \colon \int_0^1 u^2 + v^2 = 1 \right\}, \tag{3.7}$$

$$\widetilde{L_n} = \left\{ \mathbf{u} = (u,v) \in \big(E_n^- \oplus E_1^+\big) \colon \int_0^1 u^2 + v^2 \leqslant 1 \right\}. \tag{3.8}$$

First we will prove some properties of the above definitions:

Lemma 3.2. *The function* $\sigma : E^+ \to \mathbb{R} : \mathbf{u} \mapsto \sigma(\mathbf{u})$ *is continuous.*

Proof. We have, since $H^1(0,1) \subseteq \mathcal{C}^0[0,1]$ with continuous inclusion,

$$\left|\sigma(u,u) - \sigma(v,v)\right| \leqslant \|u-v\|_{L^\infty} \leqslant C\|u-v\|_{H^1} \leqslant C\left\|(u,u)-(v,v)\right\|_E. \qquad \square \tag{3.9}$$

Lemma 3.3. *The set* S_n *is homeomorphic to* T_n*, moreover* S_n *links in* E_n *with* RL_n *for any* $R > 0$.

Proof. Observe that $E_n = E_n^- \oplus E_1^+ \oplus T_n$ and denote by $P_T : E_n \to T_n$ and $P_L : E_n \to E_n^- \oplus E_1^+$ the two orthogonal projections.

The map $M : T_n \to S_n : (u,u) \mapsto (u,u) - \sigma(u)(\mathbf{1},\mathbf{1})$ is continuous by the previous lemma and has the restriction of P_T to S_n as its inverse, so it is a homeomorphism.

Now observe that the action of the map M is a translation parallel to the subspace $E_n^- \oplus E_1^+$ (in which lies $\widetilde{L_n}$) and that T_n is orthogonal to this subspace. Then we may extend the map M to the map

$$\widetilde{M} : E_n \to E_n : (u.v) \mapsto (u,v) - \sigma\big(P_T(u,v)\big)(\mathbf{1},\mathbf{1}), \tag{3.10}$$

which is still an homeomorphism and which translates each plane parallel to $\widetilde{L_n}$ by the same quantity. Since the plane containing $\widetilde{L_n}$ intersects T_n in the origin and $\sigma(0,0) = 0$, this plane is not translated and then $\widetilde{M}|_{L_n} = \mathrm{Id}$.

Finally, consider any map $\psi : \widetilde{L_n} \to E_n$ with $\psi|_{L_n} = \mathrm{Id}$ and consider the composition $\Psi = P_L \circ \widetilde{M}^{-1} \circ \psi$: Ψ is the identity on L_n and so the topological degree $\deg(\Psi, \widetilde{L_n}, 0) = \deg(\mathrm{Id}, \widetilde{L_n}, 0) = 1$, since $0 \in \widetilde{L_n}$. This implies that there exists $p \in \widetilde{L_n}$ such that $\Psi(p) = 0$, that is $\psi(p) \in \widetilde{M}(\mathrm{Ker}(P_L)) = S_n$, giving the claimed linking property. $\square$

Lemma 3.4. *Let* γ_n *be given by* (3.6)*. Then* $\gamma_n \geqslant \lambda_2/4$ (*in fact,* $\{\gamma_n\}$ *is nonincreasing and* $\gamma_n \to \lambda_2/4$).

Proof. The definition in (3.6) is analogous to that in (3.2), except for the fact that the inf is taken on S_n which is an increasing sequence of subsets of S which fill it. $\square$

Now we define, for $n > 1$ and $R_n > 0$,

$$e_n = \inf_{\gamma \in \Gamma^*_{n,R_n}} \sup_{\mathbf{u} \in \gamma(B^{n+1})} F(\mathbf{u}), \tag{3.11}$$

where now

$$\Gamma^*_{n,R_n} = \left\{\gamma \in \mathcal{C}^0\big(B^{n+1}, E_n\big) \text{ s.t. } \gamma|_{\partial B^{n+1}} \text{ is an homeomorphism onto } R_n L_n\right\}. \tag{3.12}$$

What we intend to prove is the following proposition, which in fact implies Theorem 1.1 by virtue of Lemma 3.1.

Proposition 3.5. *Under hypothesis* (H1)*, for* $\lambda = \mu \in (0, \lambda_2/4)$*,* $h_{1,2} \in L^2(0,1)$ *and suitable* R_n *large enough, the values* e_n *are critical for the restriction to* E_n *of the functional* F.

Moreover, under hypotheses (H2) *and* (H3)*, up to a subsequence,* $e_n \to e \in \mathbb{R}$ *for* $n \to \infty$ *and the critical points corresponding to the values* e_n *converge to a nontrivial solution of problem* (1.1).

