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# Horner versus Holdred: An Episode in the History



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It is well known that Horner's method for the computation of a real root of a polynomial equation was anticipated in Italy by Ruffini. In the present paper it is shown that in England the method was published by Holdred, before Horner. The resulting controversy over priority is discussed, and related letters from contemporary mathematicians are reproduced. It is concluded that the dissemination of the algorithm under the inappropriate designation "Horner's method" is mainly due to De Morgan. © 1999 Academic Press

Il est bien connu que la méthode de Horner pour calculer une racine reélle d'une équation polynôme avait été anticipée en Italie par Ruffini. Dans ce papier il est démontré qu'en Angleterre la méthode fut publiée par Holdred avant Horner. La controverse sur la priorité est discutée, et des lettres de mathématiciens contemporains sont reproduites. Il est conclu que la dissémination de l'algorithme sous la désignation inappropriée "méthode de Horner" est principalement due à De Morgan. © 1999 Academic Press

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## 1. INTRODUCTION

The algorithm for the computation of a real root of a polynomial equation with given numerical coefficients known as "Horner's method" is described in many textbooks, for example Herbert Turnbull [73, 25–26, 30–33, 83–88] and James Uspensky [74, 151–169]. In its simplest form the algorithm calculates the root digit by digit, and while somewhat slow compared with the Newton–Raphson method, is convenient for hand calculation.

William Horner's papers on root computation began in 1819. Florian Cajori [15] pointed out that the method was anticipated by Paolo Ruffini [68; 69], in Italy. Long before then related techniques were known to Chinese and Arabic writers; see for example, Martin Nordgaard [60, 54–55], Paul Luckey [48; 49], Adolf Youschkevitch [45, 41–48, 242–248; 81, 76–80, 169], Youschkevitch and Boris Rosenfeld [82], Carl Boyer [10, 268–270], Ulrich Libbrecht [46, 177–211], Roshdi Rashed [65], and Lennart Berggren [7, 48–63]. The main aim of the present paper is to show that even in England Horner was anticipated, in fact by Theophilus Holdred [36]. A further aim is to disentangle the resulting priority controversy by emphasizing that Horner's algorithm of 1819 was different from the one conventionally ascribed to him.



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## 2. A PRINCIPLE IN ROOT COMPUTATION

A technique used in many algorithms for the computation of roots is as follows. Given a polynomial equation

$$f(x) = 0, \tag{1}$$

suppose that we know an initial approximation $\alpha$ to a root. Substituting in (1)

$$x = \alpha + y, \tag{2}$$

we obtain a new polynomial equation in the error $y$:

$$g(y) = 0. \tag{3}$$

Since $y$ is small the higher-degree terms in $g(y)$ may be negligible, enabling us to find from (3) an approximate solution $\beta$ for $y$. We now repeat the procedure, substituting $y = \beta + z$ in (3), and finding an approximate solution $\gamma$ for $z$. Continuing in this way, we get a successively improved estimate $\alpha + \beta + \gamma + \cdots$ for the root.

If $x$ is a root of the original equation (1), $y$ is this root diminished by $\alpha$. Thus the above procedure consists of transformations which successively diminish the root. William Burnside and Arthur Panton [13, 1:227–241] characterized it as the principle of diminution of the roots.

Equation (2) shows that the roots of the equation in $y$ are the roots of the equation in $x$ shifted by $\alpha$. The shift is to the left if $\alpha$ is positive. We may say that the polynomial, or even that the equation, is shifted by the transformation. Thus the principle may be rephrased as that of gradually shifting the equation to make one of the roots approach zero.

## 3. THE BASIC ALGORITHM

To carry out the above technique we first need a procedure for calculating the coefficients of the shifted equation (3) in terms of those of the original equation (1). The procedure used in the so-called Horner's method is as follows. Suppose that the given equation (1) is $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$. Form an array of numbers

$$\begin{array}{ccccccc}
p_{00} & p_{01} & p_{02} & \cdots & p_{0,n-2} & p_{0,n-1} & p_{0n} \\
p_{10} & p_{11} & p_{12} & \cdots & p_{1,n-2} & p_{1,n-1} & p_{1n} \\
p_{20} & p_{21} & p_{22} & \cdots & p_{2,n-2} & p_{2,n-1} & \\
\vdots & \vdots & \vdots & & & & \\
\vdots & \vdots & \vdots & & & & \\
p_{n-1,0} & p_{n-1,1} & p_{n-1,2} & & & & \\
p_{n0} & p_{n1} & & & & & \\
p_{n+1,0} & & & & & &
\end{array} \tag{4}$$

This array is triangular except that the right-hand corner term is absent. The first row consists of the coefficients of the given equation, $p_{0j} = a_j$ $(j = 0, 1, \ldots, n)$, and each term of the first column is the leading coefficient,

$$p_{i0} = a_0 \qquad (i = 0, 1, \ldots, n + 1).$$

Any other element of the array is produced by multiplying the term on its left by $\alpha$ and adding to the result the term above the element:

$$p_{ij} = \alpha p_{i,j-1} + p_{i-1,j}. \tag{5}$$

(The second row is obtained by working from left to right, next the third row is obtained similarly, and so on.)

Then if the new equation (3) is

$$b_0 y^n + b_1 y^{n-1} + \cdots + b_n = 0, \tag{6}$$

its coefficients are given by the elements of the lowest diagonal: $b_i = p_{n+1-i,i}$ $(i = 0, 1, \ldots, n)$.

In what is called Horner's method, the first digit of a real root is found by trial and error, yielding a first approximation $\alpha$ to the root. The equation is shifted by $\alpha$, and we seek a one-digit approximate solution $\beta$ of the new equation. This will be the next digit of the root. It therefore has to be the largest digit such that $f(\alpha + \beta)$ does not have the opposite sign to that of $f(\alpha)$.

Now in array (4) the last term in the second row satisfies $p_{1n} = f(\alpha)$. In fact, evaluating $f(\alpha)$ is equivalent to evaluating $g(0)$, in view of (2), i.e., equivalent to finding the last coefficient of $g(y)$, and this is $p_{1n}$. Thus we choose the digit to be the largest of the ten possible values $0, 1, \ldots, 9$ such that the resulting shift of the equation does not reverse the sign of its last coefficient.

Some trial and error may be required in the search for this digit. But if $\beta$ is small, the higher-degree terms in the equation (6) for $y$ may be negligible, so that $\beta$ is approximately given by $\beta = -b_n/b_{n-1}$. Thus a first guess for the required digit will be the leading digit of the quantity $(-b_n/b_{n-1})$.

The procedure of shifting the equation and testing for sign change is repeated for subsequent digits of the root until a prescribed accuracy is attained. The algorithm converges if the root being computed has odd multiplicity and is not too close to any other real root.

## 4. VIÈTE'S ALGORITHM

Horner and Holdred did not state the basic sources from which they developed their ideas. (Admittedly Horner did give some references, but these were on subsidiary matters.) However, their methods belong essentially to a line of thought which goes back to François Viète and beyond. To introduce and explain these methods, a necessary preliminary will be to give a brief summary (in a modern interpretation) of Viète's technique.

