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General Free-Boundary Problems for the Heat Equation, III*

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1. INTRODUCTION

The aim of Part III is to extend the results obtained in Parts I and II to a problem of the following kind, which will be referred to once again as Problem I:

$$\mathbf{L}^{(1)}u^{(1)}(x,t) \equiv u^{(1)}_{xx} - \kappa^{(1)}u^{(1)}_t = q^{(1)}(x,t)$$

in

$$D^{(1)} = \{(x,t): 0 < x < s(t), 0 < t < T\}, \tag{1.1}$$

$$u^{(1)}(x,0) = h^{(1)}(x), \qquad 0 \leqslant x \leqslant s(0), \quad 0 < s(0) \equiv b < 1, \tag{1.2}$$

$$u^{(1)}(0,t) = \varphi^{(1)}(t), \quad 0 < t < T, \tag{1.3}$$

$$\mathbf{L}^{(2)}u^{(2)}(x,t) \equiv u^{(2)}_{xx} - \kappa^{(2)}u^{(2)}_t = q^{(2)}(x,t)$$

in

$$D^{(2)}_T = \{(x,t): s(t) < x < 1, 0 < t < T\}, \tag{1.1'}$$

$$u^{(2)}(x,0) = h^{(2)}(x), \qquad s(0) \leqslant x \leqslant 1, \tag{1.2'}$$

$$u^{(2)}(1,t) = \varphi^{(2)}(t), \qquad 0 < t < T, \tag{1.3'}$$

$$u^{(1)}(s(t),t) = u^{(2)}(s(t),t) = f(s(t),t), \qquad 0 < t < T, \tag{1.4}$$

$$\chi^{(1)}(s(t),t)\, u^{(1)}_x(s(t),t) - \chi^{(2)}(s(t),t)\, u^{(2)}_x(s(t),t) = \dot{s}(t) + \mu(s(t),t), \qquad 0 < t < T, \tag{1.5}$$

where $\kappa^{(1)}$, $\kappa^{(2)}$ are constants and $q^{(i)}(x,t)$, $h^{(i)}(x)$, $\varphi^{(i)}(t)$, $\chi^{(i)}(x,t)$, $i = 1, 2$, and $f(x,t)$, $\mu(x,t)$ are given functions.

Other kinds of boundary conditions can be considered; e.g., conditions (1.3), (1.3′) can be replaced by

$$u^{(1)}_x(0,t) = g^{(1)}(u^{(1)}(0,t),t), \qquad 0 < t < T,$$

$$u^{(2)}_x(1,t) = g^{(2)}(u^{(2)}(1,t),t), \qquad 0 < t < T$$

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(in such a case the above scheme will be called Problem II); moreover, boundary conditions of mixed type can also be introduced.

However, none of these cases will be dealt with for the sake of brevity, although the corresponding extensions of results obtained here for Problem I are troubleless.

In addition, we shall confine ourselves to the case $0 < b < 1$. The procedures of [3] can be of help in handling the cases $b = 0$, $b = 1$.

The definition of a solution $(T^*, s(t), u^{(1)}(x, t), u^{(2)}(x, t))$ to Problem I is similar to Definition 1, I. Throughout the paper the letters I and II refer to Part I and Part II, respectively.

The well-known two-phase Stefan problem is a particular case of the one we are considering. Very general results about the existence of classical solutions to two-phase Stefan problems have been obtained in [1] quite recently (see also [4]). The main regularity properties assumed there on the data are: $\varphi^{(1)}(t)$, $\varphi^{(2)}(t) \in C^1[0, T]$; $h(x) \in H^1(0, 1)$, where $h(x) = h^{(1)}(x)$ for $0 \leqslant x \leqslant b$, $h(x) = h^{(2)}(x$ for $b < x \leqslant 1)$, and $H^1(0, 1)$ is the Sobolev space endowed with the norm $\| h \|_{H^1} = \| h \|_{L^2} + \| h' \|_{L^2}$ (which is embedded in $C^{1/2}(0, 1)$); $\varphi^{(1)}(0) = h(0)$, $\varphi^{(2)}(0) = h(1)$.

In Section 4 we shall prove the existence of solutions to Problem I under no differentiability assumptions on $\varphi^{(1)}$, $\varphi^{(2)}$ and requiring only the Hölder continuity of $h^{(1)}$, $h^{(2)}$ at $x = b$ (see Section 2 for a precise statement of the hypotheses).

The continuous dependence on the data and on the coefficients is then shown in Section 5.

The main tool used, beside the techniques developed in Parts I and II, is a new reformulation of the free-boundary condition (1.5) in an integral form, which is carried out in Section 3.

In Section 6 the monotone dependence of the solutions upon the data and the coefficients is studied, including a special investigation on the dependence on the coefficients $\kappa^{(1)}$, $\kappa^{(2)}$.

The concluding section (Section 7) is devoted to some remarks on the behavior of the free boundary (Hölder continuity of $\dot{s}(t)$, asymptotic behavior as t approaches T^*, etc.).

2. List of Assumptions

The functions $q^{(i)}(x, t)$ are assumed to satisfy condition (A) of Section 2, I. In particular, Q will denote an upper bound of $| q^{(i)}(x, t)|$ in the rectangle $\Omega = \{(x, t): 0 < x < 1, 0 < t\}$.

Also assumption (B), I on $f(x, t)$ and (E), I on $\mu(x, t)$ are retained, while (C_1), I is supposed to be satisfied by both $\varphi^{(1)}(t)$ and $\varphi^{(2)}(t)$: In particular $| \varphi^{(i)}(t)| \leqslant \Phi$, $i = 1, 2$ for $t \geqslant 0$.