First, we need to estimate F on the sets defined above, in order to obtain the claimed critical points: observe that since $h_{1,2} \in L^2$ and using hypothesis (H1), we can find constants C_1, C_2 and C_3 as follows:

- $C_1(\delta, h_{1,2})$ such that

$$\left| \int_0^1 h_1 v + h_2 u \right| \leqslant \frac{\delta}{4}\left(\|u\|_{L^2}^2 + \|v\|_{L^2}^2\right) + C_1(\delta, h_{1,2}); \tag{3.13}$$

- $C_2(\delta, g_{1,2})$ such that

$$\left| \int_0^1 G_1\left(x, -v^-\right) + G_2\left(x, -u^-\right) \right| \leqslant \frac{\delta}{4}\left(\|u\|_{L^2}^2 + \|v\|_{L^2}^2\right) + C_2(\delta, g_{1,2}); \tag{3.14}$$

- $C_3(g_{1,2})$ such that

$$\int_0^1 G_1\left(x, v^+\right) + G_2\left(x, u^+\right) \geqslant -C_3(g_{1,2}). \tag{3.15}$$

Lemma 3.6. *If $\lambda = \mu > \lambda_1 = 0$, then $\forall C \in \mathbb{R}$ there exists $R > 0$ such that $F|_{\gamma(\partial B^{n+1})} \leqslant C$ for any $\gamma \in \Gamma_{n,R}^*$, $n > 1$.*

Proof. Let $\mathbf{u} = (u, v) \in L_n$. Then $\int_0^1 (u^2 + v^2) = 1$ and $\int_0^1 u'v' \leqslant \frac{\lambda_1}{2} \int_0^1 (u^2 + v^2)$ (in fact, here $\lambda_1 = 0$).

By using the above estimates, one gets (for $\rho > 0$)

$$\begin{aligned}
\frac{F(\rho\mathbf{u})}{\rho^2} &= \int_0^1 u'v' - \frac{\lambda}{2}\int_0^1 \left(v^2 + u^2\right) - \int_0^1 \frac{G_1(x, \rho v) + G_2(x, \rho u)}{\rho^2} - \int_0^1 \frac{h_1\rho v + h_2\rho u}{\rho^2} \\
&\leqslant \frac{\lambda_1 - \lambda}{2}\int_0^1 \left(v^2 + u^2\right) + \int_0^1 \left| \frac{G_1(x, -\rho v^-) + G_2(x, -\rho u^-)}{\rho^2} \right| \\
&\quad - \int_0^1 \frac{G_1(x, \rho v^+) + G_2(x, \rho u^+)}{\rho^2} + \int_0^1 \left| \frac{h_1\rho v + h_2\rho u}{\rho^2} \right| \\
&\leqslant \frac{\lambda_1 - \lambda + \delta}{2} + \frac{C_1(\delta, h_{1,2}) + C_2(\delta, g_{1,2}) + C_3(g_{1,2})}{\rho^2}.
\end{aligned} \tag{3.16}$$

Then by choosing $0 < \delta < \lambda - \lambda_1$, we have that the first part is negative and then for R large enough (namely $R^2 > 2\frac{C - C_1(\delta, h_{1,2}) - C_2(\delta, g_{1,2}) - C_3(g_{1,2})}{\lambda_1 - \lambda + \delta}$) one gets the claim for $\mathbf{u} = (u, v) \in RL_n$. $\square$

Lemma 3.7. *For $\lambda = \mu < \lambda_2/4$, there exists η such that $F|_{S_n} \geqslant \eta$ for any $n > 1$.*

Proof. For $\mathbf{u} = (u, u) \in S_n$ we have $u(x) \leqslant 0$ and $\int_0^1 (u')^2 \geqslant \gamma_n \|u\|_{L^2}^2$, then we may estimate:

$$F(\mathbf{u}) = \int_0^1 (u')^2 - \lambda \int_0^1 u^2 - \int_0^1 G_1(x,u) + G_2(x,u) - \int_0^1 h_1 u + h_2 u$$
$$\geqslant (\gamma_n - \lambda)\|u\|_{L^2}^2 - \left(\frac{\delta}{2}\int_0^1 u^2 + C_2(\delta, g_{1,2})\right) - \left(\frac{\delta}{2}\int_0^1 u^2 + C_1(\delta, h_{1,2})\right)$$
$$\geqslant (\gamma_n - \lambda - \delta)\int_0^1 u^2 - C_2(\delta, g_{1,2}) - C_1(\delta, h_{1,2}). \tag{3.17}$$

Now, if $\lambda < \lambda_2/4$, we may choose $\delta < \lambda_2/4 - \lambda$ so that the first term is non negative for any $n > 1$ by Lemma 3.4 and so $F(u) \geqslant -C_2(\delta, g_{1,2}) - C_1(\delta, h_{1,2})$. □

Lemma 3.8. *For $\lambda = \mu \in (0, \lambda_2/4)$, there exist $\zeta, \eta \in \mathbb{R}$ such that $\zeta \geqslant e_n \geqslant \eta$, for any $n > 1$.*

Proof. The bound from below is given by Lemma 3.7 and the linking property in Lemma 3.3.