First, let us recall the ordinary method for hand calculation of square roots, attributed to the Hindus [60, 8–9]. Given a numerical equation

$$f(x) \equiv x^2 - k = 0 \tag{7}$$

where $k$ is real and positive, the problem is to calculate the positive square root $x$. Suppose that the first few digits of $x$ have already been found, yielding an approximation $\alpha$ and an error $y$, so that $x = \alpha + y$. Then (7) gives $\alpha^2 + 2\alpha y + y^2 - k = 0$, which can be written as

$$y = \frac{-f(\alpha)}{2\alpha + y}. \tag{8}$$

In the denominator, the term $y$ is small; neglecting it we obtain a *trial divisor* $2\alpha$. This is divided into the numerator, the division being stopped as soon as the first digit of the quotient is found, yielding an approximate solution $y_{\mathrm{T}}$ for (8). $y_{\mathrm{T}}$ is taken as a trial value for the contribution of the next digit to the root.

This trial value is either correct or too large (because of the neglect of $y$ in forming the trial divisor). With $x$ given its new approximate value $\alpha + y_{\mathrm{T}}$, we test $f(x)$ to find its sign. If $f(\alpha + y_{\mathrm{T}}) > 0$, $y_{\mathrm{T}}$ is too large, so the new digit is decreased, one unit at a time, until it yields

$$f(\alpha + y_{\mathrm{T}}) \leq 0, \tag{9}$$

and the digit is then accepted. If (9) is an equality, the root is found exactly. If it is an inequality, further digits can be found by the same method.

François Viète [75; 76, 163–228], also known as Franciscus Vieta, used a technique for calculating a real root of a polynomial equation, which was a generalization of the above Hindu method for square roots. Suppose that the given equation is

$$f(x) \equiv a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0, \tag{10}$$

where the coefficients are all real numbers, and that the first few digits of a positive real root $x$ have already been found. This will yield an approximation $\alpha$ and an error $y$ such that $x = \alpha + y$. Substitution in (10) gives

$$f(\alpha + y) = b_0 y^n + b_1 y^{n-1} + \cdots + b_{n-1} y + f(\alpha) = 0, \tag{11}$$

where the coefficients $b_0, b_1, \ldots, b_{n-1}$ can be found by expanding the terms $(\alpha + y)^i$ appearing in $f(\alpha + y)$. From (11)

$$y = \frac{-f(\alpha)}{b_0 y^{n-1} + b_1 y^{n-2} + \cdots + b_{n-1}}, \tag{12}$$

which is a generalization of (8).

We now attempt to find a one-digit approximate solution of (12). In the treatment of (8), the term $y$ in the denominator was dropped in order to obtain a trial divisor. By analogy,

Viète dropped $b_0 y^{n-1}$ from the denominator in (12), obtaining an approximate divisor

$$b_1 y^{n-2} + b_2 y^{n-3} + \cdots + b_{n-1}. \tag{13}$$

Here $y$ is unknown, but to within one digit it will have one of the ten values $[0, 1, \ldots, 9]10^d$, where $d$ is a known integer corresponding to the displacement of the digit from the decimal point. Study of Viète's examples shows that he chose the second of these possibilities, thus replacing $y$ in (13) by $1 \cdot 10^d$, and obtaining a trial divisor

$$b_1 10^{(n-2)d} + b_2 10^{(n-3)d} + \cdots + b_{n-1}. \tag{14}$$

He divided this into the numerator in (12), obtaining a one-digit approximate quotient $y_{\mathrm{T}}$, which is taken as a trial value for the contribution $\beta$ of the new digit to the root.

The next stage is to test the new approximation $\alpha + \beta$ for the root. If it yields $\operatorname{sgn} f(\alpha + \beta) = -\operatorname{sgn} f(\alpha)$, then $\beta$ is too large, and we have to replace the new digit by a smaller value. If $\operatorname{sgn} f(\alpha + \beta) = \operatorname{sgn} f(\alpha)$, the new digit is either correct or too small. We therefore increase the digit by unity and retest. Thus the identification of the correct digit may require several tests. Further digits are obtained successively by the same technique, until a specified accuracy is reached.

Viète's statement of his method was inexplicit. He proceeded by examples, and he omitted any indication of the test procedure which is an essential part of the algorithm. However, the method was gradually elucidated by subsequent writers, especially Thomas Harriot [35, 117–180] and William Oughtred [61, 121–169]. Rashed [65] argues that Viète's method was anticipated in its essentials by the Arabic writer al-Ṭūsī; but it appears that al-Ṭūsī's divisor was different from the version (14) used by Viète.

Because of the inexplicitness of Viète's treatment, several writers have misinterpreted him. For example, Augustus De Morgan [24, 2:103] and Nordgaard [60, 28] imply that Viète's trial divisor is simply

$$b_1 + b_2 + \cdots + b_{n-1} \tag{15}$$

instead of (14). However, (15) does not agree with Viète's examples.

## 5. HORNER'S METHOD OF 1819

William George Horner (1786?–1837) was a schoolmaster who ran his own school at Bath from 1809 until his death [3]. In 1819, he published two papers on the numerical solution of equations [38; 39]. Only one of these [39] concerns us here. In this paper, he gave a tabular scheme for computing a real root of a polynomial equation, but, as will be shown, it was substantially different from the basic algorithm described in Section 3 above.

Unfortunately, Horner wrote his paper in a very obscure style. To explain his approach will require arguments that he himself did not supply. First, let us recall an economical way of evaluating a polynomial for a given value of the argument. Suppose the polynomial is $g(y) \equiv b_0 y^n + b_1 y^{n-1} + \cdots + b_n$, and we wish to evaluate it at a specified numerical value

of $y$. We write

$$g(y) = (\cdots(((b_0y + b_1)y + b_2)y + b_3))y \cdots + b_n,$$

and multiply out the brackets successively, starting with the innermost and working outwards. Thus, the following terms are evaluated successively:

$$\begin{aligned}
Q_1 &= b_0y + b_1 \\
Q_2 &= b_0y^2 + b_1y + b_2 \\
Q_3 &= b_0y^3 + b_1y^2 + b_2y + b_3 \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
Q_{n-1} &= b_0y^{n-1} + b_1y^{n-2} + \cdots + b_{n-1} \\
Q_n &= b_0y^n + b_1y^{n-1} + \cdots + b_{n-1}y + b_n.
\end{aligned}$$

The final term $Q_n$ is the required value of $g(y)$.

The method is equivalent to the iteration

$$Q_i = Q_{i-1}y + b_i \qquad (i = 1, 2, \ldots, n)$$

with initial condition $Q_0 = b_0$. This is a standard method of polynomial evaluation, and goes back at least to the manuscript writings of Isaac Newton [50; 77, 1:489–491]. For want of a better name let us call $Q_1, Q_2, \ldots, Q_n$ *Newtonian parameters* associated with the polynomial $g(y)$. Note that the penultimate Newtonian parameter $Q_{n-1}$ is the same as the divisor in the Viète equation (12).

Horner's approach can now be described. His (unstated) idea is to devise a version of Viète's iteration in which the Newtonian parameters associated with the current divisor are calculated from the Newtonian parameters associated with the previous divisor.

