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Amenable Ergodic Group Actions and an Application to Poisson Boundaries of Random Walks*

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We introduce and study the class of amenable ergodic group actions which occupy a position in ergodic theory parallel to that of amenable groups in group theory. We apply this notion to questions about skew products, the range (i.e., Poincaré flow) of a cocycle, and to Poisson boundaries.

INTRODUCTION

In this paper we introduce a new notion of amenability for ergodic group actions and use it to give a partial solution to a problem in ergodic theory concerning skew products. We also show how the notion can be used to generalize a result about Poisson boundaries of random walks on groups. Amenable ergodic actions occupy a position in ergodic theory parallel to that of amenable groups in group theory, and one therefore expects the notion to be applicable in diverse situations. Greenleaf [7] has also introduced a notion of amenability which, although related, is quite different from ours. (See Section 4.) We hope the parallel between our results and results from group theory will justify our terminology. We also remark that from Mackey's virtual subgroup point of view [11] what we will be considering are amenable virtual subgroups of locally compact groups.

There are a variety of different equivalent conditions defining amenability for groups. The condition on which we shall concentrate is the fixed point property. Thus, a group G is amenable if every continuous affine action of G on a compact convex set has a fixed point. An affine action is, of course, just a homomorphism into the group of affine automorphisms. There is a strong parallel between homomorphisms in group theory and cocycles in ergodic theory. In fact, for transitive actions the study of cocycles essentially reduces to the study of homomorphisms of the stability group. Thus, we are led to

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consider cocycles into groups of affine automorphisms and some technicalities aside, we will call a G-space S amenable if every cocycle into a group of affine homeomorphisms has a fixed element. Here, a fixed element is no longer a single point, but rather a Borel function from S into the compact convex set. With this definition of amenability the class of amenable ergodic actions behaves quite similarly to the class of amenable groups. In particular, many of the usual combinatorial properties still hold, and the parallel becomes more pronounced if one phrases everything in terms of virtual subgroups. We remark that in the transitive case, the action will be amenable if and only if the stability groups are amenable.

A well-known and useful construction in ergodic theory is the skew product construction. If S is an ergodic G-space and $\alpha: S \times G \to H$ is a cocycle into a locally compact group, then one can define a G-action on $S \times H$ by $(s, h)g = (sg, h\alpha(s, g))$. If $G = Z$, the group of integers, the action is determined by an invertible transformation T. A cocycle is determined by the function $f(s) = \alpha(s, 1)$, and the skew product takes the possibly more familiar form $\tilde{T}(s, h) = (Ts, hf(s))$. It is a natural problem, raised for example by Mackey [11], to start with a given ergodic G-space S and ask for which groups H can one find a cocycle so that the skew product action on $S \times H$ is also ergodic. Even for actions of the integers, the answer is unknown. If S is a Z-space with a finite invariant measure, then it is known that for any of the following types of groups, such a cocycle will exist: compact [15, 18]; countable discrete abelian [15]; connected nilpotent Lie [19]; discrete finitely generated nilpotent (use [19, Theorem 3.6], induction, and the fact that a subgroup of a finitely generated nilpotent group is finitely generated); and any finite product of any of these types [19]. All of these groups are amenable, and it follows from this paper that amenability is actually a necessary condition. More generally, we show that any ergodic action (even without invariant measure) of an amenable group is amenable, and that if $\alpha: S \times G \to H$ with S amenable and $S \times H$ ergodic, then H must be amenable.

A further generalization of this result is related to a generalization of the flow built under a function construction. If $\alpha: S \times G \to H$ is a cocycle, one can form an H-space X called the range of α [11] (or the Poincaré flow [5]) which reduces to the flow built under $f(s) = \alpha(s, 1)$ if $G = Z$, $H = \mathbb{R}$, and f is positive. As every $\mathbb{R}$-flow arises in this way, it is natural to ask which G-actions are the range of a cocycle on a Z-space. It follows from [19] that if there is a cocycle $\alpha: S \times Z \to H$ with an ergodic skew product, then every H-action is the range of a cocycle on some Z-space. Here, we establish the result that the range of any cocycle of an amenable action must also be an amenable action. In particular, this excludes the possibility that an action of a non-amenable group with finite invariant measure is the range of a cocycle on a Z-space. We remark that this also provides a large collection of amenable actions of non-amenable groups.

If μ is a probability measure on G, μ defines a random walk on G, and Furstenberg has associated a space to (G, μ), called the Poisson boundary, which enables one, for example, to obtain integral representation theorems for the harmonic functions of the random walk. For μ étalée, we present a different construction of the Poisson boundary which shows that it appears as the range of a naturally defined cocycle of a (semi-group) action of the non-negative integers. A modification of the proof of the result described in the preceding paragraph then shows that the Poisson boundary with a naturally defined invariant measure class is an amenable ergodic G-space. It follows in particular that if G is transitive on a Poisson space of an étalée measure (a situation which occurs in many important cases) the stability groups must be amenable.

The organization of this paper is as follows. Section 1 presents definitions, some preliminary technical material, and deals with amenability in the transitive case. In Section 2 we show that actions of amenable groups and extensions of amenable actions are also amenable. Section 3 deals with the range of a cocycle and skew products. Section 4 concerns amenable pairs, a generalization of Eymard's notion of the conditional fixed point property [4]. We also point out here a connection with Greenleaf's definition of amenability. Section 5 is devoted to the Poisson boundary.

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1. Definitions and Preliminaries

We begin by describing the concept of amenability and proving some useful auxilliary results. We shall throughout take G to be a second countable locally compact group, S a standard Borel space, and μ a probability measure on S. We suppose that there is a right Borel action of G on S and that μ is quasi-invariant and ergodic under G. Suppose that M is a Borel group (i.e., a group with a Borel structure compatible with the group operations.) A Borel function $\alpha: S \times G \to M$ is called a cocycle if for all $g, h \in G$, $\alpha(s, gh) = \alpha(s, g)\, \alpha(sg, h)$ for almost all s. The study of cocycles is playing an increasingly significant role in many aspects of ergodic theory. (See [5], [16] and the references in these papers as examples.) If S is a point, a cocycle is nothing but a Borel homomorphism $G \to M$ and many properties of homomorphisms have natural analogues as properties of cocycles for general S. Mackey has formalized this analogy in his notion of virtual groups [11], in which an ergodic G-space is considered as a virtual subgroup of G and the cocycles on $S \times G$ as the homomorphisms of this virtual group. (Although it is not necessary for a reading of this paper, the author feels that the virtual subgroup viewpoint is very suggestive.) Amenable groups can be characterized by the properties of a certain class of homomorphisms, namely the existence of fixed points in affine actions on compact convex sets. We now generalize this notion to group actions.

We first establish some preliminary facts. Let E be a separable Banach space and $\mathrm{Iso}(E)$ the group of isometric isomorphisms of E.

LEMMA 1.1. *With the strong operator topology* $\mathrm{Iso}(E)$ *is a separable metrizable group and the induced Borel structure is standard.*

Proof. That $\mathrm{Iso}(E)$ is a separable metric group can be seen exactly as in the first paragraph of [14, Lemma 8.34]. Next, note that with the strong operator topology $L_1(E)$, the unit ball in the space of bounded linear maps on E, is metrizable by a complete separable metric. To see that the Borel structure on $\mathrm{Iso}(E)$ is standard, it suffices to show that $\mathrm{Iso}(E)$ is a Borel subset of $L_1(E)$. Let $\{f_i\}$ be a countable dense subset of E_1, the unit ball in E. Then an isometry T is in $\mathrm{Iso}(E)$ if and only if each $f_i \in \mathrm{range}(T)$. Now $f \in \mathrm{range}(T)$ if and only if for all n there exists j such that $\| Tf_j - f \| \leqslant 1/n$. Thus

$$\mathrm{Iso}(E) = \bigcap_i \bigcap_n \left(\bigcup_j \left\{ T \mid \| Tf_j - f_i \| \leqslant \frac{1}{n} \right\} \right) \cap \{ T \mid T \text{ is an isometry} \}.$$

Since the set of isometries is closed in $L_1(E)$, it follows that $\mathrm{Iso}(E)$ is Borel.

COROLLARY 1.2. *The Borel structure on* $\mathrm{Iso}(E)$ *is the smallest such that all maps* $T \to Tf$ *are Borel,* $f \in E$.

Let E^* be the dual space of E and $E_1{}^*$ the unit ball in the dual. Then $E_1{}^*$ is a compact convex set when endowed with the $\sigma(E^*, E)$ topology, and since E is separable, $E_1{}^*$ is also metrizable. We denote by $\langle \ , \ \rangle$ the dual pairing of E^* and E. For $T \in L_1(E)$, we have an adjoint map $T^* \in L_1(E^*)$.

LEMMA 1.3. *The map* $\mathrm{Iso}(E) \times E_1{}^* \to E_1{}^*$, *defined by* $(T, \lambda) \to T^*(\lambda)$ *is continuous.*

Proof. Let $T_n \to T$ and $\lambda_n \to \lambda$. Then for $f \in E$,

$$\begin{aligned} |\langle T_n{}^*(\lambda_n) - T^*(\lambda), f \rangle| &= |\langle \lambda_n , T_n f \rangle - \langle \lambda, Tf \rangle| \\ &\leqslant |\langle \lambda_n , T_n f - Tf \rangle| + |\langle \lambda_n - \lambda, Tf \rangle| \\ &\leqslant \| T_n f - Tf \| + |\langle \lambda_n - \lambda, Tf \rangle|. \end{aligned}$$

As $n \to \infty$, both terms $\to 0$.