For the bound from above one may simply build a map $\tilde{\gamma} \in \Gamma^*_{n,R}$ such that $\tilde{\gamma}(B^{n+1}) = R\widetilde{L_n}$ and then the same computations in Lemma 3.6 provide the estimate

$$\sup_{\mathbf{u}\in\tilde{\gamma}(B^{n+1})} F(\mathbf{u}) \leqslant \frac{\lambda_1 - \lambda + \hat{\delta}}{2}\int_0^1 \left(u^2 + v^2\right) + C_1(\hat{\delta}, h_{1,2}) + C_2(\hat{\delta}, g_{1,2}) + C_3(g_{1,2}); \tag{3.18}$$

then again by choosing $0 < \hat{\delta} < \lambda - \lambda_1$ one gets the claimed estimate from above with $\zeta = C_1(\hat{\delta}, h_{1,2}) + C_2(\hat{\delta}, g_{1,2}) + C_3(g_{1,2})$. □

Now we may conclude:

Proof of Proposition 3.5 and Theorem 1.1. By Lemmas 3.8 and 3.6 with $C < \eta$ we can apply a linking theorem to obtain that the levels e_n are critical for the restriction of F at the finite dimensional subspace E_n, that is there exists $\mathbf{u}_n = (u_n, v_n) \in E_n$ such that Eq. (4.2) below holds.

Moreover, the estimate $\zeta \geqslant e_n \geqslant \eta$ implies (4.1) below and then we have, by Proposition 4.1, that (up to a subsequence) $\mathbf{u}_n \xrightarrow{E} \mathbf{u} = (u, v) \in E$, which is a solution of problem (1.1) (using also Lemma 3.1). □

4. Proof of the PS* condition

In this section we prove that the sequence of points in E obtained in the first part of Proposition 3.5, contains a convergent subsequence (this is known as PS* property) and that its limit is actually a critical point for F.

Proposition 4.1. *Let the sequence $\{\mathbf{u}_n\} = \{(u_n, v_n)\} \subseteq E$ with $(u_n, v_n) \in E_n$ be such that*

$$\left|F(\mathbf{u}_n)\right| = \left|\int_0^1 u_n' v_n' - \int_0^1 \frac{\lambda}{2} v_n^2 + \frac{\mu}{2} u_n^2 - \int_0^1 G_1(x, v_n) + G_2(x, u_n) - \int_0^1 h_1 v_n + h_2 u_n\right| \leqslant T, \tag{4.1}$$

$$\big\langle F'(\mathbf{u}_n),(\phi,\psi)\big\rangle=\int_0^1 u_n'\psi'+v_n'\phi'-\int_0^1 \lambda v_n\psi+\mu u_n\phi-\int_0^1 g_1(x,v_n)\psi+g_2(x,u_n)\phi$$
$$-\int_0^1 h_1\psi+h_2\phi=0\quad \forall(\phi,\psi)\in E_n. \tag{4.2}$$

Then, for $\lambda,\mu\neq 0$ *and under hypotheses* (H1)–(H3)*, there exists* $\mathbf{u}=(u,v)\in E$ *such that*

$$\int_0^1 u'\psi'+v'\phi'-\int_0^1 \lambda v\psi+\mu u\phi-\int_0^1 g_1(x,v)\psi+g_2(x,u)\phi-\int_0^1 h_1\psi+h_2\phi=0$$
$$\forall(\phi,\psi)\in E, \tag{4.3}$$

that is, (u,v) *is a solution of problem* (1.1)*.*

In fact, up to a subsequence, $\mathbf{u}_n\to\mathbf{u}$ *in* E*.*

The proof will be in most parts very close to that in [9], for the scalar problem: we sketch it here, underlining the differing parts:

(1) First one estimates (from hypothesis (H1)):

for any $\varepsilon>0$, $\bar{s}\in\mathbb{R}$ and $M\in\mathbb{R}$, there exist $C_M, C_\varepsilon\in\mathbb{R}$ (of course depending also on $\bar{s}$) such that

$$g_{1,2}(x,s)\geqslant Ms-C_M\quad \text{for } s>\bar{s}, \tag{4.4}$$
$$\big|g_{1,2}(x,s)\big|\leqslant \varepsilon(-s)+C_\varepsilon\quad \text{for } s\leqslant\bar{s}. \tag{4.5}$$

Then one supposes that the sequence $\mathbf{u}_n$ is not bounded in E and so assumes $\|\mathbf{u}_n\|_E\geqslant 1$, $\|\mathbf{u}_n\|_E\to+\infty$, defines $\mathbf{z}_n=(U_n,V_n)=\mathbf{u}_n/\|\mathbf{u}_n\|_E$, so that $\mathbf{z}_n$ is a bounded sequence in E and then we can select a subsequence such that $\mathbf{z}_n\to\mathbf{z}_0=(U_0,V_0)$ weakly in E and strongly in $[L^2]^2$ and $[\mathcal{C}^0[0,1]]^2$.

(2) Claim: $U_0, V_0\leqslant 0$.