Concerning the initial data, we assume that two constants H, $\alpha \in (0, 1]$ exist such that

$$(\bar{\mathrm{F}})\qquad \begin{aligned} |h^{(1)}(x) - f(b, 0)| &\leqslant H(b - x)^{\alpha}, \qquad 0 \leqslant x \leqslant b, \\ |h^{(2)}(x) - f(b, 0)| &\leqslant H(b - x)^{\alpha}, \qquad b \leqslant x \leqslant 1, \end{aligned}$$

with H and α subjected to the conditions

$$Hb^{\alpha} \geqslant \Phi, \qquad H(1 - b)^{\alpha} \geqslant \Phi. \tag{2.1}$$

The coefficients $\chi^{(i)}(x, t)$ in the free-boundary condition (1.5) are subjected to the condition

($\bar{\mathrm{D}}$) $\chi^{(i)}$, $\chi_x^{(i)}$, $\chi_{xx}^{(i)}$, $\chi_t^{(i)}$, $i = 1, 2$, are continuous in Ω. Moreover,

$$|\chi^{(i)}(x, t)| \leqslant X, \qquad (x, t) \in \bar{\Omega} \tag{2.2}$$

for a suitable $X \geqslant 0$.

Remark 1. We shall set $f \equiv 0$ henceforth, with no loss of generality: This needs only a proper redefinition of the data and the coefficients and leaves all the above assumptions unaffected. Actually, this transformation is the "physical" motivation for the term $\mu(x, t)$ in (1.5).

Remark 2. Some of the assumptions listed in this section can be weakened, but such refinements will not be considered here.

3. Reformulation of the Free-Boundary Condition

Let us introduce the functions $u(x, t)$, $q(x, t)$, $\chi(x, t)$, κ, which coincide with $u^{(1)}(x, t)$, $q^{(1)}(x, t)$, $\chi^{(1)}(x, t)$, $\kappa^{(1)}$, if $0 \leqslant x \leqslant s(t)$ and with $u^{(2)}(x, t)$, $q^{(2)}(x, t)$, $\chi^{(2)}(x, t)$, $\kappa^{(2)}$, if $s(t) < x \leqslant 1$. Similarly, define the function $h(x)$, equal to $h^{(1)}(x)$ if $0 \leqslant x \leqslant b$ and equal to $h^{(2)}(x)$ if $b < x \leqslant 1$. In this section we shall assume that all the assumptions listed in Section 2, except ($\bar{\mathrm{F}}$) are fulfilled. The functions $h^{(i)}(x)$ are assumed to be piecewise continuous and bounded.

Now, set:

$$\begin{aligned} \chi^{*}(x, t) &= \chi_{xx}^{(1)}(x, t) + \kappa^{(1)}\chi_t^{(1)}(x, t), \qquad 0 \leqslant x \leqslant s(t), \\ &= \chi_{xx}^{(2)}(x, t) + \kappa^{(2)}\chi_t^{(2)}(x, t), \qquad s(t) < x \leqslant 1, \end{aligned} \tag{3.1}$$

and define:

$$a = \min_{0 \leqslant t \leqslant T}\{\min[s(t), 1 - s(t)]\}. \tag{3.2}$$

We shall prove:

THEOREM 1. *Under the conditions specified above, if the triple* $(s(t), u^{(1)}(x, t), u^{(2)}(x, t))$ *solves system* (1.1)–(1.3), (1.1′)–(1.3′), (1.4), (1.5) *in a time interval* $(0, T)$, *in which* $a > 0$, *then the following relationship holds:*

$$\begin{aligned}
s(t) - b = {} & \int_0^t \int_0^1 \{\chi q - \chi^* u\}\, dx\, d\tau - \int_0^1 \kappa\chi(x, 0)\, h(x)\, dx \\
& + \int_0^1 \kappa\chi(x, t)\, u(x, t)\, dx - a^{-1} \int_0^t \int_0^a \{(a - x)\, \chi^{(1)} q^{(1)} \\
& - u^{(1)}[(a - x)\, \chi_{xx}^{(1)} - 2\chi_x^{(1)} + (a - x)\, \kappa^{(1)} \chi_t^{(1)}]\}\, dx\, d\tau \\
& - a^{-1} \int_0^t \int_{1-a}^1 \{(a + x - 1)\, \chi^{(2)} q^{(2)} - u^{(2)}[(a + x - 1)\, \chi_{xx}^{(2)} \\
& + 2\chi_x^{(2)} + (a + x - 1)\, \kappa^{(2)} \chi_t^{(2)}]\}\, dx\, d\tau \\
& - \kappa^{(1)} a^{-1} \int_0^a (a - x)\, \chi^{(1)}(x, t)\, u^{(1)}(x, t)\, dx \\
& - \kappa^{(2)} a^{-1} \int_{1-a}^a (a + x + 1)\, \chi^{(2)}(x, t)\, u^{(2)}(x, t)\, dx \qquad (3.3) \\
& + \kappa^{(1)} a^{-1} \int_0^a (a - x)\, \chi^{(1)}(x, 0)\, h^{(1)}(x)\, dx \\
& + \kappa^{(2)} a^{-1} \int_{-a}^a (a + x - 1)\, \chi^{(2)}(x, 0)\, h^{(2)}(x)\, dx \\
& - a^{-1} \int_0^t \chi^{(1)}(0, \tau)\, \varphi^{(1)}(\tau)\, d\tau - a^{-1} \int_0^t \chi^{(2)}(1, \tau)\, \varphi^{(2)}(\tau)\, d\tau \\
& + a^{-1} \int_0^t \chi^{(1)}(a, \tau)\, u^{(1)}(a, \tau)\, d\tau \\
& + a^{-1} \int_0^t \chi^{(2)}(1 - a, \tau)\, u^{(2)}(1 - a, \tau)\, d\tau - \int_0^t \mu(s(\tau), \tau)\, d\tau, \\
& \qquad 0 < t < T.
\end{aligned}$$