Letting $H(E_1{}^*)$ be the group of homeomorphisms of $E_1{}^*$ with the topology of uniform convergence, Lemma 1.3 implies that the induced map $\mathrm{Iso}(E) \to H(E_1{}^*)$ is continuous and hence Borel.

Now suppose that S is a standard Borel space and for each $s \in S$ we have a non-empty compact convex subset $A_s \subset E_1{}^*$. Then $\{A_s\}$ will be called a Borel field of compact convex sets if $\{(s, \lambda) \mid \lambda \in A_s\}$ is a Borel subset of $S \times E_1{}^*$.

We are now ready to define an amenable ergodic action. With all notation as above, suppose $\alpha: S \times G \to \mathrm{Iso}(E)$ is a Borel cocycle. Then there is an induced adjoint (Borel) cocycle $\alpha^*: S \times G \to H(E_1^*)$ defined by $\alpha^*(s, g) = (\alpha(s, g)^{-1})^*$. A Borel field of compact convex sets $\{A_s\}$ is called α-invariant if for each g, $\alpha^*(s, g)A_{sg} = A_s$ for almost all s.

DEFINITION 1.4. If S is an ergodic G-space, S is called amenable if for every separable Banach space E, cocycle $\alpha: S \times G \to \mathrm{Iso}(E)$, and α-invariant Borel field $\{A_s\}$, there is a Borel function $\varphi: S \to E_1^*$ such that $\varphi(s) \in A_s$ a.e. and for each g, $\alpha^*(s, g)\, \varphi(sg) = \varphi(s)$ a.e. We will then call φ an α-invariant section in $\{A_s\}$.

If S is a point, α becomes a Borel (and hence continuous [14, Lemma 8.28]) homomorphism $G \to \mathrm{Iso}(E)$, and amenability of the G-space S means that every compact convex G-invariant subset of E_1^* contains a fixed point. Aside from the restriction we have made regarding the separability of E, this is the condition of G being amenable. We now deal with the separability hypothesis.

PROPOSITION 1.5. *Let G be locally compact and second countable. Suppose that for every separable Banach space E, continuous homomorphism $G \to \mathrm{Iso}(E)$, and G-invariant compact convex set $A \subset E_1^*$, there is a point in A left fixed by G. Then G is amenable.*

Proof. It suffices to show that the hypothesis of the proposition is true for arbitrary (not just separable) Banach spaces. Since G is separable, we can find a collection of separable G-invariant closed subspaces $E_\sigma \subset E$, $\sigma \in I$, where I is some index set, such that $\overline{\bigcup E_\sigma} = E$. Let $\varphi_\sigma : E^* \to E_\sigma^*$ be the restriction map and $A_\sigma = \varphi_\sigma(A)$. Thus A_σ is a compact convex G-invariant set, and since E_σ is separable, the set of fixed points $F_\sigma \subset A_\sigma$ is non-empty and closed. For each $\sigma_1, \ldots, \sigma_n \in I$, the subspace $E_{\sigma_1} + \cdots + E_{\sigma_n} \subset E$ will also be separable, and hence there is a fixed point in $\varphi_{\sigma_1 \cdots \sigma_n}(A)$ where $\varphi_{\sigma_1 \cdots \sigma_n} : E^* \to (E_{\sigma_1} + \cdots + E_{\sigma_n})^*$. Denoting this set of fixed points by $F_{\sigma_1 \cdots \sigma_n}$, it is clear that

$$(\varphi^{-1}_{\sigma_1 \cdots \sigma_n}(F_{\sigma_1 \cdots \sigma_n}) \cap A) \subset \left(\bigcap_1^n \varphi^{-1}_{\sigma_i}(F_{\sigma_i}) \cap A \right),$$

and hence this latter intersection is non-empty. By the finite intersection property, it follows that $\bigcap_I \varphi_\sigma^{-1}(F_\sigma) \cap A$ is nonempty. If λ is a point in this intersection, λ is invariant when restricted to each E_σ, and since $\overline{\bigcup E_\sigma} = E$, it follows that λ is G-invariant.

COROLLARY 1.6. *The trivial G-space $\{e\}$ is amenable if and only if G is an amenable group.*

The ergodic actions with the simplest orbit structure are the essentially transitive ones. We now examine what amenability means in this case, generalizing Corollary 1.6. We preface this with a technical lemma which will be of general use.

LEMMA 1.7. *If $\{A_s\}$ is a Borel field of compact convex sets, then there is a countable collection of Borel functions $a_i : S \to E_1{}^*$ such that for all s in a conull Borel set, $A_s = \overline{\{a_i(s) \mid i = 1, 2,...\}}$. Conversely, given Borel functions a_i, then $s \to \overline{\{a_i(s) \mid i = 1, 2,...\}}$ defines a Borel field of compact convex sets if each of these sets is convex.*

Proof. (i) If $\{A_s\}$ is Borel, let $\tilde{A} = \{(s, x) \mid x \in A_s\}$, so that $\tilde{A} \subset S \times E_1{}^*$ is Borel. Fix a metric on $E_1{}^*$ and for each n choose a $1/n$-dense subset of $E_1{}^*$, say $x_1{}^n,..., x^n_{k(n)}$. Let $B_j{}^n = S \times B(x_j{}^n; 1/n) \subset S \times E_1{}^*$, so that $B_j{}^n$ is Borel. Letting $p: S \times E_1{}^* \to S$ be projection, $p(B_j{}^n \cap \tilde{A})$ will be analytic. By the von Neumann selection theorem, there is a Borel set $S_j{}^n \subset p(B_j{}^n \cap \tilde{A})$ with the same measure as $p(B_j{}^n \cap \tilde{A})$, and a Borel section of the projection $S_j{}^n \to B_j{}^n \cap \tilde{A}$. Note that $\bigcup_j S_j{}^n$ is a conull Borel set for each n. Combining these sections with just a small amount of finesse, we see that we can obtain, for each n, a finite set of Borel functions $a_j{}^n(s)$ such that $\{a_j{}^n(s)\}_j$ are $2/n$-dense in A_s for s in a conull Borel set. As n is arbitrary, the first assertion of the lemma follows easily.

(ii) We note that $x \in \overline{\{a_i(s)\}}$ if and only if $x \in \bigcap_n \bigcup_j B(a_j(s), 1/n)$. Thus

$$\{(s, x) \mid x \in A_s\} = \bigcap_n \bigcup_j \{(s, x) \mid x \in B(a_j(s), 1/n)\}$$

and so it suffices to see that each of the latter sets are Borel. But this follows from the fact that the map $S \times E_1{}^* \to \mathbb{R}$, $(s, x) \to (s, a_i(s), x) \to d(a_i(s), x)$ is Borel.

COROLLARY 1.8. *If α, β: $S \times G \to \mathrm{Iso}(E)$ are equivalent, then every α-invariant Borel field $\{A_s\}$ has an α-invariant section if and only if every β-invariant field has a β-invariant section.*

Proof. Let T: $S \to \mathrm{Iso}(E)$ be such that $T(s)\, \alpha(s, g)\, T(sg)^{-1} = \beta(s, g)$ for each g and almost all s. Suppose every β-invariant field has a β-invariant section, and let $\{A_s\}$ be an α-invariant field. Then it follows from Lemma 1.7 that $s \to T^*(s)^{-1}A_s$ agrees a.e. with a Borel field of compact convex sets, and this field will be β-invariant. If φ is a β-invariant section in $T^*(s)^{-1} A_s$ (a.e.), then $T^*(s)\, \varphi(s)$ will be an α-invariant section in A_s (a.e.).

THEOREM 1.9. *Let $H \subset G$ be a closed subgroup. Then G/H is an amenable G space if and only if H is amenable.*

Proof. (i) Suppose H is amenable. Let E be a separable Banach space and $\alpha: G/H \times G \to \text{Iso}(E)$ a cocycle. By [14, 8.24–8.28 and particularly the proof of Lemma 8.24], α is equivalent to a strict cocycle β with the following property: if $\lambda: G \to \text{Iso}(E)$ is defined by $\lambda(g) = \beta([e], g)$, then $\beta(G/H \times G) = \lambda(H)$. We remark that β can be expressed in terms of λ by $\beta([k], g) = \beta([e]k, g) = \beta([e], k)^{-1} \beta([e], kg) = \lambda(k)^{-1} \lambda(kg)$. We also note that if $h \in H$ and $g \in G$, then $\lambda(hg) = \lambda(h) \lambda(g)$ from the cocycle identity. In particular $\lambda \mid H$ is a Borel homomorphism. We let $\lambda^*(g) = \beta^*([e], g)$.