Proof of the claim. From $\frac{\langle F'(u_n,v_n),(\phi_1,\phi_1)\rangle}{\|\mathbf{u}_n\|_E}=0$ one gets (remember that in this case $\phi_1=\mathbf{1}$)

$$\int_0^1 \frac{g_1(x,v_n)}{\|\mathbf{u}_n\|_E}+\frac{g_2(x,u_n)}{\|\mathbf{u}_n\|_E}\leqslant\left|\int_0^1 \lambda V_n+\mu U_n\right|+\left|\int_0^1 \frac{h_1}{\|\mathbf{u}_n\|_E}+\frac{h_2}{\|\mathbf{u}_n\|_E}\right|. \tag{4.6}$$

Then we proceed as in [9] to obtain that, for any $\bar{x}$ such that $V_0(\bar{x})>0$, we have

$$\lim_{n\to+\infty}\frac{g_1(\bar{x},v_n)}{\|\mathbf{u}_n\|_E}=+\infty, \tag{4.7}$$

and that (for any $x\in[0,1]$)

$$\frac{g_1(x,v_n)}{\|\mathbf{u}_n\|_E}\geqslant-\varepsilon|V_n|-\frac{C_{M,\varepsilon}}{\|\mathbf{u}_n\|_E}; \tag{4.8}$$

now taking lim inf, we get

$$\liminf_{n\to+\infty} \frac{g_1(x, v_n)}{\|\mathbf{u}_n\|_E} \geqslant -\varepsilon\left|V_0(x)\right| \tag{4.9}$$

for any choice of ε and then

$$\liminf_{n\to+\infty} \frac{g_1(x, v_n)}{\|\mathbf{u}_n\|_E} \geqslant 0. \tag{4.10}$$

The analogous to (4.7) and (4.10) hold replacing g_1 with g_2 and v with u.

Since U_n, V_n are uniformly bounded (by their $\mathcal{C}^0$ convergence) and $\|\mathbf{u}_n\|_E \geqslant 1$, (4.8) implies that the functions $\frac{g_1(x,v_n)}{\|\mathbf{u}_n\|_E}$, $\frac{g_2(x,u_n)}{\|\mathbf{u}_n\|_E}$ are bounded below uniformly so that we can use Fatou's Lemma and get from (4.6), (4.7) (supposing $U_0^+ \not\equiv 0$ or $V_0^+ \not\equiv 0$) and (4.10)

$$\begin{aligned} +\infty &= \int_0^1 \liminf_{n\to+\infty} \left(\frac{g_1(x, v_n)}{\|\mathbf{u}_n\|_E} + \frac{g_2(x, u_n)}{\|\mathbf{u}_n\|_E} \right) \\ &\leqslant \liminf_{n\to+\infty} \int_0^1 \frac{g_1(x, v_n)}{\|\mathbf{u}_n\|_E} + \frac{g_2(x, u_n)}{\|\mathbf{u}_n\|_E} \\ &\leqslant \liminf_{n\to+\infty} \left(\left| \int_0^1 \lambda V_n + \mu U_n \right| + \left| \int_0^1 \frac{h_1}{\|\mathbf{u}_n\|_E} + \frac{h_2}{\|\mathbf{u}_n\|_E} \right| \right). \end{aligned} \tag{4.11}$$

The right-hand side can be estimated since the first term is bounded by $(\lambda\|V_n\|_{H^1} + \mu\|U_n\|_{H^1}) \leqslant \lambda + \mu$ and the last one clearly goes to zero; then Eq. (4.11) gives rise to a contradiction unless $U_0, V_0 \leqslant 0$. □

(3) Claim: Using hypotheses (H1) and (H3), we obtain a constant A such that

$$\int_{v_n > s_0} g_1(x, v_n)v_n \leqslant A\|\mathbf{u}_n\|_E, \qquad \int_{u_n > s_0} g_2(x, u_n)u_n \leqslant A\|\mathbf{u}_n\|_E, \tag{4.12}$$

at least for n big enough.

Proof of the claim. From $|2F(\mathbf{u}_n) - \langle F'(\mathbf{u}_n), \mathbf{u}_n\rangle| \leqslant 2T$ one gets

$$\begin{aligned} &\int_{v_n > s_0} g_1(x, v_n)v_n - 2G_1(x, v_n) + \int_{u_n > s_0} g_2(x, u_n)u_n - 2G_2(x, u_n) \\ &\quad\leqslant \int_{v_n \leqslant s_0} 2G_1(x, v_n) - g_1(x, v_n)v_n + \int_{u_n \leqslant s_0} 2G_2(x, u_n) - g_2(x, u_n)u_n \\ &\qquad + \left| \int_0^1 h_1 v_n + h_2 u_n \right| + 2T, \end{aligned} \tag{4.13}$$

and proceeding as in [9] obtains (by using hypotheses (H2) and (H3))

$$\int_{v_n > s_0} g_1(x, v_n)v_n + \int_{u_n > s_0} g_2(x, u_n)u_n \leqslant \frac{A}{2}\|\mathbf{u}_n\|_E + \frac{A}{2} \leqslant A\|\mathbf{u}_n\|_E \tag{4.14}$$

for some constant A; but by hypothesis (H2), both integrals are nonnegative, and then we obtain (4.12). □

(4) Claim:

$$\int_0^1 \frac{|g_1(x,v_n)|}{\|\mathbf{u}_n\|_E} \to 0, \qquad \int_0^1 \frac{|g_2(x,u_n)|}{\|\mathbf{u}_n\|_E} \to 0. \tag{4.15}$$

Proof of the claim. As in [9]. □

(5) Claim: $\lambda, \mu \neq 0$ implies $(U_0, V_0) = (0,0)$.

