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Nonlinear alternatives of Schauder and Krasnosel'skij types with applications to Hammerstein integral equations in L^1 spaces

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ABSTRACT

This paper is devoted to establishing new variants of some nonlinear alternatives of Leray–Schauder and Krasnosel'skij type involving the weak topology of Banach spaces. The De Blasi measure of weak noncompactness is used. An application to solving a nonlinear Hammerstein integral equation in L^1 spaces is given. Our results complement recent ones in [K. Latrach, M.A. Taoudi, A. Zeghal, Some fixed point theorems of the Schauder and the Krasnosel'skij type and application to nonlinear transport equations, J. Differential Equations 221 (2006) 256–2710] and [K. Latrach, M.A. Taoudi, Existence results for a generalized nonlinear Hammerstein equation on L_1 spaces, Nonlinear Anal. 66 (2007) 2325–2333].

1. Introduction and preliminaries

Many problems arising from various areas of natural sciences, mathematical physics, mechanics and population dynamics are modeled by mathematical equations which involve the study of nonlinear equations of the form:

$$Ay + By = y, \quad y \in M \tag{1.1}$$

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where M is a closed, convex subset of a Banach space X, and A, B are two nonlinear operators. In some special cases, a useful tool to solve problem (1.1) is the celebrated fixed point theorem proved by Krasnosel'skij in 1958 (see [23,24]):

Theorem 1.1. *Let M be a nonempty closed convex subset of a Banach space X and A, B two maps from M to X such that*

(a) *A is compact and continuous,*
(b) *B is a contraction,*
(c) *$AM + BM \subset M$.*

Then $A + B$ has at least one fixed point in M.

Recall that A compact means that $A(M)$ is relatively compact in the space X. The proof of Theorem 1.1 combines the Banach contraction mapping principle both with Schauder's fixed point theorem [21] and uses the following auxiliary lemma the proof of which is direct. The Schauder fixed point theorem is then applied to the mapping $(I - B)^{-1} \circ A$ on the set M.

Lemma 1.1. *Let E be a linear vector space and $F \subset E$ a nonempty subset. If $g : F \to E$ is a contraction, then the mapping $I - g : F \to (I - g)(F)$ is a homeomorphism.*

In 1998, Burton [12] noticed that the Krasnosel'skij fixed point theorem remains valid if the condition (a) is replaced by the following less restrictive one:

$$\forall y \in M \quad (x = Ay + Bx \Longrightarrow x \in M).$$

His result applies to a problem from stability theory and covers cases where Theorem 1.1 does not work. Subsequently, the following refinement was introduced in [8]:

$$\text{If } \lambda \in (0, 1) \text{ and } u = \lambda Bu + Av \text{ for some } v \in M, \text{ then } u \in M.$$

A is assumed weakly continuous and weakly compact while B is a contraction.

However, the mappings A and B do not in general satisfy the hypothesis of Theorem 1.1 (see for instance [9,26,27]). So, in the last couple of years, much attention has been paid to this theorem and some extensions have been obtained in many directions; we quote for instance the works [6,8,9, 12,32]. In [9], new versions of the Krasnosel'skij fixed point theorem were obtained for sequentially weakly continuous mappings (i.e. operators which map weakly convergent sequences into weakly convergent sequences) (see also [33]). Since infinite dimensional Banach spaces are not locally compact, the authors suggest a locally convex topology approach which is a weak topology of Banach spaces. One of the advantages of this locally convex topology is the fact that if a set M is weakly compact, then every sequentially weakly continuous map $T : M \to X$ is weakly continuous. This is an immediate consequence of the fact that the weak sequential compactness is equivalent to the weak compactness (Eberlein–Šmulian Theorem). As a consequence, many applications to problems with lack of compactness are solved, mainly those posed in L^1 spaces.

More recently, the Krasnosel'skij fixed point theorem is proved in the framework of Fréchet spaces in [30] with application to a general equation involving the sum of two nonlinear integrals.

In [8], Barroso established another version of Theorem 1.1 using the weak topology of a Banach space. His result only requires the weak continuity and weak compactness of A while B is a linear operator satisfying the estimate $\|B^p\| \leqslant 1$ for some integer $p \geqslant 1$. Indeed, such an assumption insures the invertibility of $I - B$ and the weak continuity of $(I - B)^{-1}$. However, when B is no longer linear, the map $(I - B)^{-1}$, which plays a key role, need not be weakly continuous.

Agarwal et al. [2] established a number of fixed point theorems and nonlinear alternatives for weakly–strongly sequentially continuous weakly compact maps in angelic spaces (a Hausdorff topological space X is said to be angelic if for every relatively countably compact set $C \subset X$, C is relatively compact and for each $x \in \overline{C}$, there exists a sequence $(x_n)_{n \geqslant 1} \subset \overline{C}$ such that $x_n \to x$). Some applications to boundary value problems with nonlinear L^p-Carathéodory nonlinearities ($p > 1$) are given. Note that in practice, the weak sequential continuity is easier to be verified than the weak continuity. A map $A : X \to Y$ is said to be weakly–strongly sequentially continuous if for every sequence $(x_n)_{n \in \mathbb{N}}$, $x_n \rightharpoonup x$ implies $A(x_n) \to A(x)$. For instance, compact linear operators on a Banach space X are weakly–strongly sequentially continuous; the converse is true if X is reflexive (see [17]).

In [14,15,31] some nonlinear alternatives are obtained for the strong topology of Banach spaces, the mappings A, B being assumed continuous and compact. Applications to nonlinear functional integral equations are discussed in [15,16].

When the mapping A is compact while B is nonexpansive with respect to the Kuratowski measure of noncompactness [7], a fixed point theory for the sum $A + B$ is developed in [32].

The Schauder and the Krasnosel'skij fixed point theorems are extended to the class of weakly compact linear operators in Dunford–Pettis spaces in [10] with application to a transport equation from Kinetic theory. The obtained Schauder fixed point theorem version relies on the fact that the closed convex hull of a weakly compact set is a convex, weakly compact set (Krein–Šmulian Theorem [17], p. 234); this property always holds true in a Banach space (see [17], p. 434). Recall that a Banach space X is said to have the Dunford–Pettis property if for each Banach space Y every weakly compact operator $A : X \to Y$ takes weakly compact sets in X into norm compact sets of Y. For instance, L^1 spaces have the Dunford–Pettis property.

