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FAULT DIAMETER OF INTERCONNECTION NETWORKS

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Abstract—We introduce the concept of fault diameter of interconnection networks. The motivation is to estimate the degradation of performance under maximally fault conditions. We derive fault diameter of specific classes of commonly used interconnection networks and various types of product graphs.

1. INTRODUCTION

A number of researchers have investigated the design of fault tolerant interconnection networks. A common notion of fault tolerance is based on the connectivity of the underlying graph. Thus, an interconnection network with fault tolerance f, is guaranteed to remain connected even if f vertices are faulty. However, while such a network would remain connected under f faulty vertices, the resulting diameter of the graph—a measure of the communication delay—might significantly increase.

In this paper, we introduce the concept of a fault diameter of interconnection networks relating to the maximum path length among all vertex disjoint paths. The motivation, of course, is to estimate the degradation of performance (with respect to communication delays of the network under maximally faulty conditions). We consider two types of associated problems:

(1) What is the fault diameter of specific classes of commonly used interconnection networks, such as the n-cubes, cube connected cycles (see [9]) and generalized Petersen graphs?

(2) How are the fault diameters of various types of product graphs related to the fault diameters of the constituent graphs? In particular, we consider Cartesian products of graphs.

One of the well studied class of interconnection networks is the n-dimensional Boolean cube. It has 2^n vertices with degree n and diameter n. Even though there exist denser graphs with more vertices for the same degree and diameter (see [5]), the n-cubes remain an attractive class of interconnection networks. We show in this paper that the fault diameter of the n-cube is $n + 1$; indicating that even under maximally faulty conditions its performance will not degrade drastically. We believe that this intrinsic property of the n-cube is its unique and attractive feature.

We then consider the cube connected cycles suggested in [9]. While at first sight the presence of cycles suggests that its fault diameter would be significantly greater than its diameter, we show that it increases only by *three*! Finally, we provide bounds for the fault diameter of Cartesian products and generalized Petersen graphs.

In Section 2 we provide the necessary definitions and notations, followed by the main results in Section 3.

2. TERMINOLOGY

We adopt standard graph theoretic terminology as in [7]. In particular:

Notation. For a graph G we denote: V_G = the vertex set; E_G = the edge set; N_G = number of vertices; d_G = the degree; k_G = the diameter; c_G = connectivity; f_G = fault diameter, whose definition is: Let C_G be the vertex connectivity of a given graph. Then there exists at least C_G vertex disjoint paths between every pair of vertices. The fault diameter is the largest diameter obtained by deleting a set of $(C_G - 1)$ vertices. We denote the cross product of two graphs G and H by $G \times H$ and the composition of G and H by $G(H)$. In $G(H)$, every vertex in G is replaced by a copy of H.

Definition. We say that a class of graphs $\{G_i\}$ is *strongly resilient* if there exists a constant t such that $f_G \leqslant k_G + t$ for all i. We say that $\{G_i\}$ is *weakly resilient* if there exists a constant t such that $f_G \leqslant t \cdot k_G$. Observe that complete multi-partite graphs and rectangular grids are examples of strongly resilient classes of graphs, whereas the n-cycles form a weakly resilient class of graphs. This is so because in complete multi-partite graphs and in rectilinear grids, the maximum path length in any vertex disjoint path is greater than the diameter of the graph by a constant amount. Similarly in n-cycles, if we take the adjacent vertices and consider the 2-vertex disjoint paths, the fault diameter is twice the diameter of n-cycle.

THEOREM 1

If, between every pair of vertices in G, there exists c_G vertex disjoint paths, each of length at most f, then $f_G \leqslant f$.

Proof. Easy.

The converse of the above Mengerian-type theorem is not true (see [3, 4, 7]).

3. MAIN RESULTS

We first derive bounds for the fault diameter of Cartesian products of graphs. Observe that $G \times H$ can be viewed as many copies of G interconnected by an H graph; or, as many copies of H interconnected by a G graph. In any case, the diameter of $G \times H$ is easily shown to be $k_G + k_H$, and its connectivity can be shown to be at least $c_G + c_H$. Furthermore,

$$c_{G \times H} \leqslant \min \{d_G + d_H,\ c_G N_H,\ c_H N_G\}.$$

THEOREM 2.

For graphs G and H, if $c_{G \times H} = c_G + c_H$, then $f_{G \times H} \leqslant f_G + f_H$.