Now suppose that $\{A_{[k]}\}$ is a β-invariant Borel field of compact convex sets in $E_1{}^*$, so that $\beta^*([k], g)A_{[k]g} = A_{[k]}$ for each g and almost all k. Then for each g, we have $\lambda^*(kg)A_{[kg]} = \lambda^*(k)A_{[k]}$ a.e., and hence, using Fubini's theorem, $\lambda^*(k)A_{[k]} = A$ a.e. where A is a fixed compact convex set. If $h \in H$, we have $\lambda^*(hk)A_{[hk]} = A$ a.e., that is $\lambda^*(h) \lambda^*(k)A_{[k]} = A$ a.e. This clearly implies $\lambda^*(h)A = A$ for all h. Since H is amenable, there is a fixed point $x \in A$, and since $\beta(s, g) \in \lambda(H)$, it follows that $\varphi(s) \equiv x$ is a β-invariant section. To show that G/H is an amenable G space, it suffices to show that $x \in A_{[k]}$ for almost all k. But this follows since $A_{[k]} = \lambda^*(k)^{-1}A$ a.e.

(ii) Conversely, suppose G/H is an amenable G-space. Let $\lambda: H \to \text{Iso}(E)$. Then there is a strict cocycle $\alpha: G/H \times G \to \text{Iso}(E)$ such that $\alpha(G/H \times G) = \lambda(H)$ and $\alpha([e], h) = \lambda(h)$ for $h \in H$. Let $A \subset E_1{}^*$ be a compact convex set and $A_s = A$ for all s. Choose an α-invariant section $\varphi(s) \in A$. Then for each g and almost all k, $\alpha^*([e], kg) \varphi([k]g) = \alpha^*([e], k) \varphi([k])$, so that for some $x \in A$, $\alpha^*([e], k) \varphi([k]) = x$ a.e. If $h \in H$, $\alpha^*([e], hk) \varphi([hk]) = x$ for almost all k, that is $\lambda^*(h) \alpha^*([e], k) \varphi([k]) = x$. It follows that $\lambda^*(h)x = x$ and hence by Proposition 1.5 that H is amenable.

The reader conversant with Mackey's virtual subgroup viewpoint will recognize that Theorem 1.9 shows that Definition 1.4 is a reasonable definition of an amenable virtual subgroup. This will be reinforced in the succeeding sections, where we exhibit many similarities between the notions of amenability for ergodic actions and for groups.

2. Actions of Amenable Groups and Extensions of Amenable Actions

In this and following sections, we examine how the property of amenability behaves under various natural operations in ergodic theory, and in the process provide a wealth of examples of amenable ergodic actions.

THEOREM 2.1. *Let G be an amenable group and (S, μ) and ergodic G-space. Then S is an amenable G-space.*

We remark that in light of Theorem 1.9, this result is a generalization of the fact that a closed subgroup of an amenable group is amenable. In Mackey's language, a virtual subgroup of an amenable group is amenable. The converse of Theorem 2.1 is not true as we shall see below. A condition under which the converse assertion will hold is given in Section 4. We begin the proof by collecting some properties of vector-valued functions.

Let E be a separable Banach space and let

$$L^1(S, E) = \left\{f\colon S \to E \mid f \text{ measurable and } \int \|f(s)\|\, d\mu < \infty\right\}.$$

We identify functions which agree on a conull set. We recall that $f\colon S \to E$ is measurable if and only if $\theta \circ f\colon S \to \mathbb{C}$ is measurable for all $\theta \in E^*$ [2, p. 149]. Then $L^1(S, E)$ is a separable Banach space [3, p. 587]. Recall that $\lambda\colon S \to E^*$ is called weakly measurable if $s \to \langle\lambda(s), x\rangle$ is measurable for all $x \in E$. If λ is weakly measurable, then $s \to \|\lambda(s)\|$ is measurable [3, 8.15.3]. We let $L^\infty(S, E^*) = \{\lambda\colon S \to E^* \mid \lambda \text{ weakly measurable and } \|\lambda(s)\| \in L^\infty(S)\}$. Then $L^\infty(S, E^*)$ is a Banach space under the essential sup norm [3, p. 578]. If $f \in L^1(S, E)$ and $\lambda \in L^\infty(S, E^*)$, we define $\langle\lambda, f\rangle = \int_S \langle\lambda(s), f(s)\rangle\, d\mu(s)$. For each λ this defines an element $\tilde{\lambda}$ of $L^1(S, E)^*$ and $\lambda \to \tilde{\lambda}$ is an isometric isomorphism of $L^\infty(S, E^*)$ with $(L^1(S, E))^*$ [3, 8.18.2]. Thus the closed unit ball in $L^\infty(S, E^*)$ is a compact metrizable space with the $\sigma(L^\infty(S, E^*), L^1(S, E))$ topology. We denote this ball by $L_1^\infty(S, E^*)$.

Now suppose $\{A_s\}$ is a Borel field of compact convex subsets of E_1^*. Let $B = \{\lambda \in L^\infty(S, E^*) \mid \lambda(s) \in A_s \text{ a.e.}\}$

PROPOSITION 2.2. *B is a closed convex subset of $L_1^\infty(S, E^*)$.*

Proof. Since E is separable, it is easy to see that there is a countable set of hyperplanes in E^* that strictly separates points in E_1^* from compact convex sets in E_1^*. For example, we may take as hyperplanes the zeroes of the functions $f_{nq}(\lambda) = \langle\lambda, x_n\rangle - q$ where $\{x_n\}$ is a countable dense set in E and q is rational. Now suppose $\lambda_j \in B$, $\lambda_j \to \lambda \in L_1^\infty(S, E^*)$, and $\lambda(s) \notin A_s$ on a set of positive measure. For each n and q, let

$$S_{nq} = \{s \in S \mid f_{nq}(A_s) < 0 \text{ and } f_{nq}(\lambda(s)) > 0\}.$$

Then $\bigcup S_{nq}$ has positive measure. If $\tilde{A} = \{(s, x) \in S \times E_1^* \mid x \in A_s\}$ and p_1, p_2 are the projections of $S \times E_1^*$ on S and E_1^* respectively, then

$$S - S_{nq} = p_1((f_{nq} \circ p_2)^{-1}([0, \infty)) \cap \tilde{A}) \cup (f_{nq} \circ \lambda)^{-1}((-\infty, 0]).$$

Since $\tilde{A}$ is Borel, the first set in the union is analytic, and it follows that S_{nq} is measurable for all n, q. It follows that S_{nq} has positive measure for some n, q. Define $f \in L^1(S, E)$ by

$$f(s) = \begin{cases} x_n & \text{if } s \in S_{nq} \\ 0 & \text{if } s \notin S_{nq}. \end{cases}$$

Then

$$\langle \lambda_j, f \rangle = \int_{S_{nq}} \langle \lambda_j(s), x_n \rangle \, d\mu \leqslant q\mu(S_{nq})$$

since $\lambda_j \in B$. On the other hand,

$$\langle \lambda, f \rangle = \int_{S_{nq}} \langle \lambda(s), x_n \rangle \, d\mu > q\mu(S_{nq}).$$

Thus we cannot have $\langle \lambda_j, f \rangle \to \langle \lambda, f \rangle$ and the proposition follows.

We now turn to the proof of Theorem 2.1.

Proof of Theorem 2.1. Suppose $\alpha: S \times G \to \mathrm{Iso}(E)$ is a cocycle and $\{A_s\}$ is a α-invariant Borel field. Let $r(s, g)$ be the Radon–Nikodym cocycle of the action of G on S, i.e., a positive Borel function such that for each g, $d\mu(sg) = r(s, g)\, d\mu(s)$. For $g \in G$, define $T(g)$ acting on $L^1(S, E)$ by $(T(g)f)(s) = r(s, g)\, \alpha(s, g)\, f(sg)$. Then

$$\| T(g)f \| = \int r(s, g) \, \| f(sg) \| \, d\mu(s)$$

$$= \int r(sg^{-1}, g) \, \| f(s) \| \, r(s, g^{-1}) \, d\mu(s) = \| f \|,$$

so that $T(g)$ is a representation of G by isometric isomorphisms on $L^1(S, E)$. To see that T is continuous, it suffices to show that $f \in L^1(S, E)$ and $\lambda \in L^\infty(S, E^*)$ implies

$$g \to \langle \lambda, T(g)f \rangle = \int \langle \lambda(s), r(s, g)\, \alpha(s, g)\, f(sg) \rangle \, d\mu(s)$$

is measurable, which it is by Fubini's theorem. We have an induced adjoint action $T^*(g) = (T(g)^{-1})^*$ on $L^1(S, E)^* = L^\infty(S, E^*)$, and one readily verifies that this is defined by $(T^*(g)\lambda)(s) = \alpha^*(s, g)\, \lambda(sg)$. Since $\{A_s\}$ is α-invariant, it follows that the set B of Proposition 2.2 is a non-empty compact convex G-invariant set, and amenability of G implies the existence of $\lambda \in B$ such that $T^*(g)\lambda = \lambda$ for all g. But then it is clear that λ is an α-invariant section in $\{A_s\}$.

COROLLARY 2.3. *If G is a countable discrete group acting freely on (S, μ) and the ergodic equivalence relation defined by the action is hyperfinite* [5], *then S is an amenable G-space.*

Proof. If the action is hyperfinite, the equivalence relation is given by an action of the integers. Cocycles of the G action will then correspond to cocycles of the Z-action: if $\alpha: S \times G \to \mathrm{Iso}(E)$, let $\beta: S \times Z \to \mathrm{Iso}(E)$ be $\beta(s, n) = \alpha(s, g(s, n))$ where $g(s, n) \in G$ is such that $s \cdot g(s, n) = s \cdot n$. Then an α-invariant section, which exists by Theorem 2.1, will also be β-invariant.