Conversely, if (1.1)–(1.3), (1.1′)–(1.3′), (1.4), *and* (3.3) *hold and if* $s(t)$ *is Lipschitz continuous and* u_x *is continuous up to* $x = s(t)$, $0 < t < T$, *then* (1.5) *is satisfied.*

For the case of Problem II, (3.3) *is replaced by:*

$$\begin{aligned} s(t) - b = {} & \int_0^t \int_0^1 \{\chi q - \chi^* u\}\, dx\, d\tau - \int_0^1 \kappa\chi(x, 0)\, h(x)\, dx + \int_0^1 \kappa\chi(x, t)\, u(x, t)\, dx \\ & + \int_0 \{\chi^{(1)}(0, \tau)\, g^{(1)}(u^{(1)}(0, \tau), \tau) - u^{(1)}(0, \tau)\, \chi_x^{(1)}(0, \tau)\}\, d\tau \\ & - \int_0^t \{\chi^{(2)}(1, \tau)\, g^{(2)}(u^{(2)}(1, \tau), \tau) - u^{(2)}(1, \tau)\, \chi_x^{(2)}(1, \tau)\}\, d\tau \\ & - \int_0^t \mu(s(\tau), \tau)\, d\tau, \qquad 0 < t < T. \end{aligned} \tag{3.4}$$

For the proof of Theorem 1 we need the following lemma.

LEMMA 1. *In the assumptions of Theorem* 1, *condition* (1.5) *is equivalent to*

$$\begin{aligned} s(t) - s(\epsilon') = {} & \int_{\epsilon'}^t \int_\epsilon^{1-\epsilon} \{\chi q - \chi^* u\}\, dx\, d\tau - \int_\epsilon^{1-\epsilon} \kappa\chi(x, \epsilon')\, u(x, \epsilon')\, dx \\ & + \int_\epsilon^{1-\epsilon} \kappa\chi(x, t)\, u(x, t)\, dx \\ & + \int_{\epsilon'}^t \{\chi^{(1)}(\epsilon, \tau)\, u_x^{(1)}(\epsilon, \tau) - u^{(1)}(\epsilon, \tau)\, \chi_x^{(1)}(\epsilon, \tau)\}\, d\tau \\ & - \int_{\epsilon'}^t \{\chi^{(2)}(1-\epsilon, \tau)\, u_x^{(2)}(1-\epsilon, \tau) - u^{(2)}(1-\epsilon, \tau)\, \chi_x^{(2)}(1-\epsilon, \tau)\}\, d\tau \\ & - \int_{\epsilon'}^t \mu(s(\tau), \tau)\, d\tau, \qquad \epsilon' < t < T, \end{aligned} \tag{3.5}$$

for both Problems I *and* II, *where* ϵ *and* ϵ' *are arbitrary positive constants such that* $\epsilon < a$ *and* $\epsilon' < T$.

Proof. Consider the well-known Green's identity:

$$\begin{aligned} & \iint_D \{V(U_{xx} - KU_t) - U(V_{xx} + KV_t)\}\, dx\, d\tau \\ & \quad = \oint_{\partial D} \{(VU_x - UV_x)\, d\tau + KUV\, dx\}, \qquad K \text{ constant}, \end{aligned} \tag{3.6}$$

which is valid for sufficiently smooth functions U and V and for sufficiently regular domain D. Use (3.6) taking $U = u^{(1)}$, $V = \chi^{(1)}$, $K = \kappa^{(1)}$, and $D \equiv \{(x, t)\colon \epsilon < x < s(\tau), \epsilon' < \tau < t\}$. Next, apply the same identity with $U = u^{(2)}$, $V = \chi^{(2)}$, $K = \kappa^{(2)}$, and with $D \equiv \{(x, t)\colon s(\tau) < x < 1 - \epsilon, \epsilon' < \tau < t\}$. Adding the resulting equations, (3.5) is easily obtained if the triple $(s, u^{(1)}, u^{(2)})$ satisfies (1.1), (1.1'), (1.4), (1.5).

Conversely, assume (1.1), (1.1'), (1.4), and (3.5) hold. It is easily proved that condition (1.5) is verified. In fact, performing the same operation described above and taking into account (3.5) we get the equation

$$s(t) - s(\epsilon') = \int_{\epsilon'}^{t} \{\chi^{(1)}(s(\tau), \tau)\, u_x^{(1)}(s(\tau), \tau) - \chi^{(2)}(s(\tau), \tau)\, u_x^{(2)}(s(\tau), \tau)\}\, d\tau,$$

from which (1.5) follows after differentiation, owing to the continuity of $u_x^{(1)}$ and $u_x^{(2)}$ (see Definition 1, I).

Proof of Theorem 1. As far as Problem II is concerned, the limits $\epsilon \to 0$, $\epsilon' \to 0$ can be performed directly in (3.5) leading to relationship (3.4).