Proof.

Case 1. $g = g'$ or $h = h'$. Without loss of generality let $h = h'$. We wish to find $c_G + c_H$ vertex disjoint paths, each of length at most $f_G + f_H$ from (g, h) to (g', h). Clearly in the copy of G labelled by h there are c_G disjoint paths of length at most f_G from (g, h) to (g', h). We now construct additional c_H disjoint paths. Since the connectivity of H is c_H, the degree of h in H is at least c_H. Let the vertices adjacent to h in H be $h_1, h_2, \ldots, h_{c_H}$. These additional c_H paths go from (g, h) to (g, h_i) for $i = 1, \ldots, c_H$. In each of the G components corresponding to (g, h_i) we take the shortest G path from (g, h_i) to (g', h_i), and finally through one additional edge to (g', h). All these c_H paths are vertex disjoint and of length at most $k_G + 2$. (Since $K_G \leqslant f_G$ and $2 \leqslant f_H$, we have the desired inequality that $K_G + 2 \leqslant f_G + f_H$.) That completes Case 1.

Case 2. $g \neq g'$ and $h \neq h'$. We need to prove that there are $c_G + c_H$ paths of length at most $f_G + f_H$. Since the fault diameter of G is f_G, there exists c_G vertex disjoint paths from g to g' in G each of length at most f_G. We select one of these paths as a designated G-path from g to g'. Similarly, we select a designated H-path from h to h' length at most f_H. Using this we construct three classes of paths.

In the first class we construct $c_G - 1$ paths from (g, h) to (g', h'). Recall that there exists $c_G - 1$ vertex disjoint paths from g to g' in$G$, other than the designated G-path. Let $g_1, g_2, \ldots, g_{c_G} - 1$ be the last but one vertices of these $c_G - 1$ paths in G. We use these paths in G to construct as many paths from (g, h) to (g_i, h), and further on to (g_i, h') using the designated H-path. Finally, with one additional edge, each of these $c_G - 1$ paths are completed to (g', h').

In the second class we construct $c_H - 1$ paths from (g, h) to (g', h'). Recall that there exists $c_H - 1$ vertex disjoint paths from h to h' in H, other than the designated H-path. Let $h_1, h_2, \ldots, h_{c_H} - 1$ be the first vertex encountered in each of these $c_H - 1$ paths respectively. We use these paths in H to construct as many paths from (g, h) to (g, h_i); on to (g', h_i) using the designated G-path; and finally to (g', h') using the remainder of each of the $c_H - 1$ vertex disjoint H paths.

In the third class we construct two additional paths using the two designated G and H paths. The first path is constructed using the designated G-path to go from (g, h) to (g', h); and then using the designated H-path to go to (g', h'). The second path is constructed by using the first edge of the designated H-path to go from (g, h) to (say) (g, h^*); then use all but the last edge of the designated G-path to go to (say) (g^*, h^*); then using the remainder of the designated H-path to reach (g^*, h'); and finally using the last edge of the G-path to reach (g', h').

These $c_G + c_H$ paths are illustrated in Fig. 1. The reader can easily verify that all of these paths are vertex disjoint.