This corollary serves to "explain" an example of A. Connes showing that a free hyperfinite action need not be the action of an amenable group. It is the action that must be amenable. We remark that the fact that for a finite type free hyperfinite action the group must be amenable [5, Proposition I.4.5] then follows from Proposition 4.4 below.

We recall that if (X, μ) and (Y, ν) are ergodic G-spaces, X is called an extension of Y if there is a Borel map $p: X \to Y$ such that $p_*\mu$ is in the same measure class as ν, and the induced map of Boolean algebras $p^*: B(Y, \nu) \to B(X, \mu)$ is a G-map. The following result generalizes Theorem 2.1 and reduces to the latter when Y is a point.

THEOREM 2.4. *If X is an extension of Y and Y is an amenable G-space, then X is an amenable G-space.*

Proof. We can assume $\nu = p_*\mu$ and by the the ergodicity assumption, we have, modulo null sets, $X = Y \times I$, where I is the unit interval, $p: X \to Y$ is projection, and $\mu = \nu \times m$ where m is a probability measure on I. Using [10] and [16, Proposition 2.1], we can reduce to the following situation. (X, μ) and (Y, ν) are ergodic G-spaces, $X \subset Y \times I$ is conull and Borel, $\mu = \nu \times m$, and the projection $p: X \to Y$ is a G-map. Denote the projection $X \to I$ by p_2. Let E be a separable Banach space, $\alpha: X \times G \to \mathrm{Iso}(E)$, and $\{A_x\}$ an α-invariant field in $E_1{}^*$. Let $F = L^1(I, m, E)$, so that F is also a separable Banach space, with dual $L^\infty(I, m, E^*)$. Define $\beta: Y \times G \to L_1(F)$, the unit ball in the bounded linear operators of F by

$$(\beta(y, g)f)(t) = r(t, y, g)\, \alpha(t, y, g) f(p_2((t, y)g)),$$

where r is the Radon–Nikodym cocycle of $X \times G$. As in the proof of Theorem 2.1 we see that β is Borel. Let $\theta(y, g): I \to I$ be $\theta(y, g)(t) = p_2((t, y)g)$, so that $\theta(y, g)$ is defined m-almost everywhere for almost all y. Then it is straightforward to check that $\theta(y, g)^{-1} = \theta(yg, g^{-1})$. Now $d(\nu \times m)((t, y)g) = r(t, y, g)$ $d(\nu \times m)(t, y)$, so that for each g and almost all y, we have $dm(\theta(y, g)t) = r(t, y, g)\, dm(t)$. We now claim that for each g, $\beta(y, g) \in \mathrm{Iso}(F)$ for almost all y. We have

$$\| \beta(y, g) f \| = \int_I r(t, y, g)\, \| f(\theta(y, g)t) \|\, dm(t),$$

and replacing t by $\theta(y, g)^{-1}t$, we obtain

$$\int_I r(\theta(yg, g^{-1})t, y, g)\, \| f(t) \|\, r(t, yg, g^{-1})\, dm(t)$$

which for each g and almost all y, $= \int_I \| f(t) \|\, dm(t)$ by the cocycle identity for r. Furthermore, for each g and almost all y, $\beta(y, g)$ is invertible and hence is in $\mathrm{Iso}(F)$. Changing β on a suitable conull set, we can then assume that $\beta: Y \times G \to$

Iso(F) is Borel, and one readily checks that β is a cocycle. For $y \in Y$, let $B_y \subset F_1{}^*$ be defined by $B_y = \{\varphi: I \to E^* \mid \varphi(t) \in A_{(t,y)}$ for almost all $t\}$. By Proposition 2.2., B_y is a compact convex set. Suppose we knew that $\{B_y\}$ is a Borel field. (We shall show this momentarily.) The α-invariance of $\{A_x\}$ is readily seen to imply the β-invariance of B_y. Amenability of Y then implies the existence of a β-invariant Borel section $\lambda: Y \to F_1{}^*$ such that $\lambda(y) \in B_y$ a.e. Let $\tilde{\lambda}(t, y) = \lambda(y)(t)$. It is straightforward to check that $\tilde{\lambda}$ is an α-invariant measurable section $X \to E_1{}^*$ proving the amenability of X. It remains only to show that $\{B_y\}$ is Borel.

Let $K = \{f: I \to I \mid f = \sum q_i \chi_{A_i}\}$, where q_i are rational and $\{A_i\}$ is a finite partition of I into intervals with rational endpoints. Thus K is countable. The fact that B_y is Borel follows from Lemma 1.7 and the following lemma, whose routine proof we omit.

LEMMA 2.5. *If $a_i : I \to E_1{}^*$ are Borel and $A_t = \overline{\{a_i(t)\}_i}$ is convex, then functions of the form $\sum_{i=1}^n f_i(t)\, a_i(t)$, where $f_i \in K$ and $\sum f_i(t) = 1$, are dense in $B = \{\lambda: I \to E_1{}^* \mid \lambda(t) \in A_t$ a.e.$\}$. We recall that $E_1{}^*$ and $B \subset L^1(I, E)^*$ are given the weak-$*$-topology.*

For a certain type of extension, a converse to Theorem 2.4 is available. If $\alpha: S \times G \to K$ is a cocycle, we can form the skew product action of G on $S \times_\alpha K/H$, where $H \subset K$ is a closed subgroup. (See [16] or the beginning of Section 3 below.)

PROPOSITION 2.6. *If K is compact and $S \times_\alpha K/H$ is ergodic and an amenable G-space, then S is an amenable G-space.*

Proof. Let $\gamma: S \times G \to$ Iso(E) be a cocycle, $\{A_s\}$ a γ-invariant Borel field in $E_1{}^*$. Define $\beta(s, x, g) = \gamma(s, g)$, so that $\beta: S \times_\alpha K/H \times G \to$ Iso(E) is a cocycle. Then $A_{(s,x)} = A_s$ defines a β-invariant field and amenability of $S \times_\alpha K/H$ implies the existence of a β-invariant section $\varphi: S \times K/H \to E_1{}^*$ in $\{A_{(s,x)}\}$. Let $\psi(s) = \int \varphi(s, x)\, dx$. Then ψ is Borel, and since for almost all s, $\varphi(s, x) \in A_s$ for almost all x, it follows that $\psi(s) \in A_s$ a.e. Finally, for almost all s,

$$\begin{aligned}\gamma^*(s, g)\, \psi(sg) &= \beta^*(s, x, g) \int \varphi(sg, x)\, dx \\ &= \beta^*(s, x, g) \int \varphi(sg, x\beta(s, g))\, dx \\ &= \int \varphi(s, x)\, dx = \psi(s).\end{aligned}$$

3. Amenability and the Range of a Cocycle

In this section we present the applications, mentioned in the introduction, of amenability to the question of ergodicity of skew products and more generally to the range of a cocycle. If S is an ergodic G-space, and $\alpha: S \times G \to H$ is a cocycle, there is an induced action of G on $S \times H$ defined by $(s, h)g = (sg, h\alpha(s, g))$. (We note that if α is not a strict cocycle then this only defines a near action; however there is an action which agrees with it a.e. for each g [16, Proposition 3.2]). We often denote $S \times H$ by $S \times_\alpha H$ when it is endowed with this action. For a given $S \times G$, it is natural to ask for which groups H is there a cocycle α for which $S \times_\alpha H$ is ergodic. (See [11] for example.) Some positive results are found in [15], [18], [19].

THEOREM 3.1. *Suppose S is an amenable G-space, $\alpha: S \times G \to H$ is a cocycle, and $S \times_\alpha H$ is ergodic. Then H is an amenable group.*

Proof. Let E be a separable Banach space, $\pi: H \to \mathrm{Iso}(E)$ a continuous representation, and $A \subset E_1^*$ compact, convex, and H-invariant. It suffices to see that A contains a fixed point (Proposition 1.5). Let $\beta = \pi \circ \alpha$, so that $\beta: S \times G \to \mathrm{Iso}(E)$ is a Borel cocycle. By amenability of S, there is a β-invariant Borel section $\varphi: S \to A$, that is for each g, $\beta^*(s, g)\,\varphi(sg) = \varphi(s)$ a.e. Define $\theta: S \times H \to A$ by $\theta(s, h) = \pi^*(h)\,\varphi(s)$. (Recall that $\pi^*(h) = (\pi(h)^{-1})^*$.) Then θ is Borel and for each g, h and almost all s,

$$\theta(sg, h\alpha(s, g)) = \pi^*(h)\,\pi^*(\alpha(s, g))\,\varphi(sg) = \pi^*(h)\,\varphi(s) = \theta(s, h).$$

By the ergodicity of $S \times_\alpha H$, $\theta(s, h)$ is essentially constant, so on a conull set we have $\pi^*(h)\,\varphi(s) = \pi^*(k)\,\varphi(t)$. In particular, for at least one h, $\pi^*(h)\,\varphi(s) = \pi^*(h)\,\varphi(t)$ for s, t in a conull set, so $\varphi(s) = \varphi(t) = a$. But then $\pi^*(h)a = \pi^*(k)a$ for h, k in a conull set. Thus, the set of points in H that leaves a invariant is a conull subgroup of H, and hence must be H itself. This completes the proof.