In order to prove (3.3), it is necessary to derive a different form for the fourth and the fifth terms in (3.5). This can be achieved using again, identity (3.4): Take $U = u^{(1)}$, $K = \kappa^{(1)}$, $V = \chi^{(1)}(a - x)/(a - \epsilon)$, and $D = (\epsilon, a) \times (\epsilon', t)$ and get

$$\begin{aligned}
&\int_{\epsilon'}^{t} \{\chi^{(1)}(\epsilon, \tau)\, u_x^{(1)}(\epsilon, \tau) - u^{(1)}(\epsilon, \tau)\, \chi_x^{(1)}(\epsilon, \tau)\}\, d\tau \\
&\quad = -(a - \epsilon)^{-1} \int_{\epsilon'}^{t}\int_{\epsilon}^{a} (a - x)\, \chi^{(1)} q^{(1)} - u^{(1)}[(a - x)\, \chi_{xx}^{(1)} \\
&\qquad - 2\chi_x^{(1)} + (a - x)\, \kappa^{(1)} \chi_t^{(1)}]\}\, dx\, d\tau \\
&\qquad + \kappa^{(1)}(a - \epsilon)^{-1} \int_{\epsilon}^{a} (a - x)\, \chi^{(1)}(x, \epsilon')\, u^{(1)}(x, \epsilon')\, dx \\
&\qquad - \kappa^{(1)}(a - \epsilon)^{-1} \int_{\epsilon}^{a} (a - x)\, \chi^{(1)}(x, t)\, u^{(1)}(x, t)\, dx \\
&\qquad - (a - \epsilon)^{-1} \int_{\epsilon'}^{t} \chi^{(1)}(\epsilon, \tau)\, u^{(1)}(\epsilon, \tau)\, d\tau + (a - \epsilon)^{-1} \int_{\epsilon'}^{t} \chi^{(1)}(a, \tau)\, u^{(1)}(a, \tau)\, d\tau.
\end{aligned} \tag{3.7}$$

Next, take $U = u^{(2)}$, $K = \kappa^{(2)}$, $V = \chi^{(2)}(a + x - 1)/(a - \epsilon)$, and $D = (1 - a, 1 - \epsilon) \times (\epsilon', t)$ and get

$$\begin{aligned}
&\int_{\epsilon'}^{t} \{\chi^{(2)}(1 - \epsilon, \tau)\, u_x^{(2)}(1 - \epsilon, \tau) - u^{(2)}(1 - \epsilon, \tau)\, \chi_x^{(2)}(1 - \epsilon, \tau)\}\, d\tau \\
&\quad = (a - \epsilon)^{-1} \int_{\epsilon'}^{t}\int_{1-a}^{1-\epsilon} \{(a + x - 1)\, \chi^{(2)} q^{(2)} - u^{(2)}[(a + x - 1)\, \chi_{xx}^{(2)} \\
&\qquad + 2\chi_x^{(2)} + (a + x - 1)\, \kappa^{(2)} \chi_t^{(2)}]\}\, dx\, d\tau \\
&\qquad - \kappa^{(2)}(a - \epsilon)^{-1} \int_{1-a}^{1-\epsilon} (a + x - 1)\, \chi^{(2)}(x, \epsilon')\, u^{(2)}(x, \epsilon')\, dx \\
&\qquad + \kappa^{(2)}(a - \epsilon)^{-1} \int_{1-a}^{1-\epsilon} (a + x - 1)\, \chi^{(2)}(x, t)\, u^{(2)}(x, t)\, dx \\
&\qquad + (a - \epsilon)^{-1} \int_{\epsilon'}^{t} \chi^{(2)}(1 - \epsilon, \tau)\, u^{(2)}(1 - \epsilon, \tau)\, d\tau \\
&\qquad - (a - \epsilon)^{-1} \int_{\epsilon'}^{t} \chi^{(2)}(1 - a, \tau)\, u^{(2)}(1 - a, \tau)\, d\tau.
\end{aligned} \tag{3.8}$$

After substituting (3.7) and (3.8) in (3.5), the limits $\epsilon \to 0$ and $\epsilon' \to 0$ can be performed and (3.3) is obtained, thus concluding the proof of Theorem 1.

Remark 3. The derivation of (3.3) is unnecessary for the proof of existence of solutions for Problem I, since (3.5) is suitable for this purpose. However, it will be a fundamental tool in proving the continuous dependence (and uniqueness) of such solutions upon the data and the coefficients.

4. Existence of Solutions

As we pointed out in the Introduction, only Problem I will be dealt with.

A sequence of approximating solutions $(T^{(k)}, s_k, u_k^{(1)}, u_k^{(2)})$ with $k = 1, 2, \ldots$ can be defined recursively as

$$\begin{aligned}
\mathbf{L}^{(i)} u_k^{(i)} &= q^{(i)}, && 0 < t < T^{(k)}, \quad 0 < x < s_k(t) \quad \text{if} \quad i = 1, \\
& && s_k(t) < x < 1 \quad \text{if} \quad i = 2, \\
u_k^{(i)}(x, 0) &= h^{(i)}(x), && 0 < x < b \quad \text{if} \quad i = 1, \qquad b < x < 1 \quad \text{if} \quad i = 2, \\
u_k^{(i)}(0, t) &= \varphi^{(i)}(t), && 0 < t < T^{(k)}, \quad i = 1, 2, \\
u_k^{(i)}(s_k(t), t) &= 0, && 0 < t < T^{(k)}, \quad i = 1, 2,
\end{aligned} \tag{4.1}$$

with the following definition of the approximating free boundaries $s_k(t)$

$$s_1(t) = b, \tag{4.2}$$

$$\begin{aligned}
\dot{s}_{k+1}(t) = {} & \chi^{(1)}(s_k(t), t)\, u_{k,x}^{(1)}(s_k(t), t) \\
& - \chi^{(2)}(s_k(t), t)\, u_{k,x}^{(2)}(s_k(t), t) + \mu(s_k(t), t),
\end{aligned} \tag{4.3}$$

for $0 < t < T^{(k+1)} \leqslant T^{(k)}$ and $s_{k+1}(0) = b$, $k = 1, 2, \ldots$.

