Cordial labeling of $mK_n$

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Abstract

Suppose $G = (V,E)$ is a graph with vertex set $V$ and edge set $E$. A vertex labeling $f: V \rightarrow \{0,1\}$ induces an edge labeling $f^*: E \rightarrow \{0,1\}$ defined by $f^*(xy) = |f(x) - f(y)|$. For $i \in \{0,1\}$, let $v_f(i)$ and $e_f(i)$ be the number of vertices $v$ and edges $e$ with $f(v) = i$ and $f^*(e) = i$, respectively. A graph $G$ is cordial if there exists a vertex labeling $f$ such that $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. This paper determines all $m$ and $n$ for which $mK_n$ is cordial.

Keywords: Cordial labeling, graceful labeling, Diophantine equation

1. Introduction

Suppose $G = (V,E)$ is a graph with vertex set $V$ and edge set $E$. A vertex labeling $f: V \rightarrow \{0,1\}$ induces an edge labeling $f^*: E \rightarrow \{0,1\}$ defined by $f^*(xy) = |f(x) - f(y)|$. For a vertex labeling $f$ and $i \in \{0,1\}$, a vertex $v$ is an $i$-vertex if $f(v) = i$ and an edge is an $i$-edge if $f^*(e) = i$. Denote the numbers of 0-vertices, 1-vertices, 0-edges, and 1-edges of $G$ under $f$ by $v_f(0)$, $v_f(1)$, $e_f(0)$, and $e_f(1)$, respectively. A vertex labeling $f$ is cordial if $|v_f(0) - v_f(1)| \leq 1$ and $|e_f(0) - e_f(1)| \leq 1$. A graph is cordial if it admits a cordial labeling. A graph is perfectly cordial if there exists a vertex labeling $f$ such that $v_f(0) = v_f(1)$ and $e_f(0) = e_f(1)$. For even $m$, $mK_n$ has an even number of vertices and an even number of edges, so cordiality is equivalent to perfect cordiality when $m$ is even.

The notion of a cordial labeling was first introduced by Cahit [3] as a weaker version of graceful labeling. See [5,10,13,14,16,17] for related results and [4,6,11] for generalizations. Cordial labelings of various families of graphs were studied in

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[1,8,9,12,15]. This paper completely determines the cordiality of $mK_n$, the disjoint union of $m$ copies of the complete graph $K_n$ of $n$ vertices.

To do this, we first transform the problem into Diophantine equations with boundary conditions. Results from number theory are then used to solve the problem.

More precisely, given a vertex labeling $f$ of $mK_n$, let $x_i$ be the number of 1-vertices in the $i$th copy of $K_n$. Restricting $f$ to this $K_n$, we have

$$v_f(0) - v_f(1) = n - 2x_i,$$

$$e_f(0) - e_f(1) = ((n - 2x_i)^2 - n)/2.$$

Setting $y_i = n - 2x_i$, we obtain the following theorem.

**Theorem 1.** $mK_n$ is cordial if and only if the following system has an integral solution $(y_1, y_2, \ldots, y_m)$:

- $|y_i| \leq n$ for $1 \leq i \leq m$, \hspace{1cm} (C1)
- $y_1 \equiv y_2 \equiv \cdots \equiv y_m \equiv n \pmod{2}$, \hspace{1cm} (C2)
- $|y_1 + y_2 + \cdots + y_m| \leq 1$, \hspace{1cm} (C3)
- $|y_1^2 + y_2^2 + \cdots + y_m^2 - mn| \leq 2$. \hspace{1cm} (C4)

**Remark.** Note that if (C4) holds and $n \geq m$, then $y_i^2 \leq mn + 2 \leq n^2 + 2 < (n + 1)^2$ for each $i$, $1 \leq i \leq m$, so (C1) holds. Therefore, when checking whether $(y_1, y_2, \ldots, y_m)$ satisfies (C1)–(C4), it is not necessary to check (C1) if $n \geq m$.

2. Cordiality of $mK_n$ for $m \leq 4$

In this section, we determine the cordiality of $mK_n$ for $m \leq 4$. The main results are contained in Theorems 2, 3, 6 and 8.

**Theorem 2 (Cahit [3]).** $K_n$ is cordial if and only if $n \in \{1, 2, 3\}$.

**Proof.** If $K_n$ is cordial, then by Theorem 1, there exists an integer $y_1$ such that $|y_1| \leq 1$ and $|y_1^2 - n| \leq 2$. We conclude that $n \in \{1, 2, 3\}$.