Motivated by a nonlinear equation arising in transport theory, Latrach et al. [26,27] established generalizations of the Schauder, Darbo and Krasnosel'skij fixed point theorems for the weak topology. Their analysis uses the concept of the Blasi measure of weak noncompactness [13]. Moreover and in contrast to previous works, to prove a new version of the Krasnosel'skij fixed point theorem (Theorem 2.3 in [27]), they neither assume the weak continuity nor the sequentially weak continuity of the operators A and B.

However in order to use all of these results, one should find a self-mapped closed convex set, i.e. to check the condition $AM + BM \subset M$ or the weaker one $(x = Ax + By, y \in M) \Rightarrow u \in M$. From an application point of view, this condition is in general quite restrictive and rather hard to come by. To avoid such a condition, we are interested in this work with fixed point theory for nonself maps. More precisely, our aim is to establish new variants of some nonlinear alternatives of Leray–Schauder and Krasnosel'skij types involving the sum of two operators A and B for the weak topology of Banach spaces. Our results rely on Theorems 2.1, 2.4 from [27] and complement them. To prove some of our results, the main tool used is again the De Blasi measure of weak noncompactness [13]. Theorem 3.1 is then applied to the following generalized Hammerstein integral equation, also discussed in [26] and in [19] when $g(t, .) = g(t)$:

$$y(t) = g\big(t, y(t)\big) + \lambda \int_{\Omega} k(t,s) f\big(s, y(s)\big)\, ds, \quad t \in \Omega \tag{1.2}$$

and posed in $L^1(\Omega, X)$, the space of Lebesgue integrable functions on a measurable domain Ω of $\mathbb{R}^n$ with values in a finite dimensional Banach space X. Note that in L^1 spaces, the weak convergence implies the strong convergence ([17], Corollary 13, p. 295). Here g is a function satisfying a contraction condition with respect to the second variable while f is a nonlinear function. k is strongly measurable and $k(s,t)$ is linear, continuous. In transport equation, k stands for an anisotropic scattering kernel; λ is a real parameter. For boundary value problems associated with ordinary differential equations, k may refer to Green's function. Our result extends previous ones and covers the cases considered in [26] and in [19] (see Corollaries 3.2, 3.3). Notice that many equations in applications fit into the general class of Eq. (1.2).

This paper is organized as follows. In Section 2, we develop abstract existence theorems of Nonlinear Alternative type; this is the content of Theorems 2.2, 2.3 and 2.5. An existence principle is then derived for the nonlinear integral equation (1.2) in Section 3 (Theorem 3.1) with applications to some particular cases (Corollaries 3.1, 3.2, 3.3). The paper ends with some comments on the obtained results and discussion on the methods used.

Throughout, the notation := means to be defined equal to. $\overline{U}$ and ∂U denote respectively the closure and the boundary of the open set U in a topological Hausdorff space (for the strong topology). Finally if Ω is an open subset of a Banach space X, $\mathcal{M}(\Omega, X)$ will refer to the set of all measurable functions $\psi : \Omega \to X$.

Let $(X, \|.\|)$ be a Banach space and let $\mathcal{T}$ be the family of semi-norms

$$\left\{\rho_f(x) = \left|\langle f, x\rangle\right|:\ f \in X^* \text{ and } \|f\|_{X^*} \leqslant 1\right\}.$$

The topology generated by $\mathcal{T}$ and denoted by $\sigma(X, X^*)$ is called the weak topology (see e.g., [11]); here X^* refers to the topological dual of X. A map $A : X \to X$ is said to be weakly continuous if for every $\varphi \in X^*$, the map $\varphi \circ A : X \to \mathbb{R}$ is continuous. Finally recall that $A \in \mathcal{L}(X)$ is said to be weakly compact if $A(B)$ is relatively weakly compact for every bounded subset $B \subset X$, where $\mathcal{L}(X)$ stands for the space of continuous linear functionals on X. If X is a reflexive Banach space, then weakly compact is equivalent to closed (for the weak topology) and bounded (for the norm topology); then every bounded sequence has a weakly converging sub-sequence and the converse is nothing but the Eberlein–Šmulian Theorem (see [11], Thm. III.28 or [20]). Moreover, the same holds true for a weakly closed subset of an arbitrary Banach space. Finally, a convex subset of a normed space is closed if and only if it is weakly closed.

2. Nonlinear alternatives

2.1. The weak MNC

Throughout this section, X denotes a Banach space; $\mathcal{B}(X)$ is the collection of all nonempty bounded subsets of X and $\mathcal{W}(X)$ is the subset of $\mathcal{B}(X)$ consisting of all weakly compact subsets of X. Let B_r denote the closed ball in X centered at 0 with radius $r > 0$. In [13], De Blasi introduced the map $\omega : \mathcal{B}(X) \to [0, +\infty)$ defined, for all $M \in \mathcal{B}(X)$ by

$$\omega(M) = \inf\left\{r > 0,\ \exists N \in \mathcal{W}(X):\ M \subseteq N + B_r\right\}.$$

Before we launch into the details, we recall for the sake of completeness, some important properties of ω needed hereafter; for further details and proofs, we refer to [13].

Lemma 2.1. *Let $M_1, M_2 \in \mathcal{B}(X)$; we have*

(a) $\omega(M_1) \leqslant \omega(M_2)$ *whenever* $M_1 \subseteq M_2$.
(b) $\omega(M) = 0$ *if and only if M is relatively weakly compact.*
(c) $\omega(\overline{M}^w) = \omega(M)$ *where $\overline{M}^w$ is the weak closure of M.*
(d) $\omega(co(M)) = \omega(M)$ *where $co(M)$ refers to the convex hull of M.*
(e) $\omega(M_1 + M_2) \leqslant \omega(M_1) + \omega(M_2)$.
(f) $\omega(M_1 \cup M_2) = \max(\omega(M_1), \omega(M_2))$.