Theorem 3.1 implies a well-known result about random walks, and can in fact be viewed as a generalization of this fact. If ν is a probability measure on G, ν defines, for each $g \in G$, a G-valued stochastic process, X_n, the random walk starting at g at time 0, with independent increments all having distribution ν [6]. We will call the random walk recurrent if for each $A \subset G$ with positive Haar measure $P(X_n \in A$ for infinitely many $n \mid X_0 = g) = 1$ for (Haar) almost all g.

COROLLARY 3.2. *If G has a recurrent random walk, then G is amenable.*

Proof. The Haar measure dg is an invariant measure for the random walk and one can form the space of (2-sided) sample sequences of this process with "initial distribution" dg. The recurrence assumption implies that the shift on the sample sequences is ergodic [9]. However, [18] shows that this space is isomorphic to a special type of skew product. Namely, if we let $S = \prod_{-\infty}^{\infty} (G, \nu)$ with the shift transformation and $\alpha: S \times Z \to G$ the cocycle with $\alpha(s, 1)$ equal to evaluation of the zero coordinate, then the space of sample sequences is isomorphic to $S \times_\alpha G$. Ergodicity then implies that G must be amenable.

The reader is referred to Section 5 for further applications to random walks.

We now wish to examine the situation in which $S \times_\alpha H$ is not ergodic. There is an action of H on $S \times_\alpha H$ defined by $(s, h_1)h_2 = (s, h_2^{-1}h_1)$. This commutes with the near action of G, so that the induced Boolean actions of G and H on $B(S \times_\alpha H)$ commute. Let $B_0 \subset B(S \times_\alpha H)$ be the σ-algebra of G-invariant sets modulo 0. Then H acts irreducibly on B_0, and by Mackey's point realization theorem [10], there is an essentially unique ergodic H-space X such that B_0 and $B(X)$ are isomorphic as Boolean H-spaces. The H-space X is called the range of the cocycle α. (See [11] for a discussion.) When $S \times_\alpha H$ is an ergodic G-space, X reduces to a point. The following is a generalization of Theorem 3.1.

THEOREM 3.3. *If S is an amenable G-space and $\alpha: S \times G \to H$ is a cocycle, then the range of α is an amenable H-space.*

We remark that for $G = Z$, the group of integers, any function $f: S \to H$ defines a cocycle with $\alpha(s, 1) = f(s)$. Hence, Theorem 3.3 provides us with an abundance of amenable actions of non-amenable groups. The proof of Theorem 3.3 is similar in spirit to that of Theorem 3.1 but the technical difficulties are significantly greater. To facilitate the discussion, we shall preface the proof by collecting some notions and facts from [13]. In [13] Ramsay works in a more general framework than we shall require, and we quote his results specified to our situation.

If $S_0 \subset S$ is a conull Borel subset of the ergodic G-space S, let $S_0 * G = \{(s, g) \mid s \in S_0, sg \in S_0\}$. Then $S_0 * G$ is called the inessential contraction of $S \times G$ based on S_0. We note that for each g, $\{s \mid (s, g) \in S_0 * G\}$ is conull in S. If $\alpha: S \times G \to M$ is a cocycle with values in the standard Borel group M, α is called strict on $S_0 * G$ if $(s, g), (sg, h) \in S_0 * G$ implies $\alpha(s, gh) = \alpha(s, g)\, \alpha(sg, h)$.

LEMMA 3.4. *If $\alpha: S \times G \to M$ is a cocycle, there is a cocycle $\beta: S \times G \to M$ such that for each g, $\beta(s, g) = \alpha(s, g)$ a.e., and β is strict on some inessential contraction $S_0 * G$.*

Proof. By [13, Theorem 5.1] there is a cocycle β such that β is strict on an inessential contraction and $\beta(s, g) = \alpha(s, g)$ a.e. Arguing as in the proof of [16, Lemma 3.6], we see that for each g, the equality holds for almost all s.

If $S_0 * G$ is an inessential contraction of S and $X_0 * H$ is an inessential contraction of the H-space X, by a strict homomorphism from $S_0 * G$ to $X_0 * H$ we mean a pair of Borel maps (q, β) where $\beta: S_0 * G \to H$ is a strict cocycle, $q: S_0 \to X_0$, $q(s)\, \beta(s, g) = q(sg)$ for all $(s, g) \in S_0 * G$, and $(q(s), \beta(s, g)) \in X_0 * H$ for all $(s, g) \in S_0 * G$. We further require that $E \subset X_0$ negligible implies $q^{-1}(E)$ is null. (We recall that E is negligible if the saturation $E \cdot H$ is null. For countable H, E negligible and E null are equivalent, but this of course is far from true for general H.) We will write (q, β) for the map $S_0 \times G \to X_0 * H$ defined by $(q, \beta)(s, g) = (q(s), \beta(s, g))$.

LEMMA 3.5. *Suppose $\alpha: S \times G \to H$ is a cocycle and let X be the range of α, so that X is an H-space. Then there is a conull set $S_0 \subset S$, a measure class preserving H-equivariant function $p: S_0 \times H \to X$, and a cocycle $\tilde{\alpha}: S \times G \to H$ such that for all g, $\tilde{\alpha}(s, g) = \alpha(s, g)$ a.e., and $(\tilde{p}, \tilde{\alpha}): S_0 * G \to X \times H$ is a strict homomorphism where $\tilde{p}(s) = p(s, e)$.*

Proof. This follows from the proof of [13, Theorem 7.8]. (The statement of [13, Theorem 7.8] is actually somewhat weaker than Lemma 3.5, but the proof in fact shows the stronger statement. The point is that the "similarity" condition in the statement of [13, Theorem 7.8] is actually shown by proving equality a.e.)

If $(\tilde{p}, \tilde{\alpha}): S_0 * G \to X \times H$ is a strict homomorphism and $\theta: S_0 \to H$ is Borel, then $(q, \beta): S_0 * G \to X \times H$ will also be a strict homomorphism where $q(s) = \tilde{p}(s)\, \theta(s)^{-1}$ and $\beta(s, g) = \theta(s)\, \tilde{\alpha}(s, g)\, \theta(sg)^{-1}$.

LEMMA 3.6. *Let α be a cocycle and choose S_0, p, and $\tilde{\alpha}$ as in Lemma 3.5. Suppose $X_0 \subset X$ is conull Borel. Then there is a conull Borel set $S_1 \subset S_0$ and a Borel function $\theta: S_1 \to H$ such that $(q, \beta)(S_1 * G) \subset X_0 * H$.*

Proof. This is just [13, Lemma 6.6]. However, as we shall need to examine the proof with some care at a later point, we present Ramsay's proof in our notation and framework. We let μ be the measure on S and ν the measure on X. Let $Z = \{(s, h) \in X \times H \mid x \in X_0 \cdot h\}$. Then Z is Borel and since $\tilde{p}(S_0)$ is not negligible, the projection of Z onto X is $\nu + \tilde{p}_*(\mu)$ conull. By the von Neumann selection theorem, we can choose a Borel function $\lambda: X \to H$ such that $(x, \lambda(x)) \in Z$ for $(\nu + \tilde{p}_*(\mu))$-almost all x. Let $\theta(s) = \lambda(\tilde{p}(s))$. Then there is a conull Borel set $S_1 \subset S_0$ such that $\tilde{p}(s)\, \theta(s)^{-1} \in X_0$ for $s \in S_1$. Furthermore, if $s, sg \in S_1$,

$$\tilde{p}(s)\, \theta(s)^{-1}(\theta(s)\, \tilde{\alpha}(s, g)\, \theta(sg)^{-1}) = \tilde{p}(sg)\, \theta(sg)^{-1} \in X_0$$

and so $(q, \beta)(S_1 * G) \subset X_0 * H$.

One further lemma we shall use is [13, Lemma 5.2].

LEMMA 3.7. *If $F \subset S \times G$ contains a conull Borel set, and $(s, g_1), (sg_1, g_2) \in F$ implies $(s, g_1 g_2) \in F$, then F contains an inessential contraction of $S \times G$.*

We are now ready to turn to the proof of Theorem 3.3.