For this purpose, Horner sets up an array of numbers which is, in essence, similar to the following:

$$\begin{array}{cccccccc}
R_{n1} & R_{n2} & R_{n3} & \cdots & R_{n,n-2} & R_{n,n-1} & R_{nn} \\
Q_1 & Q_2 & Q_3 & \cdots & Q_{n-2} & Q_{n-1} & Q_n \\
R'_{11} & R'_{12} & R'_{13} & \cdots & R'_{1,n-2} & R'_{1,n-1} & \\
R'_{21} & R'_{22} & R'_{23} & \cdots & R'_{2,n-2} & & \\
\vdots & \vdots & \vdots & & & & \\
R'_{n-2,1} & R'_{n-2,2} & & & & & \\
R'_{n-1,1} & & & & & & \\
R'_{n1} & R'_{n2} & R'_{n3} & \cdots & R'_{n,n-2} & R'_{n,n-1} & R'_{nn} \\
Q'_1 & Q'_2 & Q'_3 & \cdots & Q'_{n-2} & Q'_{n-1} & Q'_n
\end{array} \tag{16}$$

This array consists of a triangular subarray preceded by two full rows and followed by two

full rows. A primed term such as $R'_{12}$ is evaluated in the current iteration, and an unprimed term is evaluated in the previous iteration.

Thus, the terms in the first two rows are obtained from the previous iteration. The rules for assigning the other terms are as follows:

(i) The next $(n-1)$ terms in the first column are $R'_{i1} = \rho a_0$ $(i = 1, 2, \ldots, n-1)$, where $\rho$ is the contribution of the previous digit to the root estimate.

(ii) The penultimate term in the first column is $R'_{n1} = \rho' a_0$, where $\rho'$ is the contribution of the current digit to the root estimate.

(iii) Each remaining term $R'_{1j}$ in the third row is obtained by multiplying the term on its left by $\rho$ and adding the $R$-term above $R'_{1j}$:

$$R'_{1j} = \rho R'_{1,j-1} + R_{nj} \qquad (j = 2, 3, \ldots, n-1).$$

(iv) Each remaining term $R'_{ij}$ in rows $4, 5, \ldots, n-1$ is obtained by multiplying the term on its left by $\rho$ and adding the term above $R'_{ij}$:

$$R'_{ij} = \rho R'_{i,j-1} + R'_{i-1,j} \qquad (i = 2, 3, \ldots, n-2;\ j = 2, 3, \ldots, n-i). \tag{17}$$

(v) Each remaining term in the penultimate row is obtained by multiplying by $\rho'$ the $Q$-term diagonally below and to the left: $R'_{nj} = \rho' Q'_{j-1}$ $(j = 2, 3, \ldots, n)$.

(vi) Each term in the last row is obtained by summing all the terms above it in the same column, excluding the top element:

$$Q'_j = \begin{cases} Q'_j = R'_{nj} + \displaystyle\sum_{i=1}^{n-j} R'_{ij} & (j = 1, 2, \ldots, n-1) \\ Q_j + R'_{nj} & (j = n). \end{cases}$$

All the terms in the first column are found first, then all the terms in the second column, and so on.

For the first iteration the above array is inapplicable since its first two rows are undefined. Instead, we use the array:

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\ R'_{n1} & R'_{n2} & R'_{n3} & \cdots & R'_{n,n-1} & R'_{nn}. \\ Q'_1 & Q'_2 & Q'_3 & \cdots & Q'_{n-1} & Q'_n \end{array}$$

(vii) Here the first $R$-term is $R'_{n1} = \rho' a_0$, where $\rho'$ is the contribution of the first digit to the root estimate.

(viii) Each remaining term in the second row is obtained by multiplying by $\rho'$ the term diagonally below and to the left: $R'_{nj} = \rho' Q'_{j-1}$ $(j = 2, 3, \ldots, n)$.

(ix) Each term in the third row is obtained by summing the terms above it: $Q'_j = a_j + R'_{nj}$ $(j = 1, 2, \ldots, n)$.

Horner's array (16) thus evaluates the Newtonian parameters $Q'_1, Q'_2, \ldots, Q'_n$ associated with the current iteration, given the Newtonian parameters $Q_1, Q_2, \ldots, Q_n$ associated with

the previous iteration. The term $Q'_{n-1}$ corresponds to the denominator in the Viète equation (12), and $-Q_n$ corresponds to the numerator. Hence, array (16) supplies the data needed for a Viète-type iteration.

Note that the construction of the array requires knowledge of the contribution $\rho'$ of the new digit. Horner was vague on his method for finding this value, but it seems to be as follows. First, construct a preliminary version of the array assuming that $\rho' = 0$. From this, evaluate $-Q_n/Q'_{n-1}$ and take its first digit as giving a trial value for $\rho'$. Construct the array with this value of $\rho'$

In principle, the new digit ought next to be tested to see if it is the largest value such that the resulting shift of the polynomial does not reverse its sign. (It will be too large if $Q_n$ and $Q'_n$ have opposite signs.) Horner did not specifically state this test procedure, and in his examples he omitted all arithmetical details of such tests.

Horner used different notation and did not present his algorithm in the above enumerative fashion; instead he inserted the formula for each element into the array itself. Moreover, he introduced modifications into his array which are omitted here, to simplify the present exposition. He also took his leading coefficient (our $a_0$) to be unity.

Comparison of the above algorithm with the basic algorithm stated in Section 3 shows that the two are substantially different. The basic algorithm evaluates the coefficients of the shifted equation, whereas Horner's 1819 algorithm evaluates the Newtonian parameters. The basic algorithm is very simple. The Horner algorithm, on the other hand, is complicated; it requires the summation of several terms in a column, involving awkward arithmetic when these are mixed positive and negative terms.

To be sure, there are some similarities between the two algorithms. Both use tabular arrays. Also, (17), which defines some of the terms in Horner's array, is similar to (5), which defines terms in the basic method.

As already stated, Horner's paper was written in an obscure style. It made no mention of Viète, and did not give nor even imply (12), which is nevertheless central to the Horner method. He presented his arguments unconsecutively, and the paper has some unfortunate misprints. An anonymous reviewer discussed the paper twice [1; 2] and wrote:

> . . . the principles . . . are certainly much more elementary than we imagined when we [first] examined Mr. Horner's paper; who seduced us out of the plain path of analytical investigations . . . We are, however, convinced that Mr. H. arrived at his results by means of . . . plain and sober methods . . . and that all the foreign machinery, with which the article is introduced, was . . . brought forwards for the sake of parade, under the mistaken idea, but too often indulged, that importance is given to a subject by involving it in difficulty . . . . [2, 415]

George Peacock found the memoir " . . . very imperfectly developed, and in many parts of it very awkwardly and obscurely expressed . . ." [62, 330].

Few people can have actually used Horner's 1819 algorithm, partly because of the opacity of his exposition, and partly because it was superseded almost immediately by the much simpler algorithm of Holdred [36].

## 6. HOLDRED'S METHOD OF 1820

Theophilus Holdred, a London watchmaker, published in 1820 a tract entitled *A New Method of Solving Equations . . . with a Supplement . . .* [36]. It is the supplement that is of

most interest; however, a few words on the main part of the tract will help to fill in the background.

Holdred's (unstated) aim was to improve Viète's method by giving a systematic procedure for calculating the coefficients of a shifted polynomial in terms of the coefficients of the previous polynomial. Thus, he wished to calculate $b_0, b_1, \ldots$ in (11) in terms of $a_0, a_1, \ldots$ in (10). We know, as Holdred seemingly did not, that the $b$'s are simply Taylor coefficients:

$$b_i = \frac{1}{(n-i)!} f^{(n-i)}(\alpha) \qquad (i = 0, 1, \ldots, n). \tag{18}$$

However, Holdred discovered formulae equivalent to (18) by expanding separately the terms $(\alpha + y)^j$ appearing in $f(\alpha + y)$.