The map ω is called the De Blasi measure of weak noncompactness. In [5], Appel and De Pascale gave to ω the following simple form in L^1 spaces, also called measure of nonequiabsolute continuity:

$$\omega(M) = \limsup_{\varepsilon \to 0}\left\{\sup_{\psi \in M}\left[\int_D \left\|\psi(t)\right\|_X dt,\ D \subset \Omega,\ \text{meas}(D) \leqslant \varepsilon\right]\right\} \tag{2.1}$$

for all bounded $M \subset L^1(\Omega, X)$ where X is a finite dimensional Banach space, $\Omega \subset \mathbb{R}^n$ and meas(.) denotes the Lebesgue measure.

Definition 2.1. A map $f : M \subset X \to X$ is said to be ω-contractive (or an ω-contraction) if it maps bounded sets into bounded sets, and there exists some $\beta \in [0, 1)$ such that $\omega(f(V)) \leqslant \beta\omega(V)$ for all bounded subsets $V \subseteq M$.

Let N be a nonlinear operator from X into itself. Following Latrach et al. [26,27], we introduce the following conditions:

$$(\mathcal{A}1) \quad \begin{cases} \text{If } (x_n)_{n\in\mathbb{N}} \text{ is a weakly convergent sequence in } X, \text{ then} \\ (Nx_n)_{n\in\mathbb{N}} \text{ has a strongly convergent subsequence in } X. \end{cases}$$

$$(\mathcal{A}2) \quad \begin{cases} \text{If } (x_n)_{n\in\mathbb{N}} \text{ is a weakly convergent sequence in } X, \text{ then} \\ (Nx_n)_{n\in\mathbb{N}} \text{ has a weakly convergent subsequence in } X. \end{cases}$$

Regarding these two conditions, Latrach et al. noticed that (see [27], Remark 2.1):

Remark 2.1.

(a) Operators satisfying either $(\mathcal{A}1)$ or $(\mathcal{A}2)$ are not necessarily weakly continuous.
(b) Every ω-contractive map satisfies $(\mathcal{A}2)$.
(c) A map N satisfies $(\mathcal{A}2)$ if and only if it maps relatively weakly compact sets into relatively weakly compact ones (Eberlein–Šmulian Theorem).
(d) A map N satisfies $(\mathcal{A}1)$ if and only if it maps relatively weakly compact sets into relatively compact ones.
(e) The condition $(\mathcal{A}2)$ holds true for every bounded linear operator.

Moreover notice that $(\mathcal{A}1)$ is weaker than the weakly–strongly sequentially continuity of the operator N (see [2], Thm. 2.12). Regarding $(\mathcal{A}2)$, we have

Lemma 2.2. *Let X be a Banach space. Assume that a mapping $B : X \to X$ is a contraction and satisfies $(\mathcal{A}2)$; then B is ω-contractive.*

Proof. Let B be a contraction with some positive constant $\alpha \in (0, 1)$; then it maps bounded sets into bounded sets. Let A be a bounded subset of X. Let $t > 0$ and $N \in \mathcal{W}(X)$ be such that $A \subset N + B_t$. It is clear that

$$B(A) \subseteq B(N) + B_{\alpha t} \subseteq \overline{B(N)}^{\omega} + B_{\alpha t}.$$

Since in addition B satisfies $(\mathcal{A}2)$, $B(N)$ is relatively weakly compact and thus $\omega(B(A)) \leqslant \alpha t$ for all $t > 0$ such that $B \subseteq N + B_t$ with some $N \subseteq \mathcal{W}(X)$. Therefore $\omega(B(A)) \leqslant \alpha\omega(A)$, proving the lemma. □

2.2. The case C is unbounded

We will use the following variant of the Schauder fixed point theorem for weak topology to prove our first existence result which is a new version of a Leray–Schauder nonlinear alternative for the weak topology.

Theorem 2.1. *(See [27], Thm. 2.1.) Let C be a nonempty closed convex subset of a Banach space X. Assume that $F : C \to C$ is a continuous map which satisfies $(\mathcal{A}1)$. If $F(C)$ is relatively weakly compact, then there exists $x \in C$ such that $Fx = x$.*

Remark 2.2. It is easy to see that the weak relative compactness of $F(C)$ both with $(\mathcal{A}1)$ imply that F is condensing for some α (strong) measure of noncompactness. Recall that F is said to be condensing relatively to a measure of noncompactness α whenever $\alpha(F(M)) \geqslant \alpha(M)$ for some bounded subset M implies $\alpha(M) = 0$, i.e. M is relatively compact. Hence Theorem 2.1 follows directly from the Darbo–Sadovskij fixed point theorem [3]. However, in [27] the authors rather used the Schauder fixed point theorem after proving that $F|_{\mathcal{C}}$ is even compact where $\mathcal{C} = \overline{co}(FC)$; the latter set is then proved to be weakly compact by applying the Krein–Šmulian theorem ([17], p. 434). Moreover, we point out that the weak measure of noncompactness is neither used in Theorem 2.1 nor in proving the following one.

Theorem 2.2. *Let C be a nonempty closed convex set in a Banach space X and $U \subset C$ an open subset with some $p \in U$. Let $F : \overline{U} \to C$ be a continuous map which satisfies the condition $(\mathcal{A}1)$. If $F(\overline{U})$ is relatively weakly compact, then*

(i) *either the equation $Fu = u$ has a solution in $\overline{U}$,*
(ii) *or there exists an element $u \in \partial U$ such that $u = \lambda Fu + (1-\lambda)p$ for some $\lambda \in (0,1)$.*

Proof. We will follow a standard method to prove this theorem using an Urysohn function going back to Cech. Most of the proofs of nonlinear alternatives use this auxiliary function (see [1,2,14]). Suppose (ii) does not hold true and F has no fixed point on ∂U otherwise we are finished. Then

$$u \neq \lambda Fu + (1-\lambda)p, \quad \text{for all } u \in \partial U \text{ and } \lambda \in [0,1].$$