Proof of the claim. For any given $(\phi, \psi) \in E_h$ we get, from $\frac{\langle F'(\mathbf{u}_n),(\phi,\psi)\rangle}{\|\mathbf{u}_n\|_E} = 0$ with $n > h$:

$$\begin{aligned}&\left|\int_0^1 U_n'\psi' + V_n'\phi' - \int_0^1 \lambda V_n\psi + \mu U_n\phi\right| \\ &\quad\leqslant \int_0^1 \frac{|g_1(x,v_n)|}{\|\mathbf{u}_n\|_E}|\psi| + \frac{|g_2(x,u_n)|}{\|\mathbf{u}_n\|_E}|\phi| + \left|\int_0^1 \frac{h_1\psi + h_2\phi}{\|\mathbf{u}_n\|_E}\right|;\end{aligned} \tag{4.16}$$

but now the right-hand side goes to zero by Eq. (4.15), and then we get, taking limit and using weak convergence of (U_n, V_n), that

$$\int_0^1 U_0'\psi' + V_0'\phi' - \int_0^1 \lambda V_0\psi + \mu U_0\phi = 0. \tag{4.17}$$

Since $\bigcup_{h\in\mathbb{N}} E_h$ is dense in E, this remains true for arbitrary $(\phi,\psi) \in E$ and then (U_0, V_0) satisfy the system

$$\begin{cases} -U_0'' = \lambda V_0 & \text{in } (0,1), \\ -V_0'' = \mu U_0 & \text{in } (0,1), \\ U_0'(0) = V_0'(0) = U_0'(1) = V_0'(1) = 0. \end{cases} \tag{4.18}$$

Since we know that all solutions of this system with $\lambda, \mu \neq 0$ change sign (while $U_0, V_0 \leqslant 0$), this implies $(U_0, V_0) \equiv (0,0)$. □

(6) Claim: (u_n, v_n) is bounded.

Proof of the claim. From $\frac{\langle F'(\mathbf{u}_n),(v_n,u_n)\rangle}{\|\mathbf{u}_n\|_E^2} = 0$ one gets

$$\begin{aligned}\int_0^1 (U_n')^2 + (V_n')^2 &\leqslant \int_0^1 (\lambda+\mu)V_nU_n + \int_0^1 \frac{|g_1(x,v_n)||U_n| + |g_2(x,u_n)||V_n|}{\|\mathbf{u}_n\|_E} \\ &\quad + \int_0^1 \frac{h_1U_n + h_2V_n}{\|\mathbf{u}_n\|_E}.\end{aligned} \tag{4.19}$$

Using (4.15) and the fact that $(U_n, V_n) \to (0,0)$ in $[L^2]^2$ and $[\mathcal{C}^0[0,1]]^2$, (4.19) becomes

$$\int_0^1 \left(U_n'\right)^2 + \left(V_n'\right)^2 \to 0, \tag{4.20}$$

which gives contradiction since one would get $1 = \|(U_n, V_n)\|_E \to 0$. □

(7) Thus $\mathbf{u}_n$ is bounded and so there exists a subsequence such that $\mathbf{u}_n \to \mathbf{u} = (u, v)$ weakly in E and strongly in $(L^2)^2$ and $[\mathcal{C}^0[0,1]]^2$.

By taking limit in (4.2) for a given $(\phi, \psi) \in E_h$ and using the weak convergence of $\mathbf{u}_n$ one obtains (the nonlinear terms are continuous: if $v_n \to v$ in $\mathcal{C}^0$ then $g_1(x, v_n) \to g_1(x, v)$ in L^2)

$$\int_0^1 u'\psi' + v'\phi' - \int_0^1 \lambda v\psi + \mu u\phi - \int_0^1 g_1(x,v)\psi + g_2(x,u)\phi - \int_0^1 h_1\psi + h_2\phi = 0 \tag{4.21}$$

and, again, this remains true by a density argument for arbitrary $(\phi, \psi) \in E$.

(8) Finally, we prove that in fact $\mathbf{u}_n \to \mathbf{u}$ strongly too.

Let $P_n : H \to H_n = \operatorname{span}\{\phi_1, \ldots, \phi_n\}$ be the orthogonal projection map, then $P_n u \to u$ and $P_n v \to v$ in H and so $P_n u - u_n \to 0$ and $P_n v - v_n \to 0$ in L^2.