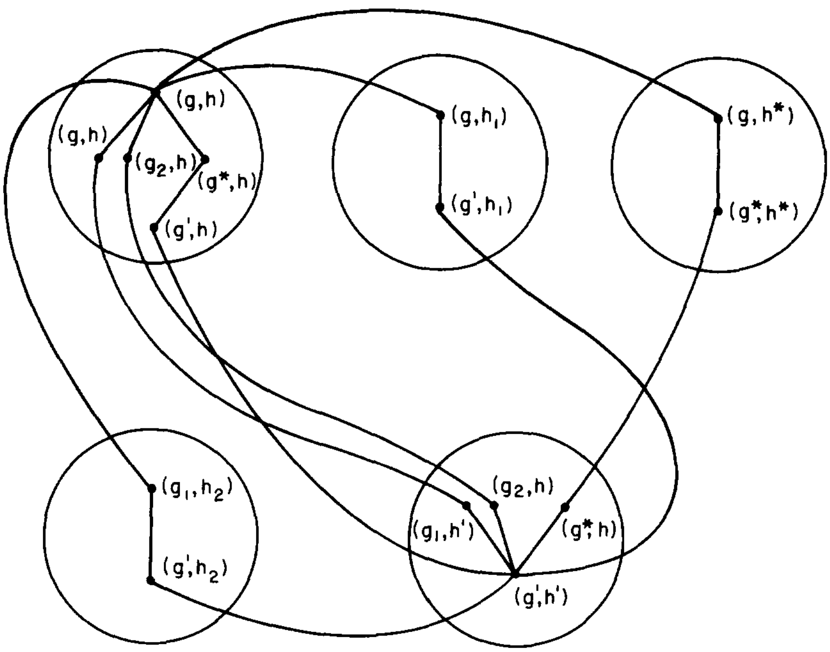

Fig. 1

The bound on the fault diameter of the Cartesian product provided by the above theorem can be improved. Note that the fault diameter of a graph establishes the existence of c_G vertex disjoint paths, each of length at most f_G, between any pair of vertices X and Y in G. However, none of these paths might correspond to the shortest path between X and Y. Consequently, each of these path lengths might be larger than the diameter.

Definition. We say that a graph G is *tight* if for every pair of vertices X and Y there exists c_G vertex disjoint paths between X and Y, each of length at most f_G, and at least one of these paths is of length at most k_G, the diameter of G.

Corollary 1

In Theorem 1, if G and H are tight then the fault diameter of $G \times H$ is at most $\max(f_G + k_H, k_G + f_H)$.

Proof. Using the definition of tight, we can choose the designated G and H paths to be of length at most k_G and k_H, respectively.

Corollary 2

The fault diameter of the n-cube is exactly $n + 1$.

Proof. By induction on n. Observe that the $(n + 1)$-cube is the Cartesian product of the n-cube (G) with the l-cube (H). (Note that the l-cube is also the complete graph on two vertices.) Furthermore, both G and H are tight, and the proof follows using Corollary 1.

We now derive the fault diameter of cube connected cycles. Recall (see [8]) that an n-dimensional cube connected cycle is the composition of an n-cube with an n-cycle; i.e. and n-cube with each of its vertices replaced by an n-cycle. The new vertices can then be labelled by a pair $(Z;i)$, where Z is an n-bit vector and $1 \leqslant i \leqslant n$. The vertex $(Z;i)$ is adjacent to $(Z;i+1)$, $(Z;i-1)$ and $(Z';i)$, where (1) $i + 1$ and $i - 1$ are computed cyclically, and (2) Z' is the bit vector that differs from Z in the ith bit only.

Observe that its connectivity is 3. The diameter of the cube connected cycle is given by the following theorem, stated without proof.

Theorem 3

The diameter of the n-dimensional CCC is $[(5n - 2)/2]$.

Theorem 4.

The fault diameter of the n-dimensional cube connected cycle is at most $[(5n + 4)/2]$.