Since $p \in U$, the set

$$\mathcal{A} := \left\{u \in \overline{U},\ u = tFu + (1-t)p,\ t \in [0,1]\right\}$$

is nonempty. Moreover, $\mathcal{A} \cap \partial U = \emptyset$ and the continuity of F implies that $\mathcal{A}$ is closed. Therefore, by Urysohn's lemma (see [21], p. 592), there exists a continuous function $\mu : \overline{U} \to [0,1]$ with $\mu(\mathcal{A}) = 1$ and $\mu(\partial U) = 0$ (take for instance $\mu(u) = \frac{d(x,\partial U)}{d(x,\partial U) - d(x,\mathcal{A})}$). Proceeding as in [1], let N be the mapping defined by

$$Nu = \begin{cases} \mu(u)Fu + (1-\mu(u))p, & \text{if } u \in \partial\overline{U}, \\ p, & \text{if } u \in C \setminus \overline{U}. \end{cases} \tag{2.2}$$

It is immediate that $N : C \to C$ is a continuous map. By Theorem 2.1, it suffices to show that the operator N verifies $(\mathcal{A}1)$ and that $N(C)$ is relatively weakly compact. To this end, let $(x_n)_{n\in\mathbb{N}}$ be a weakly convergent sequence in C. According to either or neither $(x_n)_{n\in\mathbb{N}}$ lies in $\overline{U}$ for n large enough, we distinguish between two cases:

(a) There exists some $n_0 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $(n \geqslant n_0 \Rightarrow x_n \in \overline{U})$. In this case, the sequence $(x_n)_{n\in\mathbb{N}}$ lies in $\overline{U}$ and converges weakly in U. Since F satisfies $(\mathcal{A}1)$, the sequence $(Fx_n)_{n\geqslant n_0}$ has a strongly convergent subsequence, say $(Fx_{k_n})_n$, $Fx_{k_n} \to y$ in C. Using the compactness of $[0,1]$, we can extract from $(\mu(x_{k_n}))_n$ a convergent subsequence, say $(\mu(x_{l_n}))_n$. As a result, the sequence $(\mu(x_{l_n}))_n$ verifies $Nx_{l_n} = \mu(x_{l_n})Fx_{l_n} + (1-\mu(x_{l_n}))p$ and thus the limit $ty + (1-t)p$ lies in C.
(b) If $(x_n)_n$ is such that for all $n \in \mathbb{N}$, there exists $m \in \mathbb{N}$ such that $x_{m_n} \notin \overline{U}$, then we may consider a subsequence $(x_{m_n})_n \subset C \setminus \overline{U}$ such that $Nx_{m_n} = p \to p$ in C. From (a) and (b), N verifies $(\mathcal{A}1)$. To see the weak compactness of the set $N(C)$, we use the fact that $F(\overline{U})$ is relatively weakly compact, both with an argument similar to the one used to get that N verifies $(\mathcal{A}1)$. Theorem 2.1 then guarantees the existence of some $u \in C$ with $u = Nu$. Since $p \in U$, then $u \in U$; hence $u = \mu(u)Fu + (1-\mu(u))p$. As a result, $u \in A$ and so $\mu(u) = 1$; this implies that $u = Fu$, ending the proof of the theorem. □

2.3. The case C is bounded

Notice that, in Theorem 2.2, the set C is not necessarily bounded. In case it is bounded, we obtain a more precise result:

Theorem 2.3. *Let C be a nonempty closed, convex and bounded set in a Banach space X and $U \subset C$ an open subset with some $p \in U$. Let $F : \overline{U} \to C$ be a continuous map which satisfies the condition $(\mathcal{A}1)$. If F is an ω-contraction, then*

(i) *either the equation $Fu = u$ has a solution in $\overline{U}$,*
(ii) *or there exists an element $u \in \partial U$ such that $u = \lambda Fu + (1 - \lambda)p$ for some $\lambda \in (0, 1)$.*

This theorem is the nonlinear alternative version of the following fixed point theorem due to Latrach et al. ([27], Thm. 2.2):

Theorem 2.4. *Let C be a nonempty bounded closed convex subset of a Banach space X. Assume that $F : C \to C$ is a continuous map which satisfies $(\mathcal{A}1)$. If F is ω-contractive, then there exists $x \in C$ such that $Fx = x$.*

Remark 2.3. In [27], the authors applied Theorem 2.1 to some nonempty, closed, convex, weakly compact subset of C. To this end, the Cantor intersection condition for the weak measure of noncompactness w is essential.

Proof of Theorem 2.3. Using arguments similar to those used in proving Theorem 2.2, we can see that the operator N defined by (2.2) maps continuously C onto itself and verifies the condition $(\mathcal{A}1)$. By Theorem 2.4, it is enough to check that N is an ω-contraction. To this end, let $K \subset C$; for all $u \in K$, we either have $Nu = \mu(u)Fu + (1 - \mu(u))p$, $\mu(u) \in [0, 1]$ or $Nu = p$. It follows that $N(K) \subset co(F(K) \cup \{p\})$. Making use of Lemma 2.2, we deduce that

$$\begin{aligned}\omega\big(N(K)\big) &\leqslant \omega\big(co\big(F(K) \cup \{p\}\big)\big) = \omega\big(F(K) \cup \{p\}\big) \\ &= \max\big(\omega\big(F(K)\big), \omega(\{p\})\big) = \omega\big(F(K)\big) \leqslant \alpha\omega(K).\end{aligned}$$

Appealing to Theorem 2.4, we conclude the proof. □

2.4. A Krasnosel'skij type nonlinear alternative

Making use of Theorem 2.2, we now prove a nonlinear alternative of Krasnosel'skij type for the weak topology.