Proof of Theorem 3.3. Let E be a separable Banach space, $\gamma: X \times H \to \mathrm{Iso}(E)$ a Borel cocycle and $\{A_x\}$ a γ-invariant Borel field of compact convex sets in $E_1{}^*$. By Lemma 3.4, we can assume γ is strict on some inessential contraction $X_1 * H$. For each h we have $\gamma^*(x, h) A_{xh} = A_x$ a.e. We claim that this identity holds on an inessential contraction of X. By Lemma 1.7, we can choose Borel functions $a_i : X \to E_1{}^*$ such that $\overline{\{a_i(x)\}} = A_x$ for $x \in X_2 \subset X_1$, a conull Borel set. Then if $x, xh \in X_2$, we have $\gamma^*(x, h)A_{xh} = A_x$ if and only if $\gamma^*(x, h) a_i(xh) \in A_x$ for all i and $\gamma^*(x, h)^{-1} a_i(x) \in A_{xh}$ for all i. For each i, the map $X_2 * H \to X \times E_1^* \times H$, $(x, h) \to (x, \gamma^*(x, h) a_i(xh), h)$ is Borel and injective so the image is Borel. Thus, the projection onto $X \times H$ of this image intersected with $\{(x, a, h) \in X \times E_1^* \times H \mid a \in A_x\}$ is a conull analytic subset of $X \times H$. But this set is precisely $\{(x, h) \in X_2 * H \mid \gamma^*(x, h) a_i(xh) \in A_x\}$. Hence, for each i there is a conull Borel set in $X \times H$ so that $\gamma^*(x, h) a_i(xh) \in A_x$ on this set. A similar argument shows that on a conull Borel set $\gamma^*(x, h)^{-1} a_i(x) \in A_{xh}$, and taking the intersection over all i, we see that $F = \{(x, h) \in X_2 * H \mid \gamma^*(x, h)A_{xh} = A_x\}$ contains a conull Borel set. Now if $(x, h) \in F$ and $(xh, k) \in F$, then

$$\gamma^*(x, hk)A_{xhk} = \gamma^*(x, h)\, \gamma^*(xh, k)A_{(xh)k} = \gamma^*(x, h)A_{xh} = A_x .$$

(Recall γ is strict on $X_2 * H$.) Hence by Lemma 3.7, there is an inessential contraction $X_3 * H \subset F$. For x in a conull Borel subset of X_3, $xh \in X_3$ for almost all h. Call this set X_4. Then again, for x in a conull Borel subset of X_4, $xk \in X_4$ for almost all $k \in H$. Call this set X_0. Then if $x \in X_0$, we have $x, xh, xhk \in X_3$ for almost all $(h, k) \in H \times H$.

Recall that we have the cocycle α: $S \times G \to H$. Then there is a conull Borel set $S_1 \subset S$ and a strict homomorphism (q, β): $S_1 * G \to X_0 * H$ satisfying the conditions of Lemma 3.6. We let p, $\tilde{\alpha}$, and θ be defined as in that lemma. Let $\delta = \gamma \circ (q, \beta)$, so that δ: $S_1 * G \to \mathrm{Iso}(E)$. Then δ is a cocycle which is strict on $S_1 * G$. Let $A_s = A_{q(s)}$. Then $\{A_s\}$ is a Borel field of compact convex sets and

$$\begin{aligned}\delta^*(s, g)A_{sg} &= \gamma^*(q(s), \beta(s, g))A_{q(sg)}\\ &= \gamma^*(q(s), \beta(s, g))A_{q(s)\beta(s,g)}\\ &= A_{q(s)} .\end{aligned}$$

That is, $\{A_s\}$ is a δ-invariant field. Hence there is a δ-invariant section φ: $S \to E_1^*$. Now $K = \{(s, g) \in S_1 * G \mid \delta^*(s, g)\, \varphi(sg) = \varphi(s)\}$ is Borel conull, and if (s, g_1), $(sg_1, g_2) \in K$, then $(s, g_1 g_2) \in K$. It follows by Lemma 3.7 that K contains an inessential contraction and hence, by relabelling, we may assume $\delta^*(s, g)\, \varphi(sg) = \varphi(s)$ for all $(s, g) \in S_1 * G$.

Now define ψ: $S_1 \times H \to E_1^*$ by $\psi(s, h) = \gamma^*(q(s), h^{-1})^{-1} \varphi(s)$. Suppose $(s, g) \in S_1 * G$. Then for almost all h,

$$\begin{aligned}\psi(sg, h\beta(s, g)) &= \gamma^*(q(sg), \beta(s, g)^{-1}h^{-1})^{-1}\, \varphi(sg)\\ &= \gamma^*(q(s), h^{-1})^{-1}\, \gamma^*(q(sg)\, \beta(s, g), \beta(s, g)^{-1})^{-1}\, \varphi(sg)\\ &= \gamma^*(q(s), h^{-1})^{-1}\, \gamma^*(q(s), \beta(s, g))\, \varphi(sg)\\ &= \gamma^*(q(s), h^{-1})^{-1}\, \varphi(s) = \psi(s, h).\end{aligned}$$

Now let $\omega(s, h) = \psi(s, h\theta(s)^{-1})$. Then for $(s, g) \in S_1 * G$, and almost all h,

$$\begin{aligned}\omega(sg, h\tilde{\alpha}(s, g)) &= \psi(sg, h\tilde{\alpha}(s, g)\, \theta(sg)^{-1}) \\ &= \psi(sg, h\theta(s)^{-1}\, \beta(s, g)) \\ &= \psi(s, h\theta(s)^{-1}) = \omega(s, h).\end{aligned}$$

In other words, ω is essentially G-invariant on $S \times_\alpha H$, and hence there is a map $\sigma: X \to E_1{}^*$ such that $\sigma(p(s, h)) = \omega(s, h)$ a.e.

We now show that σ is a γ-invariant section. We claim first that $\gamma^*(x, h)\, \sigma(xh) = \sigma(x)$ a.e. Now for $s \in S_1$, $q(s)h^{-1} = \tilde{p}(s)\, \theta(s)^{-1}h^{-1} = p(s, h\theta(s))$. Hence the map $(s, h) \to q(s)h^{-1}$ is measure class preserving. Thus, it suffices to show that

$$\gamma^*(q(s)k^{-1}, h)\, \sigma(q(s)k^{-1}h) = \sigma(q(s)k^{-1}) \quad \text{a.e.}$$

This is equivalent to the following almost everywhere equalities:

$$\begin{aligned}\gamma^*(q(s)k^{-1}, h)\, \sigma(\tilde{p}(s)\, \theta(s)^{-1}k^{-1}h^{-1}) &= \sigma(\tilde{p}(s)\, \theta(s)^{-1}k^{-1}) \\ \gamma^*(q(s)k^{-1}, h)\, \sigma(p(s, hk\theta(s))) &= \sigma(p(s, k\theta(s))) \\ \gamma^*(q(s)k^{-1}, h)\, \omega(s, hk\theta(s)) &= \omega(s, k\theta(s)) \\ \gamma^*(q(s)k^{-1}, h)\, \psi(s, hk) &= \psi(s, k) \\ \gamma^*(q(s)k^{-1}, h)\, \gamma^*(q(s), k^{-1}h^{-1})^{-1}\, \varphi(s) &= \gamma^*(q(s), k^{-1})^{-1}\, \varphi(s) \\ \gamma^*(q(s), k^{-1})\, \gamma^*(q(s)k^{-1}, h)\, \gamma^*(q(s), k^{-1}h^{-1})^{-1}\, \varphi(s) &= \varphi(s).\end{aligned}$$

Since $q(s) \in X_0$ for $s \in S_1$, the cocycle identity will hold for almost all (s, k, h), and the equation becomes $\varphi(s) = \varphi(s)$. Thus, we have shown $\gamma^*(x, h)\, \sigma(xh) = \sigma(x)$ for almost all (x, h). Now consider

$$H_0 = \{h \mid \gamma^*(x, h)\, \sigma(xh) = \sigma(x) \text{ for almost all } x\}.$$

Then H_0 is conull and one readily checks that it is closed under multiplication. Hence, $H_0 = H$, and we conclude that σ is a γ-invariant section.

To show that X is amenable, it remains only to show that $\sigma(x) \in A_x$ a.e., and for this it suffices to show that $\sigma(p(s, h)) \in A_{p(s,h)}$ a.e. But

$$\sigma(p(s, h)) = \psi(s, h\theta(s)^{-1}) = \gamma^*(q(s), \theta(s)h^{-1})^{-1}\, \varphi(s).$$

Now for almost all s, $\varphi(s) \in A_{q(s)}$, and for almost all (s, h),

$$\gamma^*(q(s), \theta(s)\, h^{-1})^{-1}\, \varphi(s) \in A_{q(s)\theta(s)h^{-1}} = A_{\tilde{p}(s)h^{-1}} = A_{p(s,h)}\,.$$

This completes the proof.

We shall see in Section 4 that this theorem implies that if the range has finite invariant measure and S is amenable, then H must be amenable.

4. Amenable Pairs

In [4], Eymard introduced the notion of the conditional fixed point property for a pair (H, G) where H is a closed subgroup of G. This is the condition that if G acts affinely and continuously in a compact convex set, and if there is a fixed point for H, then there is a fixed point (not necessarily the same point) for G. He relates this notion to the concept of amenable action introduced by Greenleaf [7] (i.e., the existence of a G-invariant mean on $UCB(G/H)$), and then generalizes many known results about amenable groups to amenable pairs.

We now introduce the concept of an amenable pair (S, G) where S is an ergodic G-space.

Definition 4.1. If S is an ergodic G-space, we call (S, G) an amenable pair if for every continuous homomorphism $\pi: G \to \text{Iso}(E)$, E a separable Banach space, and G-invariant compact convex set $A \subset E_1{}^*$, the existence of an α-invariant section $\varphi: S \to A$ (where $\alpha(s, g) = \pi(g)$) implies the existence of a G-fixed point in A.