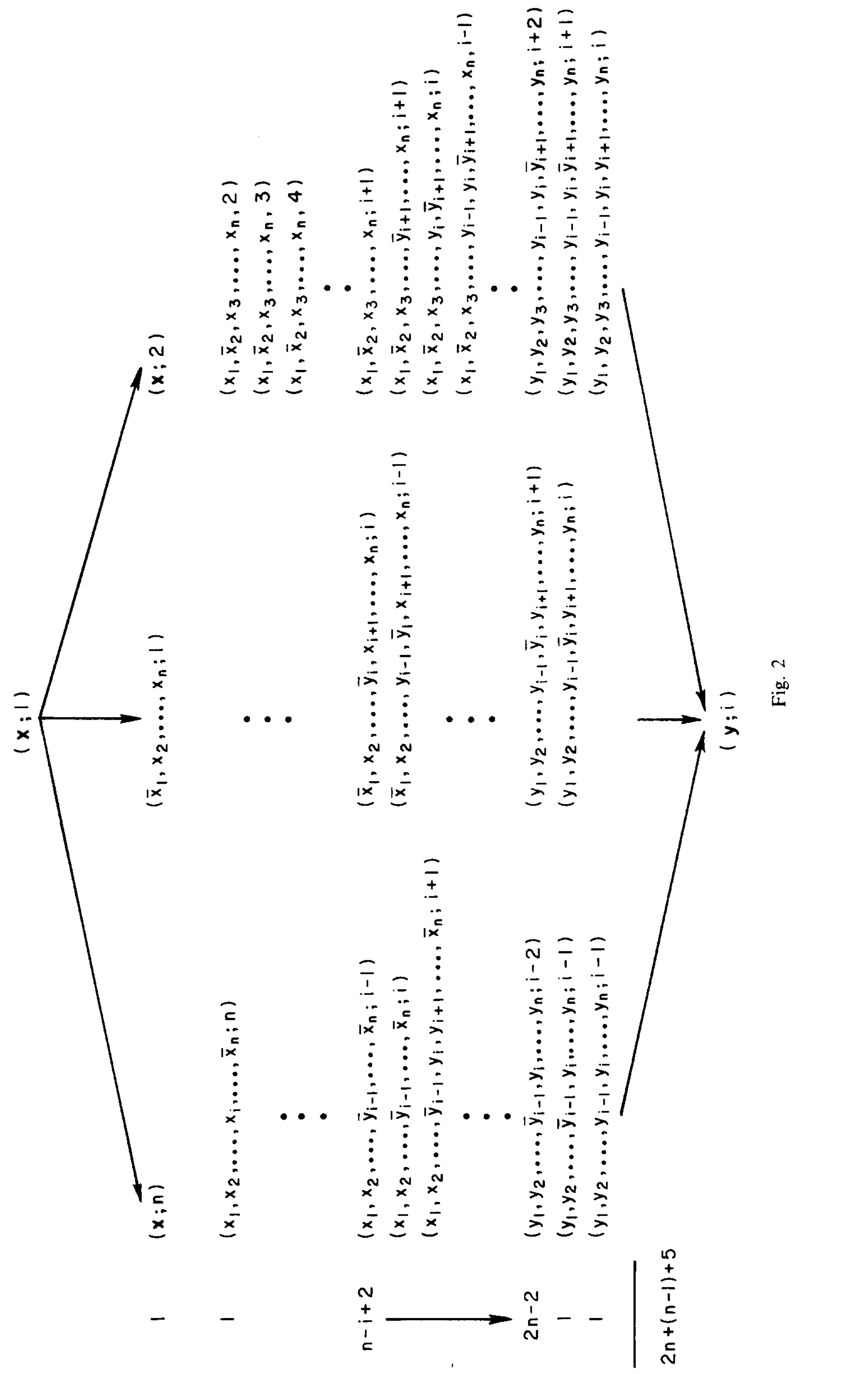

Fig. 2

Proof. Once again, we demonstrate 3 vertex disjoint paths, each of length at most $(5n+4)/2$, between any pair of vertices. Let X and Y be the pair of critical vertices for the fault diameter. Without loss of generality, we can assume that X is the form $(x; 1)$, and Y be of the form $(y; i)$. The proof of the theorem is well illustrated by the disjoint paths shown in Fig. 2. In order to assure vertex disjointedness the three paths from $(x, 1)$ must proceed to its three neighbors in the cube connected cycle; namely $(x, 2)$, (x, n) and $(x', 1)$, where x' differs from x in the first bit. This is shown in Fig. 2 which presents a complete description of the three paths together with an accurate accounting of the path lengths. The three vertex disjoint paths from $(\mathbf{x}; 1)$ to $(\mathbf{y}; i)$ in Fig. 2 are constituted by the following: Two of these paths utilize the cycle edges in $(\mathbf{x}; 1)$ and $(\mathbf{y}; i)$ and the third path is a direct path. The reader can verify that these three paths are vertex disjoint. Observe that the maximum path length is $[(5n+4)/2]$, as claimed by the Theorem.

Observe that the above theorem asserts that the cube connected cycles are strongly resilient, which is counter-intuitive: for, we would expect the cube connected cycles to be weakly resilient due to the abundance of n-cycles in the networks. This surprising result is due to the robustness of the cube, which dominates over the weakness of the cycles.

We next derive bounds for fault diameters on generalized Petersen graphs.

Definition. The generalized Petersen graph, $P(m, a)$, is defined as follows: Its vertex set is $U \cup W$, where $U = \{u_o, u, \ldots, u_{m-1}\}$, $W = \{w_0, \ldots, w_{m-1}\}$; its edges are given by (u_i, w_i), (u_i, u_{i-1}), (u_i, u_{i+1}), (w_i, w_{i+a}) and (w_i, w_{i-a}), for $0 \leqslant i \leqslant m-1$, with the addition of subscripts being performed under modulo m. We call the vertices in U as vertices in the inner circle and the vertices in W as the vertices in outer circle.