Theorem 2.5. *Let $U \ni 0$ denote an open subset of a Banach space X and $\overline{U}$ its closure. Let $A : U \to X$ and $B : X \to X$ be two mappings satisfying*:

1. *A is continuous, $A(\overline{U})$ is relatively weakly compact, and A verifies the condition $(\mathcal{A}1)$.*
2. *B is a contraction and verifies the condition $(\mathcal{A}2)$.*

Then

(i) *either the equation $Au + Bu = u$ has a solution in $\overline{U}$,*
(ii) *or there exists an element $u \in \partial U$ such that $u = \lambda Au + \lambda B(\frac{u}{\lambda})$ for some $\lambda \in (0, 1)$.*

Proof. Define the operator $L : \overline{U} \to X$ by $Lu = (I - B)^{-1} \circ Au$; L is well defined by Assumption 2. Indeed, since B is an α-contraction, $I - B$ is continuous. Moreover,

$$\big\|(I-B)u_1-(I-B)u_2\big\| \geqslant \|u_1-u_2\|-\|Bu_1-Bu_2\| > (1-\alpha)\|u_1-u_2\|.$$

This shows that $(I-B)^{-1}$ exists and is a Lipschitz map on $(I-B)(X)$. Let u be fixed in $\overline{U}$; the map which assigns to each $v \in X$ the value $Au+Bv$ defines a contraction from X into X. Then, by the Banach fixed point theorem, the equation $v = Au + Bv$ has a unique solution $v \in X$, which verifies $Au = (I-B)v$. Therefore $A\overline{U} \subset (I-B)(X)$. So the composition $(I-B)^{-1} \circ A$ is well defined. It follows from the properties of A and the continuity of $(I-B)^{-1}$ that the map L is continuous and verifies the condition $(\mathcal{A}1)$. By Theorem 2.2, it suffices to show that $L(\overline{U})$ is relatively weakly compact. Using Lemma 2.2 and the inclusion

$$(I-B)^{-1} \circ A(\overline{U}) \subset A\overline{U} + B\big((I-B)^{-1} \circ A(\overline{U})\big),$$

we can deduce from the properties of ω that

$$\omega\big((I-B)^{-1} \circ A(\overline{U})\big) \leqslant \omega\big(A(\overline{U})\big) + \alpha\omega\big((I-B)^{-1} \circ A(\overline{U})\big).$$

Since $A(\overline{U})$ is relatively weakly compact, it follows that

$$(1-\alpha)\omega\big((I-B)^{-1} \circ A(\overline{U})\big) \leqslant 0,$$

and so $\omega(L(\overline{U})) = 0$. A second appeal to Theorem 2.2 yields that either the operator equation $(I-B)^{-1} \circ Au = u$ has a solution in $\overline{U}$ or the operator equation $\lambda((I-B)^{-1} \circ Au) = u$ has a solution on the boundary ∂U for some $\lambda \in (0,1)$. This further implies that either (i) or (ii) holds true. □

3. Application to an abstract nonlinear integral equation

3.1. Preliminaries

Let Ω be a bounded domain of $\mathbb{R}^n$ and let X and Y be two separable spaces.

Definition 3.1. A function $f : \Omega \times X \to Y$ is a Carathéodory function if

(i) for all $y \in X$, the map $x \to f(x,y)$ is measurable from Ω to Y;
(ii) for almost every $x \in \Omega$, the map $y \to f(x,y)$ is continuous from X to Y.

If f is a Carathéodory function, then it defines a mapping $N_f : \mathcal{M}(\Omega, X) \to \mathcal{M}(\Omega, Y)$ by $N_f(\psi)(t) = f(t, \psi(t))$, called the Nemytskij operator associated to f, or the superposition operator. Regarding its continuity, we have

Lemma 3.1. *(See [28,29].) Let X and Y be two separable Banach spaces and $f : X \to Y$ a Carathéodory function. Then the Nemytskij operator N_f maps $L^1(\Omega, X)$ into $L^1(\Omega, Y)$ if and only if there exist a constant $\eta > 0$ and a function $\xi \in L^1_+(\Omega)$ such that $\|f(t,x)\|_Y \leqslant \xi(t) + \eta\|x\|_X$, where $L^1_+(\Omega)$ denotes the positive cone of the space $L^1(\Omega)$.*

With the conditions of Lemma 3.1, the operator N_f is obviously continuous and maps bounded sets of $L^1(\Omega, X)$ into bounded sets of $L^1(\Omega, Y)$.

Lemma 3.2. *(See [26].) Let X, Y be two finite dimensional Banach spaces and let Ω be a bounded domain in $\mathbb{R}^n$. If $f : \Omega \times X \to Y$ is a Carathéodory function and N_f maps $L^1(\Omega, X)$ into $L^1(\Omega, Y)$, then N_f satisfies the condition $(\mathcal{A}2)$.*

Remark 3.1. As noticed in Remark 2.1, N_f need not be weakly continuous. More precisely, only linear functions generate weakly continuous Nemytskij operators in L^1 spaces ([4], Thm. 2.6). The question of considering the weak sequential continuity of the Nemytskij operator acting from L^p space to L^q space ($1 \leqslant p, q < \infty$) is discussed in [29] and the answer is shown to be negative at least for $p = 2$.

Here are some hypotheses on the nonlinear functions involved in Eq. (1.2).

3.2. *Assumptions*

($\mathcal{H}1$) The function $g : \Omega \times X \to X$ is a measurable function, $g(., 0) \in L^1(\Omega, X)$ and g is a contraction with respect to the second variable, i.e. there exists $\alpha \in (0,1)$ such that $\|g(t,x) - g(t,y)\|_X \leqslant \alpha\|x - y\|_X$ for a.e. $t \in \Omega$ and all $x, y \in X$.

($\mathcal{H}2$) $f : \Omega \times X \to Y$ is a Carathéodory function and N_f acts from $L^1(\Omega, X)$ into $L^1(\Omega, Y)$.

($\mathcal{H}3$) The function $k : \Omega \times \Omega \to \mathcal{L}(Y, X)$ is strongly measurable where $\mathcal{L}(Y, X)$ refers to the space of bounded linear operators from Y to X.

($\mathcal{H}4$) For each $t \in \Omega$, the function $\rho(t) : \Omega \to \mathcal{L}(Y, X)$, $s \mapsto \rho(t)(s) = k(t,s)$ belongs to $L^\infty(\Omega, \mathcal{L}(Y, X))$, and the function

$$\begin{aligned} \rho : \Omega &\to L^\infty\big(\Omega, \mathcal{L}(Y, X)\big), \\ t &\mapsto \rho(t) \end{aligned}$$

belongs to $L^1(\Omega, L^\infty(\Omega, \mathcal{L}(Y, X))) := L^1(\Omega, L^\infty)$ for short.