Here $(0, T^{(k)})$ is the largest time interval in which $s_k(t) \in (0, 1)$ and is continuously differentiable; concerning the continuity of $u_{k,x}^{(i)}$ up to $x = s_k(t)$, see Lemma 2 of [5] and the corresponding remarks made in Parts I and II.

The techniques used in Section 5, I and developed in Section 4, II can be applied to prove the existence of a constant T_0 such that $T^{(k)} \geqslant T_0$ and to get the following uniform estimate for $t > 0$

$$|\, \dot{s}_k(t)| \leqslant A t^{-(1-\alpha)/2}. \tag{4.4}$$

The proof is based upon an induction process and the use of thermal potentials. It does not present significant differences with respect to the one displayed in Section II, apart from the fact that the same procedure is to be applied to get estimates on the x-derivatives at $x = s_k(t)$ of both $u_{k+1}^{(1)}$ and $u_{k+1}^{(2)}$. Therefore, it will not be duplicated here.

Using (4.4), the arguments of Section 4, II, and the Ascoli–Arzelà's theorem, it is easy to show that the sequence $\{s_k\}$ converges to a function $s(t)$ uniformly in $[0, T_0]$ and that this limit function is Lipschitz continuous in $(0, T_0)$.

This allows us to calculate two functions $u^{(1)}(x, t)$ and $u^{(2)}(x, t)$ by solving problems (1.1)–(1.4) and (1.1′), (1.2′), (1.3′), (1.4), specifying $s(t)$ as the limit function just found. Moreover, the same arguments used in Section 4, II show that $u^{(1)}$ and $u^{(2)}$ are the respective limits of the sequences $\{u_k^{(1)}\}$ and $\{u_k^{(2)}\}$. As a matter of fact it can be shown that a constant B exists such that (see Lemma 7, II)

$$|u_k^{(i)}(x, t)| \leqslant Bt^{-(1-\alpha)/2} |x - s_k(t)|, \qquad i = 1, 2; \quad k = 1, 2,... \tag{4.5}$$

in the respective domains of definition. The same is true for $u^{(i)}(x, t)$

$$|u^{(i)}(x, t)| \leqslant Bt^{-(1-\alpha)/2} |x - s(t)|, \qquad i = 1, 2. \tag{4.6}$$

The final step of the existence theorem consists in proving that the triple $(s, u^{(1)}, u^{(2)})$ actually verifies (1.5). This is achieved using (4.1), (4.3) to derive a relationship similar to (3.3) and then letting k go to infinity. Since (3.3) is obtained, the use of Theorem 1 completes the proof of an existence theorem in $(0, T_0)$. Taking into account (4.6), the same consideration we applied in Section 5, I can be repeated. Thus we proved:

THEOREM 2. *Under the assumption of Section 2, there exists a solution $(T^*, s, u^{(1)}, u^{(2)})$ to Problem I. Moreover, in a neighborhood of $t = 0$, it is*

$$|\dot{s}(t)| \leqslant A_0 t^{-(1-\alpha)/2}. \tag{4.7}$$

The class of solutions for which (4.7) holds will be called class $\mathscr{S}$.

It is clear that, if $T^* < \infty$, then,

$$\lim_{t \to T_-^*} \min[s(t), 1 - s(t)] = 0 \qquad \text{and/or} \qquad \limsup_{t \to T_-^*} |\dot{s}(t)| = +\infty.$$

A deeper investigation in this sense will be performed in Section 7.

5. Continuous Dependence. Uniqueness

Let $(T_1^*, s_1, u_1^{(1)}, u_1^{(2)})$ and $(T_2^*, s_2, u_2^{(1)}, u_2^{(2)})$ be two solutions of Problem I (the analysis for Problem II is even simpler) corresponding to two different sets of data and coefficients and belonging to the class $\mathscr{S}$ in which solutions have been shown to exist. Let be $T < \min[T_1^*, T_2^*]$ so that

$$|\dot{s}_i(T)| \leqslant S' \tag{5.1}$$

and

$$a = \min(a_1, a_2) > 0, \tag{5.2}$$

where the constants a_1 and a_2 are defined according to (3.2).