Each item $b_i$ in (18) is a polynomial in $\alpha$. Holdred showed how to calculate iteratively the coefficients in $b_{i+1}(\alpha)$ from those in $b_i(\alpha)$, and he set up a table for this purpose. The resulting algorithm is little more than an orderly way of carrying out the Viète procedure. According to Peter Nicholson, Holdred "was acquainted with the numeral exegesis to which his method bears a striking affinity . . ." [56, 35]; this means that Holdred did know about the Viète procedure, at least indirectly.

Apparently, Holdred completed the greatly superior method given in his supplement while the main part of the tract was being printed. In this new method, the coefficients of the shifted polynomial are found by setting up an array like (4), with terms given by the simple iteration (5). Thus, Holdred had the essential part of the basic algorithm of Section 3. Specifically, he wrote

> Let the equation
>
> $$gx^5 + hx^4 + ix^3 + kx^2 + lx = N \qquad [(19)]$$
>
> be proposed; write $r$ for an assumed root . . . write the coefficients in a line . . .
>
> $$g \quad h \quad i \quad k \quad l$$
>
> . . . multiply it [the coefficient of the first term] by $r$, and write the product under the coefficient of the second term ($h$); add it thereto, and write the sum underneath; then multiply that sum by $r$, and write the product under the coefficient of the third term ($i$); add it thereto, and write the sum underneath; and so on. [36, 47–49]

Here, Holdred is using the Newtonian method of polynomial evaluation to obtain the lefthand side of (19) for $x = r$. He called this the subtrahend (it is to be subtracted from the righthand side). Suppose the coefficients of the shifted polynomial are $A, B, C, D, E, F$, where $F$ is the coefficient of the highest power. Holdred continued

> . . . $B$ is formed from the numbers which are produced in forming the subtrahend in the same manner as the subtrahend is formed from the coefficients of the given equation. In like manner, $C$ is formed out of the numbers produced in forming $B$; and $D$ is formed from the numbers produced in forming $C$, and so on . . . there are never more than two numbers to be added together, in this method . . . . [36, 49, 55]

To illustrate the technique Holdred set up his table for the 9th-degree equation $275x^9 - 450x^7 + 378x^5 - 140x^3 + 15x - 1 = 0$ using the shift value $r = 0.06$. He also treated the example $x^4 - 80x^3 + 1998x^2 - 14937x + 5000 = 0$ and obtained the root estimate $x = 0.350\ 987\ 045\ 8$.

Holdred verified his new approach for a general fifth-degree equation by evaluating the terms in his array algebraically rather than numerically and comparing the resulting expressions for the coefficients of the shifted polynomial with those found in the main part of the tract. While this is not a general proof, it does provide rather convincing evidence of the validity of the method.

Ideally, Holdred should have rewritten the tract to bring the supplementary part to the forefront. But he was a poor man, and presumably would not have been able to afford to have a new version printed.

While Holdred's style was clearer than Horner's, he still had some obscurities. He did not refer to Viète. He did mention Harriot, but only to dismiss him with the following words (by involutions here he means polynomial evaluations): "The method which has (by some) been called Mr. HARRIOT's Universal Method, is so extremely tedious, on account of the tedious involutions, as to exhaust all human patience" [36, iii].

Holdred seemed to misunderstand slightly the Viète method since he used, in place of the trial divisor (14), the sum of the coefficients (15). However, both expressions are often dominated by $b_{n-1}$, so both will often yield the same trial digit.

In discussing the main part of his tract, Holdred said "My first discovery of this method was made when I was about twenty years of age now forty years ago . . ." [36, iii]. This indicates that he was born in about 1760. He also discussed how he came to discover the method in his supplement:

> The discovery which is the subject of the Supplement was made in the following manner. A Subscriber to this Work sent me an equation to solve, which I did accordingly, and went to him with two of the roots; but . . . I committed an error . . . My friend instantly shewed me, that after the sixth figure in decimal parts I differed from the author whence he had taken the equation: at which I diligently sought to find where the error had taken place. When satisfied on that subject, I next considered by what means I might have proved the root with less trouble to myself. I soon thought of the method with which the Supplement begins . . .
>
> T. HOLDRED
>
> No.2, Denzel Street, Clare Market [London]
> June 1, 1820. [36, vi–vii]

## 7. PETER NICHOLSON

Peter Nicholson (1765–1844) was an architect, a mathematics teacher or coach, and a prolific writer of textbooks [47; 64]. He was another competitor in the search for root computation algorithms. His work on these algorithms does not show much originality, but it will be worthwhile to give some excerpts since they provide information on Holdred.

In a paper published in 1818, Nicholson spoke of " . . . a Mr. Holdred, who is now about to publish a small tract, in which he has shown by an original and ingenious method, how the roots of equations as well as the roots of numbers may be accurately and easily extracted . . . His principle appears to be new, and is more general than any thing of the kind that has yet appeared in this country" [52, 348]. This confirms that Holdred was working on root computation well before Horner's paper of 1819 appeared.

By 1820, Nicholson and Holdred had quarrelled. Nicholson decided to issue his own version of Holdred's and Horner's algorithms, in the form of an essay [54]. This was published in 1820 just before Holdred's tract appeared. Here, Nicholson wrote:

> Mr. Theophilus Holdred, a gentleman but little known in the mathematical world, some time since submitted for my inspection and opinion an original tract, containing a method of finding the roots

> of equations of all degrees in numbers; but from the obscurity, want of connexion, and the antiquated manner in which the subject was treated, I was able to form but a very imperfect idea of the principles upon which his method was founded . . . After I had pointed out many defects and obscurities in his manuscript, he agreed to write the whole anew . . . The improvements produced, at last, an entirely new form in his practical operations . . . I found that Mr. Holdred, . . . had been persuaded by an acquaintance, an utter stranger to algebraic operations, to publish his own manuscript; rejecting my improvements . . . I resolved to insert what I had done in a new work entitled Rudiments of Algebra, which I was then preparing for publication . . . My Rudiments were published early in July 1819 . . . Mr. Horner's paper was published Dec. 1st 1819. I read the article attentively, and, although my mind had been so long devoted to the subject, I did not at that time fully comprehend the force of his reasoning . . . as the demonstration can be understood only by the very few . . . I resolved to try . . . to derive all that he had done from principles well known to every algebraist . . . .
>
> Mr. Holdred I have known about ten years. At the commencement of our acquaintance he showed me his general method . . . and, if I remember rightly, he said that he had been in possession of it for many years.
>
> I have had no communication with him since his determination to leave out the Appendix which I had prepared . . . by publishing the method, I have secured his credit as an original inventor . . . It is certainly remarkable that Mr. Holdred, who is a watch-maker by trade, and has spent but little of his time in the study of mathematics, and who, not being conversant with the modern improvements in the science, is unacquainted with some of the elementary principles of algebra, should have been the first (as appears probable from his age, and from the time I have known him) to have made the discovery of a general method of extracting the roots of equations . . . He was not acquainted with any work on algebra later than Ward or Ronaynes; and so partial was he to the old forms of notation, viz. that of employing the vowels $a$, $e$, $i$, $o$, $u$ &$c$. instead of $x$, $y$, $z$, that I found it difficult to persuade him to introduce the final letters for the unknown quantities . . . to a reader of modern works nothing can appear more uncouth than his manner of treating the subject . . . . [54, iii–xii]

A review of this essay and two more of Nicholson's books was given by the anonymous commentator mentioned above [2]. The essay mentioned Nicholson's *Rudiments of Algebra* of 1819; in this book [53, v–vii] he claimed to have improved on the method in the main part of Holdred's tract, which was then unpublished (he misspelled Holdred as Holdroid).