Proposition 4.2. *If $S = G/H$, then $(G/H, G)$ is an amenable pair if and only if (H, G) has the conditional fixed point property of Eymard described above* [4, p. 11].

Proof. (i) Suppose (H, G) has the conditional fixed point property. If $\pi: G \to \text{Iso}(E)$ and φ is an α-invariant section, it suffices to show there is an H-fixed point. For each g and almost all k, $\pi(g)\, \varphi([k]g) = \varphi([k])$, or $\pi(kg)\, \varphi([k]g) = \pi(k)\, \varphi([k])$. It follows that for almost all k, $\pi(k)\, \varphi([k]) = a$, for some $a \in A$. If $h \in H$, we have $\pi(hk)\, \varphi([k]) = a$ a.e., which clearly implies $\pi(h)a = a$ for all h.

(ii) Conversely, suppose $(G/H, G)$ is amenable. Let E be a separable Banach space, $\pi: G \to \text{Iso}(E)$, and $a \in A \subset E_1{}^*$ an H-fixed point. There is a strict cocycle $\beta: G/H \times G \to \text{Iso}(E)$ such that β is equivalent to α, the equivalence is implemented by elements of $\pi(G)$, and $\beta(G/H \times G) \subset \pi(H)$ [14, 8.23–8.27]. It follows that $\varphi([k]) = a$ is a β-invariant section, and hence there exists an α-invariant section in A. Therefore G has a fixed point in A. The case of E nonseparable follows by an argument similar to Proposition 1.5.

Proposition 4.3. (i) *If G is amenable, (S, G) is an amenable pair for every G-space S.*

(ii) *If S is an amenable G-space and (S, G) is an amenable pair, then G is amenable.*

When S is a transitive G-space, amenability of (S, G) is equivalent to the existence of a G-invariant mean on $L^\infty(S)$ [4, p. 28] which is in turn equivalent to the existence of a G-invariant mean on $CB(S)$, the continuous bounded functions on S [7, Theorem 3.3]. When S is not transitive, the latter equivalence is no longer implied.

PROPOSITION 4.4. *If there is a G-invariant mean on $L^\infty(S)$ (in particular if S has a finite G-invariant measure), then (S, G) is an amenable pair.*

Proof. Let $\pi: G \to \mathrm{Iso}(E)$ be a continuous representation, $\alpha(s, g) = \pi(g)$, and $A \subset E_1{}^*$ a G-invariant compact convex set. Suppose $\varphi: S \to A$ is an α-invariant section. Then one can form, as in [4, p. 8], $\int_S \varphi(s)\, dm(s) = a \in A$, where m is a G-invariant mean on $L^\infty(S)$. Since $\pi(g)$ acts affinely on $E_1{}^*$, we have by [4, p. 9], $\pi(g)a = \int_S \pi(g)\, \varphi(s)\, dm(s) = \int \pi(g)\, \varphi(sg)\, dm$ by G-invariance, $= \int \varphi(s)\, dm = a$.

COROLLARY 4.5. *If $\alpha: S \times G \to H$ is a cocycle and S is an amenable G-space, then the existence of an H-invariant mean on $L^\infty(X)$, where X is the range of α, implies that H is amenable.*

Proof. Theorem 3.3 and Proposition 4.4.

A partial converse to Proposition 4.4 is provided by the following.

PROPOSITION 4.6. *Suppose S is a compact metric space, G acts continuously on S and μ is quasi-invariant and ergodic. If (S, G) is an amenable pair, then there is a G-invariant probability measure on S.*

Proof. Let $E = C(S)$, $A \subset E_1{}^*$ the set of probability measures and $\pi(g)$ translation by g in E. For each $s \in S$, let $\varphi(s)$ be the Dirac measure at s. Then φ is Borel and clearly $\pi^*(g)\, \varphi(sg) = \varphi(s)$. Hence φ is an α-invariant section, and so there is a G-fixed point in A.

EXAMPLE 4.7. Suppose $\theta: G \to H$ is a continuous homomorphism with dense range. Then H is an amenable group if and only if (H, G) is an amenable pair, where H is the G-space defined by θ.

Proof. If H is amenable, there is a G-invariant mean on $L^\infty(H)$, so (H, G) is amenable by Proposition 4.4. Conversely, suppose (H, G) is an amenable pair; let $\pi: H \to \mathrm{Iso}(E)$ and $A \subset E_1{}^*$ compact, convex, and H-invariant. Since $\theta(G)$ is dense in H, it suffices to show there is a point in A fixed by G. Let $\alpha(h, g) = \pi(\theta(g))$. Choose $a \in A$ and let $\varphi(h) = \pi^*(h)^{-1}a$. Then $\alpha(h, g)\, \varphi(h\theta(g)) = \varphi(h)$, so that φ is an α-invariant section. By the amenability of (H, G), there is a point left fixed by G, completing the proof.

COROLLARY 4.8. *If $\theta: G \to H$ has dense range and H is an amenable group and an amenable G-space, then G is amenable.*

We remark that dense embeddings of groups are the "normal virtual subgroups" [17], and Corollary 4.8 can be viewed as the virtual group analogue of the statement that if an amenable normal subgroup has an amenable quotient, then the entire group is amenable. This last statement is of course also a consequence of Corollary 4.8.

We remark that one can also define amenable pairs of G spaces (X, Y), where X is an extension of Y, by a conditional invariant section property. We shall not pursue this here however.

5. An Application to Poisson Boundaries

For any random walk on a group G, Furstenberg has shown how to embed G in a larger space $G \cup P$ in such a way that almost every path of the random walk in G converges to a unique point of P, and every bounded harmonic function on G has an essentially unique representation as a certain type of integral over P. P is called the Poisson boundary and can be used as well for a study of the harmonic functions on homogeneous spaces of G. In a special case, the theory reduces to the classical theory of the Poisson integral representation for harmonic functions in the unit disk. In this section, we present a different method for constructing the Poisson boundary, at least when the law of the walk is étalée, showing it can be obtained by a slight modification of the range construction for a cocycle. A small modification of the proof of Theorem 3.3 will then enable us to show that the Poisson boundary, with a certain naturally defined measure class, is an ergodic amenable G-space.

Let μ be any probability measure on G. Let $\Omega = \prod_{-\infty}^{\infty} G$, $\Omega_0 = \prod_0^{\infty} G$, and $r\colon \Omega \to \Omega_0$ the projection. Give Ω and Ω_0 the product measures $\eta = \prod_{-\infty}^{\infty} \mu$ and $\eta_0 = \prod_0^{\infty} \mu$. Let $T\colon \Omega \to \Omega$ be the shift, $(T\omega)(i) = \omega(i+1)$, and T_0 the shift on Ω_0 defined by the same formula. Define $\alpha\colon \Omega_0 \to G$ by $\alpha(\omega) = \omega(0)$, which we also consider as a function defined on Ω. Then, as we indicated in Corollary 3.2 (see [18] for details), if we let $X = \Omega \times_\alpha G$ with the product measure $\eta \times dg$, then the skew product transformation $\tilde{T}$ is isomorphic to the shift on the space of sample sequences of the (2-sided) random walk. If we let A be the range of α, then Borel functions on A will correspond to invariant functions on the sample sequence space of the 2-sided walk. Harmonic functions on G are in correspondence with the invariant functions on the 1-sided random walk, and so we proceed to construct the range of the 1-sided shift. Thus, if we let $B_0 \subset B(\Omega_0 \times G)$ be the Boolean subalgebra of $\tilde{T}_0$-invariant elements (where $\tilde{T}_0(\omega, g) = (T_0\omega, g\alpha(\omega))$, then B_0 becomes a Boolean G-space, and hence is isomorphic to $B(P, m)$ where (P, m) is an ergodic G-space [10]. By discarding a T_0-invariant Borel null set in Ω_0, we will have a G-map $p\colon \Omega_0^1 \times G \to P$ which preserves measure class and induces the Boolean isomorphism $p^*(P) \cong B_0$. Now $p \circ \tilde{T}_0$ and p are both G maps from $\Omega_0^1 \times G \to P$ which induce the same map of Boolean algebras. Hence, they agree on a conull G-invariant Borel set in $\Omega_0^1 \times G$, which we will continue to denote $\Omega_0^1 \times G$. Thus, we now have that p is $\tilde{T}_0$-invariant and G-equivariant. Let ν be the measure on P defined by $\nu = (p \circ i)_* (\eta_0)$ where $i: \Omega_0^1 \to \Omega_0^1 \times G$ is given by $i(\omega) = (\omega, e)$. We note that the measure class of m can be recovered from ν by $m = \int_G (\nu \cdot g)\, d\xi(g)$ where ξ is any probability measure in the class of Haar measure. We recall that a measure μ on G is called étalée if the convolution power μ^n has a non-singular component with respect to Haar measure, for some n.

THEOREM 5.1. *(P, ν) is a boundary of (G, μ)* [6, *Definition* 3.4] *and if μ is étalée, (P, ν) is the Poisson boundary* [6, *Theorem* 3.1].

Thus, this theorem says that the Poisson boundary is the "range" of a naturally defined cocycle of a semi-group action (of the non-negative integers).