Consider Eq. (4.2) with $\psi = u_n - P_n u$ and $\phi = 0$:

$$\int_0^1 u_n'(u_n - P_n u)' - \int_0^1 \lambda v_n(u_n - P_n u) - \int_0^1 g_1(x, v_n)(u_n - P_n u)$$
$$- \int_0^1 h_1(u_n - P_n u) = 0; \tag{4.22}$$

$g_1(x, v_n)$ is bounded in L^2, $(u_n - P_n u) \to 0$ in L^2 and then

$$\int_0^1 u_n'(u_n - u + u - P_n u)' \to 0, \tag{4.23}$$

which implies $u_n \to u$ strongly in H.

The same argument gives $v_n \to v$ strongly in H.

5. Proof of Theorem 1.2: The resonant cases

5.1. The resonance in $\lambda_2/4$

Since we may make a change of unknowns as in Lemma 3.1, assume $\lambda = \mu = \lambda_2/4$ and (HR1).

Since Lemma 3.6 and Proposition 4.1 still hold in this case, the only difference arises in Lemma 3.7, where one has to exploit (HR1) to obtain

$$F(\mathbf{u}) = \int_0^1 (u')^2 - \frac{\lambda_2}{4}\int_0^1 u^2 - \int_0^1 G_1(x,u) + G_2(x,u) - \int_0^1 h_1 u + h_2 u$$
$$\geqslant \left(\gamma_n - \frac{\lambda_2}{4}\right)\|u\|_{L^2}^2 - M_0; \tag{5.1}$$

actually, we assumed without loss of generality that $\rho_0 = 0$, since the integral

$$\int_{u\in[-\rho_0,0]} G_1(x,u) + G_2(x,u) + h_1 u + h_2 u$$

is bounded.

5.2. *The resonance in zero*

We observe that the resonance in zero is more complicated: we may no longer proceed as in Lemma 3.1, that is suppose $\lambda = \mu$; however, we may exploit the same kind of change of unknowns to assume, without loss of generality, $\lambda, \mu < \lambda_2/4$. This implies that the conclusions of Lemma 3.7 still hold, by simply replacing the term $\lambda \int_0^1 u^2$ with $\frac{\lambda+\mu}{2}\int_0^1 u^2$ in (3.17).

So consider first the case $\lambda = \mu = 0$ and assume (1.1) holds for $i = 1, 2$.

Modifications in the proof of Lemma 3.6. We will estimate (for $\delta, M > 0$)

$$-\int_0^1 h_1 v \leqslant \left|\int_0^1 h_1 v^+\right| + \int_0^1 h_1 v^- \leqslant \delta \int_0^1 \left(v^+\right)^2 + C_\delta - d\int_0^1 v^-, \tag{5.2}$$

$$\int_0^1 \left|G_1\left(x, v^-\right) + G_2\left(x, u^-\right)\right| \leqslant \delta \int_0^1 \left(v^- + u^-\right) + C_\delta, \tag{5.3}$$

$$\int_0^1 G_1\left(x, v^+\right) + G_2\left(x, u^+\right) \geqslant M\int_0^1 \left[\left(v^+\right)^2 + \left(u^+\right)^2\right] - C_3(g_{1,2}, M), \tag{5.4}$$

where we used (HR0) in the first two lines (and the same holds with h_2 and u in place of h_1 and v).

Then we may join the above estimates to obtain, in place of (3.16) (recall that $\lambda = \mu = \lambda_1 = 0$):

$$F(\rho\mathbf{u}) \leqslant K_{M,\delta} - \left(M\rho^2 \int_0^1 \left[\left(v^+\right)^2 + \left(u^+\right)^2\right]\right) + \left(\delta\rho\int_0^1 v^- + u^-\right)$$
$$+ \left(-d\rho\int_0^1 \left(v^- + u^-\right) + \delta\rho^2\int_0^1 \left(\left(v^+\right)^2 + \left(u^+\right)^2\right)\right)$$
$$\leqslant K_{M,\delta} + (-M+\delta)\rho^2\int_0^1 \left(\left(v^+\right)^2 + \left(u^+\right)^2\right) + (-d+\delta)\rho\int_0^1 \left(v^- + u^-\right), \tag{5.5}$$

where we collected all the constants in $K_{M,\delta}$.

Now, by choosing $\delta < d < M$, we obtain a negative contribution from both the positive and the negative part of the functions; however, $\int_0^1[(v^+)^2 + (u^+)^2] + \int_0^1(v^- + u^-)$ is bounded away from zero in L_n but not uniformly with respect to n: this implies that we may find the claimed R but depending on n; however this is not a problem since in the proof of Proposition 3.5 R may depend on n. □

Modifications in the proof of Proposition 4.1. From Eq. (4.18) we now obtain that U_0 and V_0 are two independent nonpositive constants.