Under the assumptions of Section 2, a and S' can be estimated a priori. Next, define:

$$\Delta\varphi = \sum_{i=1,2} \int_0^t |\varphi_1^{(i)}(\tau) - \varphi_2^{(i)}(\tau)| (t-\tau)^{-1/2} \, d\tau, \tag{5.3}$$

$$\Delta h = \int_0^1 | h_1(x) - h_2(x)| \, dx, \tag{5.4}$$

$$\Delta\kappa = \sum_{i=1,2} | \kappa_1^{(i)} - \kappa_2^{(i)} |, \tag{5.5}$$

$$\Delta q = \sum_{i=1,2} \int_0^T \int_0^1 | q_1^{(i)}(x,t) - q_2^{(i)}(x,t)| \, dx \, dt, \tag{5.6}$$

$$\begin{aligned} \Delta\chi = \sum_{i=1,2} \Big\{ & \int_0^T \int_0^1 (| \chi_{1,t}^{(i)} - \chi_{2,t}^{(i)} | + | \chi_{1,xx}^{(i)} - \chi_{2,xx}^{(i)} | + | \chi_{1,x}^{(i)} - \chi_{2,x}^{(i)} |) \, dx \, dt \\ & + \int_0^T | \chi_1^{(i)}(i-1,\tau) - \chi_2^{(i)}(i-1,\tau)| \, d\tau \\ & + \int_0^1 | \chi_1^{(i)}(x,0) - \chi_2^{(i)}(x,0)| \, dx \Big\}, \end{aligned} \tag{5.7}$$

$$\Delta\mu = \max_{\substack{0 \leq x \leq 1 \\ 0 \leq t \leq T}} | \mu_1(x,t) - \mu_2(x,t)| . \tag{5.8}$$

THEOREM 3. *Under the assumptions of Section* 2 *and if, in addition, the derivatives appearing in* (5.7) *are bounded a constant N can be determined such that for any pair of solutions of Problem* I *in the class $\mathscr{S}$*

$$| s_1(t) - s_2(t)| \leq N(\Delta\varphi + \Delta h + \Delta\kappa + \Delta q + \Delta\chi + \Delta\mu + | b_1 - b_2 |). \tag{5.9}$$

The constant N depends on the bounds of the functions entering (5.3)–(5.7), *on a, S' and on the Lipschitz constants of the μ_j.*

Proof. Subtract Eqs. (3.3) for the two problems considered, in which the constant a is taken according to (5.2). The proof of (5.9) is then achieved by means of arguments similar to the ones employed in Section 8, I and Section 6, II. The main difference consists in the fact that $\int_0^\infty w_1(x,t) \, dx$ must be estimated instead of $\int_0^\infty x w_1(x,t) \, dx$ (see (8.20), I). This results in a different definition of $\Delta\varphi$.

6. Monotone Dependence

In order to investigate the monotone dependence of solutions of Problem I upon the data and the coefficients, we confine our attention to problems in which *the coefficients* $\chi^{(1)}$, $\chi^{(2)}$ *do not change their sign in a time interval* $(0, T')$. In such a case we can assume that $\chi^{(1)}$ and $\chi^{(2)}$ are nonpositive, since this condition can always be fulfilled by means of a proper transformation.

A further remark concerns condition (6.3). Proving the same theorem under the condition $b_1 = b_2$ requires an application of Theorem 3 and needs the simultaneous fulfillment of the assumptions of Theorem 3 and 4.

Theorem 4. *Let* $(T_1^*, s_1, u_1^{(1)}, u_1^{(2)})$ *and* $(T_2^*, s_2, u_2^{(1)}, u_2^{(2)})$ *be two solutions of Problem* I *corresponding to the respective data* $h_j^{(i)}$, $\varphi_j^{(i)}$, $i = 1, 2, j = 1, 2$ *and to the coefficients* $\kappa_j^{(i)}$, $q_j^{(i)}$, $\chi_j^{(i)}$, μ_j, $i = 1, 2, j = 1, 2$. *Define* $\bar{T} = \min(T_1^*, T_2^*, T')$ *and assume:*

$$u_2^{(1)}(x, t) \geqslant 0, \qquad 0 \leqslant x \leqslant s_2(t), \quad 0 \leqslant t \leqslant \bar{T}, \tag{6.1}$$

$$u_1^{(2)}(x, t) \leqslant 0, \qquad s_1(t) \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant \bar{T}, \tag{6.2}$$

and

$$0 \leqslant b_1 < b_2 \leqslant 1, \tag{6.3}$$

$$h_1^{(1)}(x) \leqslant h_2^{(1)}(x), \qquad 0 \leqslant x \leqslant b_1, \tag{6.4}$$

$$h_1^{(2)}(x) \leqslant h_2^{(2)}(x), \qquad b_2 \leqslant x \leqslant 1, \tag{6.5}$$

$$\varphi_1^{(i)}(t) \leqslant \varphi_2^{(i)}(t), \qquad 0 \leqslant t \leqslant \bar{T}, \quad i = 1, 2, \tag{6.6}$$

$$q_1^{(i)}(x, t) \geqslant q_2^{(i)}(x, t), \qquad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant \bar{T}, \quad i = 1, 2, \tag{6.7}$$

$$\mu_1(x, t) \leqslant \mu_2(x, t), \qquad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t \leqslant \bar{T}. \tag{6.8}$$

Then, if

$$\chi_2^{(1)} = \chi_1^{(1)} < 0, \qquad \chi_2^{(2)} \leqslant \chi_1^{(2)} < 0, \tag{6.9}$$

and

$$\kappa_1^{(i)} = \kappa_2^{(i)}, \qquad i = 1, 2, \tag{6.10}$$

it is

$$s_1(t) < s_2(t), \qquad 0 \leqslant t \leqslant \bar{T}. \tag{6.11}$$

Moreover, if it is assumed that $u_1^{(1)} \leqslant 0$ *and/or* $u_2^{(2)} \geqslant 0$, *then* (6.11) *is valid irrespective of the relative magnitude of* $\chi_1^{(1)}$, $\chi_2^{(1)}$, *and/or* $\chi_1^{(2)}$, $\chi_2^{(2)}$.