The above extracts are taken from Nicholson's preface. In the body of the essay, he expounded first a slightly modified version of the algorithm in the main part of Holdred's tract, and then the algorithm in Horner's 1819 paper. At this stage, then, Nicholson, although somewhat aggrieved by Holdred's rejection of his proposed improvements, was prepared to give him some praise for his originality.

Nicholson encountered the published version (1820) of Holdred's tract soon afterwards. He became seriously annoyed by the dismissal of his contributions. Furthermore, the new method in Holdred's supplement threatened to supersede the work in Nicholson's essay. Nicholson thereupon produced a second edition of his essay (published in the same year, 1820, as the first edition), with a 30-page postscript appended [55]. Much of this postscript is devoted to denigrating Holdred.

Holdred's reply to these comments of Nicholson included the following:

> I was master of the figurate method in theory in 1780, but poverty kept me from publishing or practising it . . . The method I have given at the end of my Supplement, Mr. Nicholson calls Mr. Horner's method; and he says I have been anticipated by himself in point of publication: but it resembles Mr. Horner's in no respect, but there being no regard had to the figurate numbers; so that I have been anticipated by no one in this method . . .
>
> In June or July 1819 Mr. Robert Gibson told Mr. Nicholson that I had made an improvement in the method of solving equations . . . I showed the method to a subscriber, Mr. Jonathan Horn, of Bowes in Yorkshire, in the house of Mr. Bayles, in the Strand, on Monday, January 3, where Mr. Horn was on a visit . . .

> In June 1819, a printer had the manuscript in hand, who declined printing it, because I could not put ten pounds into his hands before he commenced the work. An advance of money was also asked by the printer who did print it at last; and Mr. Robert Gibson prevailed upon him to begin without, by saying he would see it paid . . .
>
> I had the Supplement complete in theory, in the month of May 1819 . . . . [37, 375–378]

Figurate numbers are what are nowadays called binomial coefficients [31, 7]. By "figurate method," Holdred meant the method in the main part of his tract, i.e., an algorithm making use of binomial coefficients. Assuming the truth of his last remark, in arriving at the basic method, Holdred could not have been influenced by Horner's paper, which was read on July 1 and published in December 1819.

Nicholson published a reply [56] to this note of Holdred's. Here, he rightly criticized Holdred for expressing the trial divisor in the Viète procedure as a sum of coefficients (see (15)), and for the incompleteness of Holdred's proof. He also stated that with respect to the method in Holdred's supplement "Mr. Holdred has no claim to style himself the original inventor of it, as I was the first to give him any hint of it . . ." [56, 39]. In subsequent papers [57–59], Nicholson gave a somewhat labored proof of the Holdred method of shifting a polynomial. He did not call it that, but he adopted a more conciliatory tone:

> The world has been much indebted to Mr. Holdred, as being . . . the first who invented a general method of extracting the roots of equations of all orders . . . I have never made any pretensions to the discovery of Mr. Holdred, or of any other individual; but I solemnly affirm that upon my seeing his figurate method, I discovered the non-figurate mode from a consideration of my general method of transforming functions, published in my Combinatorial Analysis in the year 1818 . . . . [59, 434]

In this book, *Essays on the Combinatorial Analysis* [51], Nicholson treated identities like

$$\begin{aligned} &a_0(x+\alpha_1)(x+\alpha_2)(x+\alpha_3)+a_1(x+\alpha_1)(x+\alpha_2)+a_2(x+\alpha_1)+a_3 \\ &\quad\equiv b_0(x+\beta_1)(x+\beta_2)(x+\beta_3)+b_1(x+\beta_1)(x+\beta_2)+b_2(x+\beta_1)+b_3. \end{aligned}$$

Here, $a_0, a_1, a_2, a_3; \alpha_1, \alpha_2, \alpha_3; \beta_1, \beta_2, \beta_3$ have given numerical values, and the problem is to calculate the coefficients $b_0, b_1, b_2, b_3$. For this purpose, Nicholson set up a tabular algorithm which is very similar to Holdred's algorithm for shifting a polynomial; indeed it reduces to Holdred's algorithm in the special case where $\alpha_1, \alpha_2, \ldots$ are all zero and $\beta_1, \beta_2, \ldots$ are all equal. However, in the book Nicholson did not mention this case; furthermore, he applied his algorithm to the summation of series rather than to root computation. Thus, his claim to have anticipated Holdred has little justification.

## 8. J. R. YOUNG

John Radford Young (1799–1885) was a mathematician, largely self-educated, who first taught at an establishment for the deaf and dumb in London, and later became a professor of mathematics at Belfast College [16]. He wrote some 18 books and over 40 papers.

His first book was *An Elementary Treatise on Algebra* (1823), and on pages 198–202 he gives the method of Holdred's supplement, i.e., what we have called the basic method. He also writes of Holdred: "When this gentleman's tract on equations first appeared in 1820, I was much struck with the simplicity of his method of solving the higher equations, as contained in the supplement of that tract . . ." [79, x]. Young also mentions Horner's method of 1819, but makes no attempt to describe it.

However, in his subsequent book, *Theory and Solution of Algebraical Equations* (1835), Young gives the credit for discovering the basic algorithm to Horner. He says: "Mr. Holdred's work contains two modes of solving a numerical equation, one a tedious method . . . and the other printed as a supplement . . . Between this latter method and the one published by Mr. Horner, there is a remarkable resemblance . . ." [80, viii–ix]. We can only speculate on what caused Young's change of mind in favor of Horner. Presumably, he had failed to understand Horner's paper of 1819, and assumed it to contain the basic method as given in Horner's subsequent paper [41], which will be discussed below.

## 9. HORNER'S SUBSEQUENT PAPERS

After Holdred's tract appeared, Horner wrote papers in which he claimed for himself the method in Holdred's supplement. These papers, written in the early 1820's, were not published until much later.

A letter from Horner to Thomas Leybourn, editor of the *Mathematical Repository*, was dated 20 January, 1821 but not published until 1830; it makes clear that Horner had read Holdred's tract: " . . . I am obliged to you, Sir, for a sight of Mr. Holdred's late publication . . . and am exceedingly surprised to see annexed . . . a supplement detailing, as the latest discovery of the author, MY NONFIGURATE METHOD!!! . . ." [40, 39] (Horner's emphasis). Horner is taking over Holdred's method here, although, as we have seen, it is substantially different from that in Horner's 1819 paper.