However, by using Eq. (4.2), with test functions the couples $(\phi_1, 0)$ and $(0, \phi_1)$ we get, respectively

$$\int_0^1 g_1(x, v_n) + \int_0^1 h_1 = 0, \qquad \int_0^1 g_2(x, u_n) + \int_0^1 h_2 = 0 \tag{5.6}$$

where (if $U_0, V_0 \not\equiv 0$), $u_n, v_n \to -\infty$ uniformly and so we get, by (HR0), the contradiction $\int_0^1 h_{1,2} \to 0$; then as before $U_0 \equiv V_0 \equiv 0$. □

Finally, the case in which only one of the parameters is zero is similar: let $\lambda = 0$, $\mu > 0$ (and, without loss of generality as observed above, $\mu < \lambda_2/4$) and assume (HR0) only for $i = 1$: then in (5.5) one has also a term $-\mu\int_0^1 u^2$ which may be exploited as in Eq. (3.16), so that it is no more necessary to assume (HR0) for $i = 2$, while from system (4.18) one obtains $U_0 \equiv 0$ and $V_0 \leqslant 0$ constant, and proceeds as above to show that in fact $V_0 \equiv 0$ too by (HR0).

Remark 5.1. By comparing hypothesis (HR0) and Proposition 7.1 below, one sees that if in addition to (HR0) we have also $g_i > 0$, then the sufficient condition $h_i < -d < 0$ and the necessary one $\int_0^1 h_1 \leqslant -\inf_{x\in[0,1],s\in\mathbb{R}}(g_i(x,s)) = 0$, become similar enough.

6. Proof of Theorem 1.3

To deal with this problem, we may exploit a change of unknown as done in Lemma 3.1; in this case we will assume $\lambda = \mu^- \in (0, \lambda_2/4)$ and $\mu^+ > 0$.

Observe that the right-hand side of the second equation may be rewritten as $\mu^- u + (\mu^+ - \mu^-)u^+ + g_2(x, u)$ and that the term $\tilde{g}_2(x, u) = (\mu^+ - \mu^-)u^+ + g_2(x, u)$ satisfies the estimates (3.14) and (3.15) since $\mu^+ > \mu^-$. Then Lemmas 3.6 and 3.7 still hold.

Modifications in the proof of Proposition 4.1. Estimate (4.4) now holds only for g_1, while g_2 satisfies an estimate as (4.5) also for $s > \bar{s}$; then (4.11) becomes

$$\begin{aligned}\int_0^1 &\liminf_{n\to+\infty} \frac{g_1(x, v_n)}{\|\mathbf{u}_n\|_E} \\ &\leqslant \liminf_{n\to+\infty}\left(\left|\int_0^1 \lambda V_n + \mu^+ U_n^+ - \mu^- U_n^-\right| + \left|\int_0^1 \frac{h_1 + h_2 + g_2(x, u_n)}{\|\mathbf{u}_n\|_E}\right|\right)\end{aligned} \tag{6.1}$$

and implies $V_0^+ \equiv 0$.

Later, in (4.13), one passes the whole term containing g_2 and G_2 to the right-hand side and estimates it with (H3) and (H3*), and so obtains (4.12) (and (4.15) later) for g_1 only.

Finally, in place of (4.18) one gets

$$\begin{cases} -U_0'' = \lambda V_0 \quad \text{in } (0,1), \\ -V_0'' = \mu^+ U_0^+ - \mu^- U_0^- \quad \text{in } (0,1), \\ U_0'(0) = V_0'(0) = U_0'(1) = V_0'(1) = 0; \end{cases} \tag{6.2}$$

and again deduces $(U_0, V_0) \equiv (0,0)$, actually since $\lambda \neq 0$ and V_0 does not change sign, U_0 may only be a constant and so $V_0 \equiv 0$, but then the second equation implies that $U_0 \equiv 0$ too since $\mu^\pm \neq 0$.

The rest of the proof follows straightforward. □

7. The case $\lambda, \mu < 0$ and an analogous result for the Dirichlet problem

As anticipated in the introduction, we will show here (Proposition 7.1) that when λ or μ is below the first eigenvalue $\lambda_1 = 0$, no result like Theorem 1.1 may hold, since it is always possible to find forcing terms h_1 or h_2 for which no solution exists.

This result has an analogue for the Dirichlet problem, which will be given in Proposition 7.2.

Proposition 7.1. *For $\lambda < 0$ (respectively $\mu < 0$), under hypothesis* (H1)*, the problem* (1.1) *has no solution if $\int_0^1 h_1\phi_1 > -\min_{x\in[0,1],s\in\mathbb{R}}[\lambda s + g_1(x,s)]$ ($\int_0^1 h_2\phi_1 > -\min_{x\in[0,1],s\in\mathbb{R}}[\mu s + g_2(x,s)]$, respectively).*

Proof. Consider the case $\lambda < 0$: by testing the first equation against $\phi_1 = \mathbf{1}$ one gets

$$0 = \int_0^1 \lambda v + g_1(x,v) + \int_0^1 h_1 \tag{7.1}$$

$$\geqslant \min_{x\in[0,1],s\in\mathbb{R}} \bigl[\lambda s + g_1(x,s)\bigr] + \int_0^1 h_1, \tag{7.2}$$

where the minimum above is well defined by the continuity of g_1 and the hypotheses (H1) and $\lambda < 0$. Then we obtain the necessary condition $\int_0^1 h_1 \leqslant -\min_{x\in[0,1],s\in\mathbb{R}}[\lambda s + g_1(x,s)]$.