($\mathcal{H}5$) There exists a constant $M > 0$ independent of $\lambda^* \in (0,1)$ such that any solution of the integral equation

$$y(t) = \lambda^* g\left(t, \frac{1}{\lambda^*} y(t)\right) + \lambda^* \int_\Omega k(t,s) f\big(s, y(s)\big)\, ds, \quad t \in \Omega,$$

satisfies $\|y\|_{L^1(\Omega, X)} \neq M$.

3.3. *A fixed point formulation*

First notice that Eq. (1.2) may be written in the abstract form

$$y = Ay + By$$

where B is the Nemytskij operator associated to the function g (i.e. $B \equiv N_g$):

$$\begin{aligned} &B : L^1(\Omega, X) \to L^1(\Omega, X), \\ &\quad y \mapsto By : \Omega \to X; \quad By(t) = g\big(t, y(t)\big) \end{aligned}$$

and A appears as the composition of the Nemytskij operator associated to f with the linear integral operator λC where C is the Fredholm operator defined by

$$\begin{aligned} &C : L^1(\Omega, Y) \to L^1(\Omega, X), \\ &\quad \psi \mapsto C\psi : \Omega \to X; \quad C\psi(t) = \int_\Omega k(t,s)\psi(s)\, ds. \end{aligned}$$

Our aim is now to prove that the sum $A+B$ has a fixed point in $L^1(\Omega, X)$. Before starting solving problem (1.2), we give some remarks.

Remark 3.2.

(a) Apart from $(\mathcal{H}5)$, assumptions $(\mathcal{H}1)$–$(\mathcal{H}4)$ are the same as conditions (a)–(d) in [26], p. 2331. Connection between assumption $(\mathcal{H}5)$ and the condition (e) in [26] ($(\mathcal{H}8)$ in this paper) will be made more precise later on through Corollaries 3.2 and 3.3.

(b) It should be noted that assumptions $(\mathcal{H}3)$ and $(\mathcal{H}4)$ lead to the estimate

$$\forall \psi \in L^1(\Omega, Y), \quad \left\| \int_\Omega k(t,s)\psi(s)\,ds \right\|_X \leqslant \left\| \rho(t) \right\|_{L^\infty(\Omega, \mathcal{L}(Y,X))} \|\psi\|_{L^1(\Omega,Y)}$$

and so

$$\|C\psi\|_{L^1(\Omega,X)} = \int_\Omega \left\| \int_\Omega k(t,s)\psi(s)\,ds \right\|_X dt \leqslant \|\rho\|_{L^1(\Omega,L^\infty)} \|\psi\|_{L^1(\Omega,Y)}.$$

This shows that the linear operator C is continuous, hence weakly continuous from $L^1(\Omega, Y)$ into $L^1(\Omega, X)$ and that $\|C\| \leqslant \|\rho\|_{L^1(\Omega,L^\infty)}$.

(c) Using assumption $(\mathcal{H}1)$, we get

$$\left\| g(t,u) \right\|_X \leqslant \left\| g(t,0) \right\| + \alpha \|u\|_X, \quad \text{for every } u \in X \text{ and a.e. } t \in \Omega,$$

where $\|g(t,0)\| \in L^1_+(\Omega)$. This shows that the Nemytskij operator N_g is continuous and maps bounded sets of $L^1(\Omega, X)$ into bounded sets of $L^1(\Omega, X)$. According to Lemma 3.2, the operator B satisfies the condition $(\mathcal{A}2)$.

3.4. Existence results

The main result in this section is

Theorem 3.1. *Let X and Y be two finite dimensional Banach spaces and let Ω be a bounded domain of $\mathbb{R}^n$. Assume that assumptions $(\mathcal{H}1)$–$(\mathcal{H}5)$ hold true. Then, Eq.* (1.2) *has at least one solution in $L^1(\Omega, X)$.*

Proof. We apply Theorem 2.5 with

$$U = \left\{ y \in L^1(\Omega, X)\colon \|y\|_{L^1(\Omega,X)} < M \right\}.$$

Claim 1. Let $\varphi, \psi \in L^1(\Omega, X)$. It follows from assumption $(\mathcal{H}1)$ that

$$\begin{aligned} \left\| B(\psi) - B(\varphi) \right\|_{L^1(\Omega,X)} &= \int_\Omega \left\| g\big(t, \psi(t)\big) - g\big(t, \varphi(t)\big) \right\|_X dt \\ &\leqslant \alpha \int_\Omega \left\| \psi(t) - \varphi(t) \right\|_X dt = \alpha \|\psi - \varphi\|_{L^1(\Omega,X)}. \end{aligned}$$

So, B is a strict contraction mapping on $L^1(\Omega, X)$ and from Remark 3.2(c), B satisfies the condition $(\mathcal{A}2)$.

Claim 2. Clearly A is continuous (see Lemma 3.1 and Remark 3.2(a)). Now we check that A satisfies the condition $(\mathcal{A}1)$. For this, let $(\psi_n)_n$ be a weakly convergent sequence of $L^1(\Omega, X)$. Using the fact that N_f satisfies $(\mathcal{A}2)$, $(N_f(\psi_n))_{n\in\mathbb{N}}$ has a weakly convergent subsequence, say $(N_f(\psi_{k_n}))_{n\in\mathbb{N}}$. Moreover, the continuity of the linear operator C implies its weak continuity on $L^1(\Omega, Y)$. Thus the sequence $((C \circ N_f)(\psi_{k_n}))_{n\in\mathbb{N}}$, i.e. $(A(\psi_{k_n})_{n\in\mathbb{N}})$ converges pointwisely for almost every $t \in \Omega$. Using Vitali's Convergence Theorem, we conclude that $(A\psi_{k_n})_{n\in\mathbb{N}}$ converges strongly in $L^1(\Omega, X)$. Then A satisfies $(\mathcal{A}1)$.