The proof of this theorem is strictly similar to the proof of Theorem 9, I (cases (iv) and (v)) and for this reason it is omitted here.

Concerning the case in which (6.10) is not satisfied, first we remark that if we define

$$r_1 = \kappa_2^{(1)}/\kappa_1^{(1)}, \qquad r_2 = \kappa_2^{(2)}/\kappa_1^{(2)}, \tag{6.12}$$

then the case in which

$$r_1 = r_2 = r \tag{6.13}$$

can be reduced to the previous one by means of the transformation

$$\tau = rt, \qquad \hat{u}_1^{(i)}(x, t) = u_1^{(i)}(x, \tau/r), \qquad \hat{s}_1(\tau) = s_1(\tau/r).$$

Hence we are led to compare problems satisfying (6.10) and if the assumptions of Theorem 4 are fulfilled by the transformed data and coefficients, we have

$$\hat{s}_1(t) < s_2(t),$$

i.e., $s_1(t/r) < s_2(t)$. Therefore, if $r > 1$ ($r < 1$) and s_1 is nonincreasing (nondecreasing), (6.11) is still valid.

The general case in which (6.13) does not hold is much more complicated (recall that even in the ordinary heat conduction problems the monotone dependence of the temperature upon the thermal coefficients occurs only in special cases). We can prove the following comparison theorem.

THEOREM 5. *Suppose that* (6.1), (6.2), (6.3), (6.8) *are valid, that* $r_1 \geqslant 1$, $r_2 \leqslant 1$ *and that* $s_1(t)$, $s_2(t)$ *are nonincreasing in* $(0, \bar{T})$. *Next, assume that two functions* $\bar{h}(x)$, $\bar{\bar{h}}(x)$ *exist such that*

$$h_1^{(1)}(x) \leqslant \bar{h}(x) \leqslant h_2^{(1)}(x), \qquad \bar{h}(x) \geqslant 0, \quad 0 \leqslant x \leqslant b_1, \tag{6.14}$$

$$h_1^{(2)}(x) \leqslant \bar{\bar{h}}(x) \leqslant h_2^{(2)}(x), \qquad \bar{\bar{h}}(x) \leqslant 0, \quad b_2 \leqslant x \leqslant 1, \tag{6.15}$$

and that

$$\bar{h}(0) \geqslant \sup \varphi_1^{(1)}(t), \qquad \bar{\bar{h}}(1) \geqslant \sup \varphi_1^{(2)}(t), \tag{6.16}$$

$$\bar{h}''(x), \bar{\bar{h}}''(x) \leqslant -Q, \tag{6.17}$$

where Q *is an upper bound for* $|\, q_j^{(i)} \,|$, $i = 1, 2, j = 1, 2$.

If

$$\varphi_2^{(1)}(t'') \geqslant \varphi_1^{(1)}(t'), \qquad 0 \leqslant t' \leqslant t'' \leqslant \bar{T}, \tag{6.18}$$

$$\varphi_2^{(2)}(t'') \geqslant \varphi_1^{(2)}(t'), \qquad 0 \leqslant t' \leqslant t'' \leqslant \bar{T}, \tag{6.19}$$

and

$$q_2^{(1)}(x, t'') \leqslant q_1^{(1)}(x, t'), \qquad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t' \leqslant t'' \leqslant \bar{T}, \tag{6.20}$$

$$q_2^{(2)}(x, t'') \leqslant q_1^{(2)}(x, t'), \qquad 0 \leqslant x \leqslant 1, \quad 0 \leqslant t' \leqslant t'' \leqslant \bar{T}, \tag{6.21}$$

then we can assert that

$$s_1(t) < s_2(t), \qquad 0 \leqslant t \leqslant \bar{T}, \tag{6.22}$$

under the same assumptions on $\chi_j^{(i)}$ $(i = 1, 2, j = 1, 2)$ *as in Theorem* 4. *A similar result holds if* $r_1 \leqslant 1$, $r_2 \geqslant 1$, *and* $s_1(t)$, $s_2(t)$ *are nondecreasing.*

Proof. If $\bar{t}$ is the first instant in which $s_1(t)$ and $s_2(t)$ are supposed to be equal, all we have to prove is that:

$$u_2^{(i)}(x, \bar{t}) \geqslant u_1^{(i)}(x, \bar{t}), \qquad i = 1, 2, \quad 0 \leqslant x \leqslant 1. \tag{6.23}$$

Then (6.22) follows by standard arguments.

For this purpose, let us consider, in the domain $0 < x < s_1(t)$, $0 < t < \bar{t}$ the transformation

$$\begin{aligned} \tau = \bar{t} - r_1(\bar{t} - t), \qquad & U_1(x, \tau) = u_1^{(1)}(x, \bar{t} - (\bar{t} - \tau)/r_1), \\ S_1(\tau) = s_1(\bar{t} - (\bar{t} - \tau)/r_1). & \end{aligned} \tag{6.24}$$

Note that τ varies from $\tau_1 = -\bar{t}(r_1 - 1)$ to $\bar{t}$ as t varies from 0 to $\bar{t}$ and that $\tau \leqslant t$ in this interval. In order to prove (6.23) for $i = 1$ it is enough to show that

$$U_1(x, \bar{t}) \leqslant u_2^{(1)}(x, \bar{t}), \qquad 0 \leqslant x \leqslant S_1(\bar{t}) = s_1(\bar{t}), \tag{6.25}$$

since $U_1(x, \bar{t}) = u_1^{(1)}(x, \bar{t})$.