Horner also published a series of notes entitled "Horae mathematicae" (nevertheless written in English). The first of these was dated July 1820, but they were not published until 1830 [41]. In the fourth of these notes, dated 11 June, 1821, he gives without acknowledgment the Holdred method of obtaining the coefficients of a shifted polynomial. However, he provides an improved and simple proof of the technique and subsequently calls it *synthetic division*. He claims that the rule is a "strict verbal interpretation of the Theorem in Art. 14 of my original Essay" [41, 42]; but his article 14 has no theorem as such and has no obvious connection with the rule in question.

Horner also submitted a further paper to the Royal Society of London, where it was read on 19 June, 1823, but was not accepted for publication. This again gives the method of synthetic division, without referring to Holdred. Thomas Davies eventually published the paper in *The Mathematician*, of which he was an editor [20; 43]. In introducing the paper Davies wrote:

> Mr. Horner's first paper on equations was printed in the Philosophical Transactions for 1819; and Mr. Horner has often stated to me, that much demur was made to the insertion of it in that publication . . . The *elementary character* of the subject was the professed objection; his recondite method of treating it, was the professed passport for its admission . . . the mode in which it is drawn up . . . may be considered unfortunate . . . Mr. Horner . . . immediately attempted a simplification of the principles. The consequence of this attempt was the paper now about to be submitted to the public for the first time. [43, 108]

The paper [43] is still difficult to read. However, a readable exposition of synthetic division appears in the same journal, under the authorship of "D. V. S." [70]. (Query: Is D. V. S. a disguised form of the name Davies?)

T. S. Davies also edited a reprint of Horner's 1819 paper in the *Ladies' Diary* [42]. This reproduces without comment the misprints of the original, so it seems likely that Davies did not fully understand Horner's arguments.

Thomas Stephens Davies (1795–1851) was a mathematics master in the Royal Military Academy at Woolwich [8]. (James Joseph Sylvester had a chair there subsequently, see [67].) At one time Davies resided in Bath, and was presumably an acquaintance of Horner's. After Horner died, his papers were entrusted to Davies, who intended to produce a collected version of Horner's works. He did not succeed in this. However, as mentioned by De Morgan [28, 2:151–153], a few letters from Horner's correspondence were passed from Davies to De Morgan, and they are now in the library of London University. Some extracts are reproduced in the Appendix.

## 10. DE MORGAN

Augustus De Morgan (1806–1871) was a prominent figure in 19th-century English mathematics. He was professor of mathematics at University College, London from 1828 to 1831 and from 1836 to 1866, twice resigning from the chair on matters of principle. His main original work was in the field of logic; but he also gave much attention to the history of mathematics [66], and in particular to the history of root computation. He was a prolific writer, contributing around 850 articles to the *Penny Cyclopaedia*, and writing regularly for at least 15 periodicals [30].

It is mainly due to De Morgan that the basic method (Holdred's) caught on in England; however, De Morgan ascribed it to Horner. In 1839, he gave a history of the problem of root extraction, or "evolution" as it was then called. There he noted that "... a tract was published by Mr. Theophilus Holdred ... with a supplement ... being neither more nor less than the method which Mr. Horner had put in print more than six months before" [22, 42]. This remark indicates that De Morgan had failed to understand Horner's 1819 paper, and that he had probably assumed the equivalence of Horner's algorithm with that given in Horner's 1830 paper [41]. Curiously enough, in 1836 De Morgan had approved of Holdred sufficiently to refer to him for a method of evaluating cube roots [21, 51–57].

In his article on "Involution and Evolution" in the *Penny Cyclopaedia*, De Morgan [23] gave a fairly clear account of the basic method of root computation. He again credited the method to Horner (1819), and did not mention Holdred. De Morgan returned to the history in an article in the *Supplement to the Penny Cyclopaedia*. He wrote: "... unfair attempts were made by claimants who had no title whatever to deprive the author [Horner], who was a man of real genius, of his rights over his own discovery. We refer to MM. Holdred and Nicholson, though we do not believe the second was knowingly unfair ... the first elementary writer who brought Horner's method into instruction was Mr. (now Professor) Young, in his 'Elements of Algebra,' published in 1823" [24, 106]. De Morgan failed to mention that Young then called it Holdred's method, not Horner's.

In the *English Cyclopaedia*, De Morgan [27] repeated his exposition and history of what he calls Horner's method. He also gave, as an indication of what can be done with the method, a solution of Newton's example, $x^3 - 2x - 5 = 0$ to 152 decimal places, obtained by John Power Hicks, a student of De Morgan's [28, 2:66–68]. (Actually Hicks's last 12 digits are incorrect—they should be 173 965 531 394.)

De Morgan took every opportunity to publicize Horner's work [28, 2:187–189]. He even included Horner's 1819 paper in his list of arithmetical books [25, 89], although Horner's article was not a book. At the same time, he omitted Holdred's and Nicholson's tracts, which actually could be considered as books. Yet he wrote: "The only reason for a work being in

this list, is that it has come my way; the only reason for one being out of it is, that it has not come my way . . .” [25, x].

## 11. SOME TWENTIETH-CENTURY COMMENTATORS

Cajori performed a useful service in drawing attention to Ruffini’s work. However, he accepted without question De Morgan’s version of the history, calling it “the standard reference to early English research on the subject” [15, 410]. Earlier, he thought that the arrangement in Horner (1819) was “not quite the same as the one explained in text-books . . . but the difference is slight” [14, 248]. Nordgaard [60, 53–57] also accepted that in England Horner invented the basic method, and referred to De Morgan for the history.

Edmund Whittaker and George Robinson [78, 86, 100–106] called the basic method the Ruffini–Horner method, and said incorrectly that Horner had discovered the rule in 1819.

David Smith, in his valuable *Source Book in Mathematics* [71, 1:232–252], gave a slightly abridged version of Horner’s 1819 paper. Margaret McGuire edited the paper for Smith. Neither Smith nor McGuire seem to have understood it properly, since they did not correct or mention the misprints. To her credit, though, McGuire indicated her unease with the article, for she wrote: “ . . . The modern student of mathematics will notice at once the length and difficulty of Horner’s treatment when comparing it with the simple, elementary explanation in modern texts” [71, 1:232].

Julian Coolidge included Horner in his *The Mathematics of Great Amateurs* [19, 186–194], but felt the need to apologize for this, saying: “ . . . A great mathematician he certainly was not . . . He offers a fine example of what an amateur can accomplish by dogged industry, and his method is surely the best we have for solving numerical equations . . .” [19, 186]. Coolidge gave some extracts from Horner (1819), but soon tired of this, commenting: “ . . . From this he finally gets an elaborate rule which I will not write, as it does not seem to be of much practical use . . .” [19, 188]. Thus, Coolidge seems to be one of the few commentators (who include Harold Jeffreys and Bertha Jeffreys [44, 696–697]) to realize that the algorithm in Horner’s 1819 paper is not substantially the same as the basic method. But Coolidge was inconsistent on this, since later he implied that in Horner [43] “we get the fundamental method” of Horner (1819). He added that Horner “ . . . becomes excited over the accusation that a certain Mr. Holdred had anticipated his work . . .” [19, 192]. However, Coolidge made no attempt to investigate the controversy.

## 12. CONCLUSIONS

In England, the basic algorithm conventionally called Horner’s method was first given by Holdred in 1820, and was not published by Horner until 1830. The method in Horner’s 1819 paper uses a different and more complicated algorithm. Horner pretended that the basic method could be found in his 1819 paper, but this is not the case. He was able to make this claim because the paper was written so obscurely that few people had the patience to study it thoroughly.