Claim 3. $A(\overline{U})$ is relatively weakly compact. For this, we show that

$$\omega\big(A(\overline{U})\big) = \limsup_{\varepsilon\to 0}\left\{\sup_{y\in\overline{U}}\left[\int_D \|Ay(t)\|_X\,dt,\ |D| < \varepsilon\right]\right\} = 0,$$

where $|D| = \mathrm{meas}(D)$. For all $D \subseteq \Omega$, and every $y \in U$, we have

$$\begin{aligned}
\int_D \|Ay(t)\|_X\,dt &= \int_D \left\|\lambda\int_\Omega k(t,s)f\big(s,y(s)\big)\,ds\right\|_X dt\\
&\leqslant |\lambda|\left[\int_\Omega \big\|f\big(s,y(s)\big)\big\|_Y\,ds\right]\int_D \|\rho(t)\|_{L^\infty}\,dt\\
&\leqslant \ell\int_D \|\rho(t)\|_{L^\infty}\,dt,
\end{aligned}$$

where $\ell := |\lambda|[\|\xi\|_{L^1_+(\Omega)} + \eta M]$ and the function ξ is defined in Lemma 3.1. Since $\rho \in L^1(\Omega, L^\infty(\Omega, \mathcal{L}(Y, X)))$, we have that the mapping $t \mapsto \|\rho(t)\|_{L^\infty}$ lies in $L^1_+(\Omega)$. From Corollary 11 in [17], p. 294, we deduce that

$$\limsup_{\varepsilon\to 0}\left\{\int_D \|\rho(t)\|\,dt,\ |D| < \varepsilon\right\} = 0.$$

Therefore, $A(\overline{U})$ is relatively weakly compact. Thanks to assumption $(\mathcal{H}5)$, possibility (ii) in Theorem 2.5 cannot occur, and so the sum $A + B$ has a fixed point in $\overline{U}$; equivalently Eq. (1.2) has a solution in $\overline{U}$. □

Remark 3.3. The requirement that X and Y should be finite dimensional Banach spaces comes from the usage of the relation (2.1) proved in [5] for bounded subsets in the space of Lebesgue integrable functions with values in a finite dimensional Banach space.

Remark 3.4.

(a) To prove Theorem 3.1, we can use the following Dunford–Pettis theorem instead of the measure of weak noncompactness. The proof is the same as above and is omitted.
Theorem [17,18]. Let $\mathcal{F} \subset L^1(\Omega)$ be a bounded subset. Then $\mathcal{F}$ is relatively weakly compact if and only if

$$\forall\varepsilon > 0,\ \exists\delta > 0,\quad \int_A \|f(x)\|\,dx < \varepsilon,\quad \forall f \in \mathcal{F} \text{ and } \forall A \subset \Omega,\ |A| < \delta.$$

(b) If $k \in L^\infty(\Omega \times \Omega)$, then another alternative proof for Theorem 3.1 is to start from the estimate

$$\left\| Ay(t) \right\|_X \leqslant |\lambda| \left\| \rho(t) \right\|_{L^\infty}, \quad \forall y \in U.$$

Since $\rho \in L^1(\Omega, L^\infty)$, then we deduce that the family $AU = \{Ay\}_{y \in U}$ is integrably bounded (recall that $\mathcal{F} \subset L^1(\Omega)$ is integrably bounded if there exists $g \in L^1_+(\Omega)$ such that for every $f \in \mathcal{F}$, $\|f(x)\| \leqslant g(x)$, for each $x \in U$). Moreover $\{AU(t)\}_{t \in \Omega}$ is relatively compact in the finite dimensional space X. Therefore AU is semi-compact, hence weakly relatively compact by the Dunford–Pettis theorem (see [22], Proposition 4.2.1).

Remark 3.5. As mentioned in the introduction, Eq. (1.2) was motivated by the transport equation from Rotenberg's model in [25] where the authors first considered the problem in L^p-spaces ($1 < p < +\infty$). Now, the closed, bounded, convex ball $\overline{U}$ in the proof of Theorem 3.1 is weakly closed. Since L^p spaces are reflexive for $1 < p < +\infty$, then $\overline{U}$ is weakly compact. Finally, A satisfies $(\mathcal{A}1)$, then A is relatively compact and hence the classical Krasnosel'skij theorem applies. Unfortunately, this does no longer hold true in L^1-spaces. However, since $A = \lambda C \circ N_f$, A is relatively compact whenever either one of the Fredholm operator C or the Nemytskij operator is relatively compact.

Our existence principle, namely Theorem 2.2 is now used to get another existence criteria for Eq. (1.2). Similar assumptions to $(\mathcal{H}6)$ and $(\mathcal{H}7)$ were given in [1].

Corollary 3.1. *Let X and Y be two finite dimensional Banach spaces and let Ω be a bounded domain of $\mathbb{R}^n$. Assume that in addition to $(\mathcal{H}1)$–$(\mathcal{H}4)$, the following assumptions hold true*:

$(\mathcal{H}6)$ *There exists a continuous function $h : [0, +\infty) \to [0, +\infty)$ such that $h(u) > 0$ whenever $u > 0$ and*

$$|\lambda| \int_\Omega \left\| \int_\Omega k(t,s) f\big(s, y(s)\big)\, ds \right\|_X dt \leqslant h\big(\|y\|_{L^1(\Omega,X)}\big), \quad \forall y \in L^1(\Omega, X).$$

$(\mathcal{H}7)$

$$\sup_{\theta \in [0,+\infty)} \left(\frac{(1-\alpha)\theta}{\|\zeta\|_{L^1_+} + h(\theta)} \right) > 1,$$

where $\zeta(t) := \|g(t,0)\|_X$ and α is the contraction constant of g.