Note that $S_1(\tau_1) = b_1$ and that $\dot{S}_1(t) = \dot{s}_1(t)/r_1 \geqslant \dot{s}_1(t)$; from the last inequality and from the fact that $S_1(\bar{t}) = s_1(\bar{t})$ it follows that

$$S_1(t) \leqslant s_1(t), \qquad 0 \leqslant t \leqslant \bar{t}. \tag{6.26}$$

The function $U_1(x, t)$ is defined in the domain $0 < x < S_1(t)$, $\tau_1 < t < \bar{t}$, where it satisfies the equation

$$U_{1,xx} - \kappa_2^{(1)} U_{1,t} = q_1^{(1)}(x, \bar{t} - (\bar{t} - t)/r_1). \tag{6.27}$$

Let us compare $U_1(x, t)$ with the function $\bar{h}(x)$ in the time interval $(\tau_1, 0)$. It is:

$$\bar{h}(x) \geqslant U_1(x, \tau_1) = h_1^{(1)}(x), \qquad 0 \leqslant x \leqslant b_1,$$

$$\bar{h}(S_1(t)) \geqslant 0, \qquad \bar{h}(0) \geqslant U_1(0, t), \qquad 0 \leqslant t \leqslant \bar{t},$$

$$d^2\bar{h}/dx^2 \leqslant U_{1,xx} - \kappa_2^{(1)} U_{1,t}$$

owing to (6.14), (6.16), (6.17). Therefore,

$$U_1(x, 0) \leqslant \bar{h}(x), \qquad 0 \leqslant x \leqslant S_1(0) \leqslant b_1. \tag{6.28}$$

Now, (6.1), (6.14), (6.18), (6.20), (6.28) and the maximum principle yield (6.25). The proof of (6.23) for $i = 2$ follows a parallel path; so the proof of Theorem 5 is complete.

7. Remarks on the Behavior of the Free Boundary

First, we state the following theorem which can be proved by arguments similar to the ones used in [4], with minor changes.

THEOREM 6. *Suppose the data and coefficients are bounded measurable functions and that two positive constants $\bar{X}$ and $\nu \in (0, 1)$ exist such that*

$$|\chi^{(i)}(x', t') - \chi^{(i)}(x'', t'')| \leqslant \bar{X}\{|x' - x''|^\nu + |t' - t''|^\nu\},$$

for (x', t') and $(x'', t'') \in \Omega$.

Let $(T^*, s, u^{(1)}, u^{(2)})$ be a solution of Problem I *and consider an arbitrary time interval $[t_1, t_2]$ included in $(0, T^*)$. Then, for any $\nu' \in (0, \nu] \cap (0, \frac{1}{2})$, a constant $\bar{S}$ exists, depending on ν', t_1, t_2, $\bar{X}$, on the minimum values of $s(t)$ and $1 - s(t)$ in $[t_1, t_2]$, on the maximum value of $|\dot{s}(t)|$ in the same interval and on the bounds on the data and the coefficients, such that:*

$$|\dot{s}(t') - \dot{s}(t'')| \leqslant \bar{S}\,|t' - t''|^{\nu'}, \qquad t', t'' \in [t_1, t_2].$$

Remark 4. If the assumptions of Theorem 2 are satisfied, the constant $\bar{S}$ can be estimated a priori.

Theorem 7 below states sufficient condition for global existence in the case of Stefan problems.

THEOREM 7. *As far as the ordinary two-phase Stefan problem is concerned, under the assumption of Theorem 2 and if the functions $\varphi^{(1)}$ and $\varphi^{(2)}$ never vanish, $T^* = +\infty$.*

Proof. By means of standard techniques (see [2]) it can be proved that in any time interval $(0, T)$ a constant $a(T)$ exists such that $0 < a \leqslant s(t) \leqslant 1 - a$. Thus the proof of the theorem consists in showing that $|\dot{s}(t)|$ is bounded for finite $t > 0$.

Recall (4.6) and consider any interval (t_0, T^*), with t_0 fixed in $(0, T_0)$. The procedures introduced in [4] lead to an estimate of the integral of $[u_x^{(1)}(x, t)]^2$ between 0 and $s(t)$ and of $[u_x^{(2)}(x, t)]^2$ between $s(t)$ and 1 and this estimate is independent of t in (t_0, T^*). This implies uniform Hölder estimate of $u^{(i)}(x, T)$, $i = 1, 2$ for any $T < T^*$ and hence the fact that the solution can be continuated beyond T over a time interval whose amplitude does not depend on T.

Consequently, $T^* = +\infty$.

Remark 5. If in Problem I the following sign specifications are given (recall $f = 0$, see Remark 1): $\text{sign}(\chi^{(1)}) = \text{sign}(\chi^{(2)}) = \text{const}$, $(-1)^i \chi^{(i)} h^{(i)} \geqslant 0$, $(-1)^i \chi^{(i)} \varphi^{(i)} \geqslant 0$, $(-1)^i \chi^{(i)} q^{(i)} \leqslant 0$, the method displayed in the proof of Theorem 8, I is still applicable leading to the conclusion that if $T^* < +\infty$ and $0 < \liminf_{t \to T\_^*} s(t) \leqslant \limsup_{t \to T\_^*} s(t) < 1$, then: $\liminf_{t \to T\_^*} \dot{s}(t) = -\infty$ and $\limsup_{t \to T\_^*} \dot{s}(t) = +\infty$.

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