Analogous computations give the result for $\mu < 0$. □

The same kind of nonexistence result may be proved for the Dirichlet problem, with some more complicated computation: in the following λ_1 and φ_1 will denote the first eigenvalue and eigenfunction of the Dirichlet problem.

Proposition 7.2. *For $\lambda < 0$, or $\mu < 0$, or $\sqrt{\lambda\mu} < \lambda_1$, under hypotheses* (H1)*, there exist two constants $C \in \mathbb{R}$ and $m > 0$, such that if $m\int_0^1 h_1\phi_1 + \int_0^1 h_2\phi_1 > C$, then problem* (1.1) *has no solution.*

Proof. Let $\xi > 0$, test the equations against φ_1, multiply the first by ξ, integrate by parts and sum them: this gives

$$0 = \int_0^1 (\xi\lambda - \lambda_1) v\varphi_1 + \xi g_1(x, v)\varphi_1 + \int_0^1 (\mu - \xi\lambda_1) u\varphi_1 + g_2(x, u)\varphi_1 + \int_0^1 \xi h_1\varphi_1 + h_2\varphi_1. \tag{7.3}$$

Now, if $(\xi\lambda - \lambda_1)$ and $(\mu - \xi\lambda_1)$ were both negative, then as in the proof of Proposition 7.1 one could get the minimum obtaining the necessary condition

$$\xi \int_0^1 h_1\varphi_1 + \int_0^1 h_2\varphi_1$$

$$\leqslant -\left(\min_{\substack{x\in[0,1]\\ s\in\mathbb{R}}} \big[(\xi\lambda - \lambda_1)s + \xi g_1(x, s)\big] + \min_{\substack{x\in[0,1]\\ s\in\mathbb{R}}} \big[(\mu - \xi\lambda_1)s + g_2(x, s)\big] \right) \int_0^1 \varphi_1. \tag{7.4}$$

But this may always be obtained: for $\lambda, \mu > 0$, $\sqrt{\lambda\mu} < \lambda_1$ one may choose $\xi = \sqrt{\mu/\lambda}$, while if $\lambda < 0$ (respectively $\mu < 0$), then a good choice is ξ sufficiently large (respectively sufficiently small). □

Remark 7.3. Observe that these two nonexistence results may be extended straightforward to any spatial dimension, whenever the usual conditions (on the superlinearities $g_{1,2}$) which allow to use variational techniques are satisfied.

Moreover, the hypothesis (H1) was used just in order to guarantee that the functions $\lambda s + g_1(x, s)$, etc., were bounded from below; then superlinearity is not necessary, one could simply ask $\liminf_{s\to+\infty} \lambda + \frac{g_1(x,s)}{s} > 0$ and an analogous condition for g_2, in the Neumann case, and a some more complicated condition (since the two equations remain coupled) for the Dirichlet case.

References

[1] V. Benci, P.H. Rabinowitz, Critical point theorems for indefinite functionals, Invent. Math. 52 (1979) 241–273.

[2] D.G. de Figueiredo, P.L. Felmer, On superquadratic elliptic systems, Trans. Amer. Math. Soc. 343 (1994) 99–116.

[3] J. Hulshof, R. van der Vorst, Differential systems with strongly indefinite variational structure, J. Funct. Anal. 114 (1993) 32–58.

[4] D.G. de Figueiredo, J. do Ó, B. Ruf, Critical and subcritical elliptic systems in dimension two, Indiana Univ. Math. J. 53 (2004) 1037–1054.

[5] A. Ambrosetti, G. Prodi, On the inversion of some differentiable mappings with singularities between Banach spaces, Ann. Mat. Pura Appl. (4) 93 (1972) 231–246.

[6] E. Massa, On the Fučík spectrum and superlinear elliptic equations, PhD thesis, Università degli Studi di Milano, Italy, 2003.

[7] D.G. de Figueiredo, B. Ruf, On a superlinear Sturm–Liouville equation and a related bouncing problem, J. Reine Angew. Math. 421 (1991) 1–22.

[8] S. Villegas, A Neumann problem with asymmetric nonlinearity and a related minimizing problem, J. Differential Equations 145 (1998) 145–155.

[9] E. Massa, On a variational characterization of the Fučík spectrum of the Laplacian and a superlinear Sturm–Liouville equation, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004) 557–577.

[10] K. Perera, Existence and multiplicity results for a Sturm–Liouville equation asymptotically linear at $-\infty$ and superlinear at $+\infty$, Nonlinear Anal. 39 (2000) 669–684.

[11] E. Massa, Superlinear equations and a uniform anti-maximum principle for the multi-Laplacian operator, Electron. J. Differential Equations 97 (2004) (electronic).