At first sight, Horner’s plagiarism seems like direct theft. However, he was apparently of an eccentric and obsessive nature (see Appendix). Such a man could easily first persuade himself that a rival method was not greatly different from his own, and then, by degrees, come to believe that he himself had invented it.

De Morgan was Horner's chief protagonist, and it is surprising that he did not see through Horner's claim; one can only assume that he, like others, was baffled by the obscurity of the 1819 paper. According to Rupert Hall [34] "... De Morgan, though ever painstaking in the pursuit of information, and ever with a critical eye open, tended to base his decision on judgement rather than certain knowledge..." For example, De Morgan "... became the personal friend and ardent champion of ... Guglielmo Libri whose zeal in recruiting his own splendid collections of manuscripts from the public libraries of France is now widely admitted ...." Again, the biography of De Morgan written by his wife Sophia quoted Herbert Spencer to the effect that "many ... have remarked on the perversity of Professor De Morgan's judgements ..." [29, 162–163]. It is fair to add that Spencer had been stung by an implicit criticism from De Morgan.

## APPENDIX. LETTERS FROM CONTEMPORARY MATHEMATICIANS

London University Library holds a number of letters which show something of the characters of Horner and Holdred, and also illustrate the circumstances in which mathematicians worked in the 19th century. The letters and some related items are filed under L2[Horner]fol. Some of the letters are reproduced below; here unreadable words are represented by [...], and doubtful words are indicated thus: [Souter?].

*Letter from T. Leybourn to W. G. Horner, 6 September, 1821*

To Mr. W. G. Horner
Grosvenor Place
Bath

R. M. College [Royal Military College]
Bagshot
6th September 1821

My dear Sir,

The day before the Coronation I sat down intending to reply to your letter of the 11th June: but before pen, ink & paper were ready some thing or other came in the way and prevented me from going on.

In the Review to which I referred you through Mr. Glendinning you would find Notices of several of Mr. Nicholsons Books in the course of which Notices your paper was referred to more than once. The Rev. I alluded to was that for Dec$^{r}$ 1820. When I was in London in July I understood from Mr. Nicholson that he was preparing a reply to your observations as related to himself—I informed him that I had just before received a letter from [name omitted] on the subject and that I expected another; in which it was probable that you would say all you intended on the matter in question—

When I was in London Mr. Edward Riddle of Newcastle informed me that Mr. Henry Atkinson of that place, ten years ago, had made similar discoveries, and that he had read a paper on the subject to the Philosophical Society of that place and which paper is still in existence and that some time ago he had transmitted a copy of it to Mr. Tilloch for insertion in the Phil$^{l}$. Magazine, but who had some difficulty as to whether he should print it or not. Mr. Riddle and myself made Mr. Holdred acquainted with the nature of Mr. Atkinsons paper—he, Holdred, seems a harmless inoffensive person. It appears to me that he is not a likely person to have made any use of your paper—I am persuaded he is not in a condition to read it; but I may be mistaken. I forgot to mention above that at the time Mr. Atkinson made his discoveries he communicated them to Dr. Hutton who recommend [*sic*] their being published in some of the periodical journals: adding however that he did not think them of much importance—this last observation of the Dr. induced Mr. A$^{n}$. to think he had overstated matters and of course he thought no more about the matter till very lately. I expect Mr. Glendinning will soon be in a state to proceed on the 19th [...] of the Repository. It would be better I think if the Horae articles [...] were completed before any other article [...] upon—if you are of the same opinion perhaps you will take an early opportunity of sending me the remd$^{r}$. [remainder] of your articles on the subject. I expected to [...] on [...] months ago, but Glen. has for a long time had "an old house about his ears"—being obliged to take down his house & rebuild it. A horrible job—

I shall take this opportunity of saying that I have no doubt whatever as to your right to the invention of the new modes of approximations which has been so much discussed of late. But I do not think that Holdred could have taken from you, being persuaded in my own mind from what I have seen of him that [. . .] was not in a condition to do so—indeed from what seems to have passed between him and Mr. Nicholson, his performance in the first instance was a very lame one.

Y$^{rs}$ very truly
T Leybourn

According to Holdred's acct. [account] he has been in possession of his method for 40 years—after such a length of time to appear in such a rough uncouth state as when Nicholson first saw it, to say the least of it is very surprizing.

Thomas Leybourn (1770–1840) was professor of mathematics at the Royal Military College, Sandhurst; see Charles Platts [63] and Niccolò Guicciardini [33, 114–116, 157]. Glendinning was the printer of the *Mathematical Repository* which Leybourn edited. Henry Atkinson's tract [4] was read at Newcastle in 1809 and published in 1831. It made no attempt to treat what we have called the basic method of root computation, so is not of immediate interest to us.

*Part of a Letter from W. G. Horner to T. S. Davies, Undated*

[The main part of this letter is missing.]

P.S. You wished to "call me out," you say. I hope I have "given you complete satisfaction," as the phrase is. But you will perceive throughout the whole affair, that "through your sides" I have been [shooting?] at the public, when you wish to be spectators of the "affair." *Peacock* himself could not complain of the "imperfect development" of my views; although he might find them "awkwardly expressed," and you may feel the lack of profundity which pervades my scribble; no $\Delta$s, nor $\partial$s, nor $f$s nor $\phi$, nor even $\sqrt{-1}$; all smooth superficiality, as a skating pond. But to continue this "[. . .]," I think it will bear you, if you break through, you'll be glad to get to the surface again. I learn from William, that your *conundrum* is in Carrington's [last?]. I have not thought requisite to make any allusion to it, as the enquiry is general, and [think?] as well left out of sight. If you consider the paper to be suitable for publication, I leave it with you to "transfer it" to Phil. [. . .]. Strike out any expressions which you may think [. . .] [. . .]—any such, I mean, as seem to have the air of teaching *you*. Do not let me appear to the world as a coxcomb.

Had you not thrown this apple my way, I should very probably have sent Phillips a few strictures on [Fourier?]. But that business is itself but the apple of discord, and I regret any delay in putting my "improvements" out of my hands. Have you sounded [Souter?].

The curious style of this note suggests that Horner was somewhat eccentric; the indirectness is characteristic also of his papers. The William referred to is presumably Horner's son, and the fact that he did not explain this to Davies suggests that they were close acquaintances. William took over his father's school after Horner senior died. Souter might be John Souter, the London publisher of some of J. R. Young's books.

*Letter from P. Barlow to (Presumably) A. De Morgan, 10 December, 1845*

Woolwich [London]
10$^{th}$ Dec$^{r}$ 1845

My dear Sir

I have a distinct recollection of receiving the letter alluded to in your note but I cannot find it nor can I give you the date; it may [be] soon after the publication of my tables. In it M$^{r}$ Horner stated in very obscure terms the nature of his solution which he illustrated in some way by continued fractions. It in no way resembled in form [or indeed?] in principle what he afterwards published in the Phil. Trans. I could make nothing of his illustration and I suppose set little store by the letter—

M$^{r}$ Horner seems to have suspected that by some means M$^{r}$ Peter Nicholson had seen this letter and that he then acquired the first ideas of his method but this was certainly not the case—I know very little

of P Nicholson and had no correspondence with him till the dispute between him and M$^{r}$ Horner except so far that he taught my two sons Architecture and Perspective Drawing, taking their lessons three times per week in [town?].