Proof. First, we construct a P-valued μ-process in the sense of [6, p. 16]. Define $z_k : \Omega_0^1 \to P$ by $z_k(\omega) = p(T_0^k\omega, e)$, where $\omega = (x_0, x_1, \ldots)$. It is clear that z_k is a function of x_j, $j \geqslant k$, and since T_0^k preserves measure on Ω_0, z_k all have the same distribution. By invariance of p, and the fact that it is a G-map, we have

$$z_{k+1}x_k^{-1} = p(T_0^{k+1}\omega, x_k) = p(T_0^k\omega, e) = z_k ,$$

verifying that z_k is a μ-process. (The difference between our relation $z_{k+1}x_k^{-1} = z_k$ and Furstenberg's condition (d) on [6, p. 16] is only a matter of converting from a right to a left action on P.) Since ν is the common distribution of z_k, it follows that (P, ν) is a boundary [6, Definition 3.4].

For any bounded measurable function φ on P, there corresponds a harmonic function h_φ on G defined by $h_\varphi(g) = \int_P \varphi(gz)\, d\nu(z)$ (where, to conform to Furstenberg's notation, we write the G-action on the left: $gz = zg^{-1}$.) Equivalently, $h_\varphi(g) = \int_{\Omega_0^1} (\varphi \circ p)(\omega, g)\, d\omega$. To see that (P, ν) is the Poisson boundary it suffices to see that every bounded harmonic function can be so represented [6, Theorem 3.1]. But for any bounded harmonic h, there is a bounded function H on the sample sequence space of the 1-sided walk, invariant under the 1-sided shift, for which $h(g) = E_g(H)$ [12, Proposition I.4.2]. Now the proof of [18, Theorem 3] shows that the map

$$\Phi: \Omega_0 \times G \to \Omega_0 , \quad (\Phi(\omega, g))(n) = g\omega(0) \cdots \omega(n-1)$$

intertwines $\tilde{T}_0$ and the shift on Ω_0, and that $\Phi(\eta_0 \times \xi) = P_\xi$ for any probability measure ξ on G, where P_ξ is the probability measure for the random walk with law μ and initial distribution ξ. Thus $H \circ \Phi$ is invariant on $\Omega_0 \times_\alpha G$ and

$$h(g) = E_g(H) = \int_{\Omega_0 \times G} (H \circ \Phi)\, d(\eta_0 \times \delta_g) = \int_{\Omega_0} (H \circ \Phi)(\omega, g)\, d\omega.$$

Since $H \circ \Phi$ is invariant, there is a function $\varphi\colon P \to \mathbb{C}$ such that $H \circ \Phi = \varphi \circ P$ on a conull set. Thus, for almost all g, $h(g) = \int (\varphi \circ p)\, d\omega$. If μ is étalée, every harmonic function is continuous [1, p. 23], and hence two harmonic functions agreeing almost everywhere must be identical.

THEOREM 5.2. *If μ is a measure on a group H, then the boundary (P, m) constructed above is an amenable ergodic H-space.*

Proof. Recall we have $\alpha\colon \Omega \to H$ and A is the range of α considered as a cocycle. We clearly have that P is a factor of A, and after discarding some H-invariant conull Borel sets, we have a commutative diagram of measure class preserving Borel H-maps:

$$\begin{array}{ccc} \Omega' \times H & \xrightarrow{\ p\ } & A \\ {\scriptstyle r \times \mathrm{id}}\downarrow & & \downarrow{\scriptstyle t} \\ r(\Omega') \times H & \xrightarrow{\ p_0\ } & P \end{array}$$

where Ω', $r(\Omega')$ are Borel, conull, and shift invariant, and p, p_0 are invariant under the shifts. Suppose $\gamma_0 : P \times H \to \mathrm{Iso}(E)$ is a cocycle and $\{A_z\}$ is a γ_0-invariant field in $E_1{}^*$. Let γ be the restriction of γ_0 to $A \times H$, i.e., $\gamma(a, h) = \gamma_0(t(a), h)$. In the proof of Theorem 3.3 a γ-invariant section σ was constructed. To prove Theorem 5.2, it suffices to show that in the proof of Theorem 3.3, σ can be chosen so that it actually factors (modulo 0) to a function on P, and in the notation of the proof of Theorem 3.3, it suffices to see that $\omega\colon \Omega' \times H \to E_1{}^*$ factors to a function on $r(\Omega') \times H$. We have

$$\begin{aligned} \omega(s, h) &= \gamma^*(\tilde{p}(s)\, \theta(s)^{-1}, \theta(s)h^{-1})^{-1}\, \varphi(s) \\ &= \gamma_0{}^*(t(\tilde{p}(s))\, \theta(s)^{-1}, \theta(s)h^{-1})^{-1}\, \varphi(s). \end{aligned}$$

Now $t(\tilde{p}(s))$ factors to $r(\Omega')$, so it suffices to see that we can choose $\varphi(s)$ and $\theta(s)$ so that they factor to $r(\Omega')$.

The map $\theta(s)$ is constructed in the proof of Lemma 3.6, where X_0 is the conull Borel set constructed in the proof of Theorem 3.3. Since the cocycle γ is the restriction of γ_0 which is defined on $P \times H$, it is straightforward to check that X_0 can be chosen to be $t^{-1}(P_0)$ for some conull Borel $P_0 \subset P$. Then the map $\lambda\colon A \to H$ in the proof of Lemma 3.6 can be chosen so as to factor to a map on P, and hence $\theta(s)$ factors to a map on $r(\Omega')$.

The map $\varphi\colon \Omega' \to E_1{}^*$ was chosen in the proof of Theorem 3.3 to be an invariant section in $\{A_s\}$ of the cocycle

$$[\gamma \circ (q, \beta)](s, n) = \gamma_0(t(\tilde{p}(s))\, \theta(s)^{-1}, \theta(s)\, \tilde{\alpha}(s, n)\, \theta(sn)^{-1}).$$

Recall that we knew such a section existed by the amenability of the Z-space Ω', which in turn followed from the fact that Z is amenable (Theorem 2.1). In the proof of that theorem, φ was chosen to be a fixed point under the induced action of Z on $L^\infty(\Omega', E^*)$. For $n \geqslant 0$, $(\gamma \circ (q, \beta))(s, n)$ factors to a function on $r(\Omega')$, and hence under the induced Z-action on $L^\infty(\Omega', E^*)$, $L^\infty(r(\Omega'), E^*) \subset L^\infty(\Omega', E^*)$ will be invariant under Z^+, the non-negative integers. Since Z^+ is an amenable semi-group, there will be a Z^+-invariant section in A_s, and this section will then also be Z-invariant. Thus, we can choose φ so that it factors to $r(\Omega')$, completing the proof.

COROLLARY 5.3. *If μ is étaleé, the Poisson boundary with its natural quasi-invariant measure, is an amenable ergodic H-space.*

In many important cases, H is transitive on the Poisson boundary.

COROLLARY 5.4. *If H is transitive on the Poisson boundary of an étalée probability measure, then the stability groups are amenable.*

Proof. Corollary 5.3 and Theorem 1.9.

6. CONCLUDING REMARKS

(a) When $E = C(X)$ for X a compact metric space, the condition of amenability ensures the existence of relatively invariant measures for certain extensions whose fiber is X. More precisely, suppose Y and $Y \times X$ are Borel G-spaces, that the projection $p: Y \times X \to Y$ is a G-map, and that X is compact metric. For each y and g, the action defines a map $\{y\} \times X \to \{yg\} \times X$ which can be identified with a map $\alpha(y, g): X \to X$. Suppose further that these maps are homeomorphisms. Suppose that ν is a quasi-invariant measure on Y such that (Y, ν) is an amenable ergodic G-space. Then there is a probability measure μ on $Y \times X$ that is relatively invariant over ν. That is, one of the two following equivalent conditions holds: (i) if $\mu = \int^{\oplus} \mu_y \, d\nu$ is a decomposition over the fibers of p, then for each g and almost all y, $\alpha(y, g)_* \mu_y = \mu_{yg}$; and (ii) if $r: Y \times G \to \mathbb{R}$ is the Radon–Nikodym cocycle of Y, then $\tilde{r}$ defined by $\tilde{r}(y, x, g) = r(y, g)$ is the Radon–Nikodym cocycle of $Y \times X$.

To see this, by amenability it suffices to show that the induced cocycle

$$\beta: Y \times G \to \mathrm{Iso}(C(X)), \quad (\beta(y, g)f)(x) = f(\alpha(y, g)x)$$

is Borel. But for this it suffices to see that if $\lambda \in M(X)$, then $(y, g) \to \int f(\alpha(y, g)x) \, d\lambda(x)$ is Borel. Since $(y, g, x) \to \alpha(y, g)x$ is Borel, this follows.

(b) One can also define the notion of amenability for a countable ergodic equivalence relation R on a space S using the cohomology of such an object [5] just as we have used the cohomology of $S \times G$. If $R = R_G$ for some amenable action of a countable discrete group G, then R will be amenable. Amenability of R implies amenability of S_G if the action of G is free, but this is not true if the freeness condition is dropped.

(c) There are many other known properties of groups related to amenability and to amenable pairs [4], [8], and one can try to extend these notions and results to our framework. In some cases this is easy, in others it seems more difficult. It would be interesting to determine just how much of the theory does extend.

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