Then, Eq. (1.2) *has a solution in $L^1(\Omega, X)$.*

Proof. Thanks to Theorem 3.1, it suffices to show that $(\mathcal{H}6)$ and $(\mathcal{H}7)$ imply $(\mathcal{H}5)$. Let $M > 0$ satisfy

$$\frac{(1-\alpha)M}{\|\zeta\|_{L^1_+} + h(M)} > 1. \tag{3.1}$$

Assumption $(\mathcal{H}7)$ both with the property of the supremum ensure the existence of such an M. Let $y \in L^1(\Omega, X)$ be any solution of the operator equation

$$y = \lambda^* Ay + \lambda^* B\left(\frac{y}{\lambda^*}\right), \quad \lambda^* \in (0,1). \tag{3.2}$$

Then, for $t \in \Omega$, we have the estimates

$$\big\|y(t)\big\|_X \leqslant \alpha\big\|y(t)\big\|_X + \lambda^*\big\|g(t,0)\big\|_X + \lambda^*|\lambda|\bigg\|\int_\Omega k(t,s)f\big(s,y(s)\big)\,ds\bigg\|_X$$

and so

$$(1-\alpha)\int_\Omega \big\|y(t)\big\|_X\,dt \leqslant \|\zeta\|_{L^1_+} + h\big(\|y\|_{L^1(\Omega,X)}\big).$$

Therefore

$$\frac{(1-\alpha)\|y\|_{L^1(\Omega,X)}}{\|\zeta\|_{L^1_+} + h(\|y\|_{L^1(\Omega,X)})} \leqslant 1. \tag{3.3}$$

Assuming $\|y\|_{L^1(\Omega,X)} = M$, (3.3) implies $\frac{(1-\alpha)M}{\|\zeta\|_{L^1_+}+h(M)} \leqslant 1$, contradicting (3.1). Finally, every solution of (3.2) satisfies $\|y\|_{L^1_+} \neq M$. Therefore Theorem 3.1 guarantees that Eq. (1.2) has a solution $y \in L^1(\Omega, X)$. □

As a consequence of Theorem 3.1, we also recover the following two existence results obtained by Latrach and Taoudi (see [26], Thm. 3.1 and Cor. 3.1). The second corollary is also the main result obtained in [19], p. 609.

Corollary 3.2. *Let X and Y be two finite dimensional Banach spaces, and let Ω be a bounded domain of $\mathbb{R}^n$. Assume assumptions $(\mathcal{H}1)$–$(\mathcal{H}4)$ hold together with*

$(\mathcal{H}8)$ $\alpha + \eta|\lambda|\|C\| < 1$

(*the constant η was introduced in Lemma* 3.1 *and $\|C\|$ denotes the norm of the operator C*). *Then Eq.* (1.2) *has a solution $y \in L^1(\Omega, X)$ for every $\lambda \in \mathbb{R}$.*

Corollary 3.3. *Let X and Y be two finite dimensional Banach spaces, and let Ω be a bounded domain of $\mathbb{R}^n$. Assume assumptions $(\mathcal{H}2)$–$(\mathcal{H}4)$ hold true together with $(\mathcal{H}8)$ and let $g \in L^1(\Omega, X)$. Then the equation*

$$y(t) = g(t) + \lambda\int_\Omega k(t,s)f\big(s,y(s)\big)\,ds$$

has a solution $y \in L^1(\Omega, X)$.

Proof of Corollary 3.2. Let $y \in L^1(\Omega, X)$. From $(\mathcal{H}3)$ and $(\mathcal{H}4)$, we have

$$\begin{aligned}|\lambda|\int_\Omega\bigg\|\int_\Omega k(t,s)f\big(s,y(s)\big)\,ds\bigg\|_X dt &= |\lambda|\big\|C\circ N_f(y)\big\|_{L^1(\Omega,X)}\\ &\leqslant |\lambda|\|C\|\big\|N_f(y)\big\|_{L^1(\Omega,X)}\\ &\leqslant |\lambda|\|C\|\big(\|\xi\|_{L^1_+} + \eta\|y\|_{L^1(\Omega,X)}\big).\end{aligned}$$

On the other hand, arguing as in the proof of Corollary 3.1, we get the estimate

$$\|y\|_{L^1(\Omega,X)} \leqslant \alpha\|y\|_{L^1(\Omega,X)} + \lambda^*\|\zeta\|_{L^1_+} + \lambda^* k\big(\|y\|_{L^1(\Omega,X)}\big),$$

where $k(\theta) := |\lambda|\|C\|(\|\xi\|_{L^1_+} + \eta\theta)$. Hence

$$(1-\alpha)\|y\|_{L^1(\Omega,X)} \leqslant \|\zeta\|_{L^1_+} + |\lambda|\|C\|\big(\|\xi\|_{L^1_+} + \eta\|y\|_{L^1(\Omega,X)}\big). \tag{3.4}$$

Let

$$M > \frac{\|\zeta\|_{L^1_+} + |\lambda|\|C\|\|\xi\|_{L^1_+}}{1-\alpha-|\lambda|\|C\|\eta}.$$

If $\|y\|_{L^1(\Omega,X)} = M$, then (3.4) implies that

$$M \leqslant \frac{\|\zeta\|_{L^1_+} + |\lambda|\|C\|\|\xi\|_{L^1_+}}{1-\alpha-|\lambda|\|C\|\eta},$$

which is a contradiction. Assumption $(\mathcal{H}5)$ is then satisfied and the result follows from Theorem 3.1. □

4. Concluding remarks

(a) In addition to Ref. [26], Eq. (1.2) was treated in [19] in the particular case $g(t, y(t)) = g(t)$. An existence result was obtained under assumptions $(\mathcal{H}1)$–$(\mathcal{H}4)$ together with the condition $(\mathcal{H}8)$.

(b) The particular case of the model integral equation

$$y(t) = g\big(y(t)\big) + \int_0^t f\big(s, y(s)\big)\,ds, \quad y \in C\big([0,T], E\big),$$

where E is a reflexive Banach space is widely considered in the literature (see also Corollary 3.3). Obviously this equation falls into the class covered by Eq. (1.2). In [9], the function f and $g(t,.)$ are sequentially weakly continuous. The existence of a solution is then proved under a Nagumo-type growth condition on the nonlinearity g.

(c) It should be emphasized that one of the advantage of working with nonlinear alternatives rather than with fixed points is not to check an operator maps a closed convex subset onto itself. In this paper, this is illustrated by the fact that the constraint $(\mathcal{H}8)$ is removed in Theorem 3.1 and replaced by assumption $(\mathcal{H}5)$ which deals with a priori estimates of the sought solutions. We believe that this work could make a contribution in fixed point theory for the sum of nonlinear operators, which is well developed in the literature (see for instance [1,34,21] and the references therein).

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