I should observe that in saying I remember M$^{r}$ Horners letter—I allude to his illustration referred to above, but I do not remember whether or not it had reference to Budans work. I remember M$^{r}$ Davies while residing in Bath writing to me for the loan of Budans book but I was not aware of M$^{r}$ Horner having done so.

I am happy to say I am enjoying excellent health, but my pursuits are and have been for some time more practical than abstract; at present the Broad and Narrow Gauges are the hobbies I am riding.

I am Dear Sir
Yours very truly
Peter Barlow

Peter Barlow (1776–1827) was a professor of mathematics at the Royal Military Academy, Woolwich [18; 33, 108–121, 157]. He compiled a set of mathematical tables which were first published in 1814 [5]. They were subsequently published in various editions, and were still being printed in 1971 [6]. It is unfortunate that when De Morgan edited the 1840 edition, he omitted some of Barlow's introductory matter, which included a useful account of the history of root computation.

François-Désiré Budan de Boislaurent (1761–1840) published in 1807 an algorithm for root computation [11], which was related to the basic method. In his technique, all polynomial shifts are by unity. Thus, when the units digit of the root is sought, the values $0, 1, \ldots, 9$ are tested successively until the polynomial value is found to change sign. To find the next digit of the root, all roots are first multiplied by 10 (thus the coefficients are multiplied by powers of 10), and then the new units digit is sought. Further digits can be found by the same technique. Budan used array (4), but to construct its terms he did not simply apply (5) with $\alpha = 1$. Instead, he used the equivalent but lengthier relation $p_{ij} = p_{i-1,0} + p_{i-1,1} + \cdots + p_{i-1,j}$.

Budan's algorithm has the advantage that no multiplications are required (apart from the trivial multiplications by powers of 10). However, each evaluation of a root digit requires up to nine shifts to be carried out, and this makes the method somewhat tedious. In his 1819 paper, Horner referred to Budan's method, saying "its extremely slow operation renders it perfectly nugatory" [39, 311]; but later he thought that it was useful for preliminary numerical exploration of root locations [43, 141–142].

In 1822, Budan published a revised version [12] of his book. Here, he presented the tabular method (see (4), (5)) for calculating the coefficients of a shifted polynomial, this time for a general shift value [12, 99–100], thus generalizing his earlier treatment. However, he seemed to regard this transformation as of academic interest only, and for purposes of root computation reverted to his previous transformation using unity shift values [12, 102].

For further discussion of Budan's work, see Jacques Borowczyk [9], Ivor Grattan-Guinness [32, 1:244–248], and Jean-Luc Chabert *et al*. [17, 263–270].

*Letter from J. Horner to A. De Morgan, 20 October, 1862*

Everton Vicarage, N$^{r}$ S$^{t}$ Neots
Oct$^{r}$. 20. 1862

Dear Sir,

Returning home on Saturday last from a week's holiday, I was pleased and perhaps a little elated at receiving your kind letter and its accompaniments. I will reply [briefly?] now with the intention of [. . .] the advantage of your correspondence [. . .] at length on future occasions.

William George Horner (George being the surname of a relation of his mother's) was the eldest of ten children of whom I was the ninth. 'The relative into whose hands you returned his papers many years ago' must have been his son William Horner, The Hermitage, Bath, who has a flourishing school there, and will I am [sure?] be glad to answer any questions you may at any time wish to make about his Father, or his scientific remains. At the death of W. G. Horner, my nephew aforesaid was induced to entrust all his Math$^{l}$. papers and books to the T. S. Davies you mention, that he might examine and if desirable put them in the way of publication. Poor Davies I suppose found Mathesis not a very gainful mistress, and got into difficulties in consequence of which I suppose my brother's papers and books got dispersed or brought under the hammer. My nephew repeatedly wrote to his widow but c$^{d}$. [could] get no reply. This will explain the fact of your having met with some books that have the W.G.H. mark upon them. I particularly remember a Bonnycastle's Algebra in two volumes interleaved, which contained some of his operations in ink, but I have not been able to trace the copy. My own impression has been that beyond his published papers there was not much behind that w$^{d}$. [would] have been of particular service to science; but I have no *especial* grounds for that opinion.

My eldest brother's widow still survives at 33, Grosvenor Place, Bath, where she resides with her only surviving daughter M$^{rs}$. Harrison. My brother had a family of six daughters & two sons of whom only the pair to whom I have alluded are living. I have read in some publications [of yours?] which I do not this moment identify a wish expressed, of the same kind that your letter conveyed, that more were known of W. G. Horner and I felt half disposed to open a communication with you on the subject. Now that you make a direct request, I will endeavour to put together a few facts that may be interesting to studious men. I have been rather deterred, from the feeling that one cannot shew a lion without also shewing the showman.

My brother was a very peculiar man, and though my early association with him was long, it was neither very [reserved?] while it lasted nor very happy [at?] its close. Consequently I know really less about him than might readily be supposed. He died at the latter end of the year 1837, having about completed his 53$^{d}$. year. I had not known much of his pursuits for about $6\frac{1}{2}$ years previous to that event.

During that time, I had gone up late in life to Cambridge, and was fighting my way amidst every difficulty to a degree. The attainment of that much altered my position & my prospects. A fellowship soon followed and a small living here succeeded which was tenable with it; but I vacated the fellowship by marriage & have ever since united the cares of a small parish with those of preparing young men for the Universities. You will think I am giving you my history instead of my brother's, you may however consider these facts as a useful preliminary to any thing that may by [. . .] follow.

You mention a paper of mine on the numerical solution of Equations in a series of [. . .] aliquot parts. There is another in the 14$^{th}$ &15$^{th}$ N$^{os}$ Oct 1860 & Feb 1861, which I should like to ask your attention, if you have not looked it through. In it I had occasion to refer several times to y$^{r}$ Diff$^{l}$. & Int$^{l}$. Calculus [your *Differential and Integral Calculus*], and in one short note to make my acknowledgments to it as supplying a cup of water to set my pump at work. The reference to my kindred with W. G. H. which caught your attention last week was made from finding that our common friend M$^{r}$. Todhunter of S$^{t}$. John's [College, Cambridge] had taken me formerly to be that very person, and as I had occasion to use my brother's method of transformation, I thought it w$^{d}$. [would] be unjust to his memory and to my own claims to simplicity of character if I allowed an opportunity of removing that confusion about names to pass without availing myself of it.

I thank you very much for the printed matter you have sent me. I have not yet made myself fully master of it. For some months I have been gradually concocting some crude ideas on the general theory of symmetrical $\phi^{ns}$. [functions] of the roots of Equations, which I hope a period of temporary leisure will soon enable me to offer to the Quarterly Journal for publication. Believe me

Dear Sir,

Your much obliged

Joseph Horner

Here, Joseph Horner's material has been split into paragraphs, to improve readability. It is significant that the only description of W. G. Horner given by his brother Joseph is that he "was a very peculiar man." Joseph Horner published about ten papers in the *Quarterly Journal of Pure and Applied Mathematics* during 1860–1873. Isaac Todhunter, who is

mentioned above, gives in his *Theory of Equations* (1867) a clear account of the basic method of root computation [72, 150–164]. But he ascribed this to Horner, referring to De Morgan [22] for the history.

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