This generalization of Petersen graph is due to Coxeter [6] and has also been studied by Bannai [2]. However, the analysis of their diameter and, more interestingly, their fault diameter is new. Observe that Petersen graph is $P(5, 2)$. It is fairly straightforward to see that the connectivity of $P(m, 1)$ is 3, the diameter of $P(m, 1)$ is $m/2 + 1$ if m is even, $[m/2]$ if m is odd; the fault diameter is $m - 1$ (this is achieved by deleting u_m and w_m). It is clear that the connectivity of $P(m, 2)$ is 3, as three vertex disjoint paths could be exhibited between any pair of vertices.

LEMMA 1

The diameter of $P(m, 2) = O(m/4)$, m is odd.

Proof. If we consider only the outer cycle, the maximum distance between u_1 and $u_{(m+1)/2}$ is $[(m-1)/2]$. However, we can reach $u_{(m+1)/2}$ by first going to w_1 and going to steps of 2 to $w_{(m+1)/2}$ and then going to $u_{(m+1)/2}$. Clearly this distance is $O(m/4)$. (This is the case when $(m+1)/2$ odd.) If $(m+1)/2$ is even, go from u_1 to w_1, then to $w_{(m-1)}$ and through a W-path to $w_{(m+1)/2}$ and then to $u_{(m+1)/2}$.

This path strategy is fairly general to go from any vertex to any other vertex. The idea is to employ a W-path as much as possible, because it jumps in steps of 2. In order to show that this is, in fact, a lower bound, all we need to observe is that in a distance s less than $O(m/4)$, the maximum vertex number we would have reached would be $2s$ and this implies that from u_1, we would not have reached $w_{(m+1)/2}$ and this contradicts the claim for diameter.

THEOREM 5

The fault diameter of generalized Petersen graph $P(m, a)$ is

$$\min\{\max\{c^*(m/a + 2a + 2),\ m/2 + 2\},\ m - 1\}.$$

Proof. We will exhibit 3 vertex disjoint paths of the specified fault diameter between any two points x and y. Before going to proof we will make an observation that for every specified vertex in the inner circle, there is at least one vertex in every consecutive group of a vertices in the outer circle which is at a distance of at most $[m/a] + 1$ from it. We will consider the following three cases.

Case 1. Both x and y are in the outer circle. As a first path, take a shortest path from x to y in the outer circle. This distance is at most $m/2$. As a second path, travel from x to y_1 in the "opposite" direction at a distance of at most $2a$. From y_1 go to y_1' in the inner circle. In the inner circle at distance of m/a reach y' and then reach y in the outer circle. (We abuse a notation that i and i' are connected.) As a third path, go from x to x' and travel in the inner circle, travel a distance of at most m/a to reach a y_1', such that y_1 is in the "opposite" direction.

Case 2. Both x' and y' are in the inner circle. As a first path, take the shortest path from x' to y' in the inner circle. This distance is at most $m/2$. As a second path, travel a distance at most $[m/a]$ from x' to y_1' in the inner circle (so that y_1 is closest to y at a distance of at most a). Go to y_1, and travel in the outer circle to y and finally go to y'. As a third path, go from x' to x and travel a distance of a in the outer circle to reach a vertex p, such that p' is at most $[m/a]$ from y'. After travelling to p' go to y'.

Case 3. Without loss of generality x is the outer circle and y' is in the inner circle. As a first path, take the shortest path from x to y in the outer circle and then go to y'. Clearly this distance is at most $m/2 + l$. As a second path, go from x to x' and travel a distance of at most m/a in the inner circle to go to a vertex p'. From p travel a distance of a to come to a vertex q closest to y'. From that vertex go to q' and from q' go to y'. This distance is at most $m/a + a + 2$. As a third path, go from x to x_1 in the outer circle at a distance of at most a in the "opposite" direction. From x_1, go to x_1' and travel a distance of at most $[m/a]$ to reach y'. This distance is at most $[m/a] + a + 1$. This completes the proof.

4. CONCLUDING REMARKS

The notion of fault diameter is doubtless a useful measure in the design and analysis of fault tolerant communication networks. In this paper we provide some tools for performing this analysis and point out that some familiar networks (in particular, the cube connected cycles) have surprising fault diameters. The analysis of generalized Petersen graphs also suggests techniques for improving the fault diameter.

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