Conversely, for $n \in \{1, 2, 3\}$, $y_1 = (n \mod 2)$ is a solution to (C1)–(C4), so $K_n$ is cordial. \hfill $\square$

**Theorem 3.** $2K_n$ is cordial if and only if $n = k^2$ for some integer $k$.

**Proof.** Suppose $2K_n$ is cordial. Let $(y_1, y_2)$ be a solution to (C1)–(C4). By (C2), $y_1$ and $y_2$ are of the same parity, i.e., $y_1 + y_2$ is even. This together with (C3) imply
\( y_1 + y_2 = 0. \) (C4) then gives \(|y_1^2 - n| \leq 1. \) As \( y_1 \) and \( n \) are of the same parity, we have \( n = y_1^2. \)

Conversely, if \( n = k^2 \) for some integer \( k \), then \((y_1, y_2) = (-k, k)\) is a solution to (C1)–(C4). Hence \( 2K_n \) is cordial. \( \square \)

**Theorem 4.** (a) If \( n \) is even, then \( 3K_n \) is cordial if and only if \( n \equiv 0 \) or \( 2 \) (mod 8) and \( x^2 + 3y^2 = a/2 \) has an integral solution, where \( a \) is the only number in \( \{3n, 3n + 2\} \) satisfying \( a \equiv 0 \) (mod 8).

(b) If \( n \) is odd, then \( 3K_n \) is cordial if and only if \( n \equiv 1, 3 \) or 7 (mod 8) and \( x^2 + 3y^2 = 6a - 2 \) has an integral solution, where \( a \) is the only number in \( \{3n - 2, 3n, 3n + 2\} \) satisfying \( a \equiv 3 \) (mod 8).

**Proof.** (a) Suppose \( n \) is even. If \( 3K_n \) is cordial, then (C1)–(C4) has a solution \((y_1, y_2, y_3)\), where \( y_1, y_2, \) and \( y_3 \) are even. So (C3) implies \( y_1 + y_2 + y_3 = 0. \) Substituting \( y_3 = -y_1 + y_2 \) into (C4), we obtain

\[
2y_1^2 + 2y_2^2 + 2y_1y_2 - 3n = -2, 0 \text{ or } 2.
\]

However, \( 2y_1^2 + 2y_2^2 + 2y_1y_2 \equiv 0 \text{ or } 2 \) (mod 3) only. Thus

\[
a = 2y_1^2 + 2y_2^2 + 2y_1y_2 = 3n \text{ or } 3n + 2.
\]

Since \( 2y_1^2 + 2y_2^2 + 2y_1y_2 \equiv 0 \) (mod 8), \( n \equiv 0 \) or \( 2 \) (mod 8), and \( a \) is the unique integer amongst \( 3n \) and \( 3n + 2 \) satisfying \( a \equiv 0 \) (mod 8). Consequently, \( x^2 + 3y^2 = a/2 \) has an integral solution because \((y_1 + y_2/2)^2 + 3(y_2/2)^2 = a/2. \)

Conversely, suppose \( x^2 + 3y^2 = a/2 \) has an integral solution \((x, y)\). Let \((y_1, y_2, y_3) = (x - y, 2y, -x - y)\). To verify (C1), by the remark after Theorem 1, we only need to consider the case of \( n = 2. \) In this case, \( a = 8 \) and \((x, y) = (\pm 1, \pm 1) \) or \((\pm 2, 0)\); thus (C1) holds. Since \( a \equiv 0 \) (mod 8), \( x \) and \( y \) are of the same parity, hence (C2) holds. (C3) and (C4) follow from a straightforward calculation. Therefore \( 3K_n \) is cordial.

(b) Suppose \( n \) is odd. If \( 3K_n \) is cordial, then (C1)–(C4) has a solution \((y_1, y_2, y_3)\) where \( y_1, y_2, \) and \( y_3 \) are odd. So (C3) implies \( y_1 + y_2 + y_3 = -1. \) Substituting \( y_3 = -y_1 + y_2 \) into (C4), we obtain

\[
2y_1^2 + 2y_2^2 + 3y_1y_2 - 3n = -2, 0 \text{ or } 2.
\]

However, \( 2y_1^2 + 2y_2^2 + 3y_1y_2 \equiv 0 \) (mod 8), \( n \equiv 1, 3 \) or 7 (mod 8) and \( a \) is the unique integer in \( \{0, 3n \pm 2\} \) satisfying \( a \equiv 3 \) (mod 8). Using the solution \((-y_1, -y_2, -y_3)\) if necessary, we may further assume \( y_1 + y_2 + y_3 = -1, \) which leads to \( a = 2y_1^2 + 2y_1y_2 + 2y_2^2 + 2y_1 + 2y_2 + 1, \) or equivalently, \( (3y_1 + 1)^2 + 3(y_1 + 2y_2 + 1)^2 = 6a - 2. \) Thus \( x^2 + 3y^2 = 6a - 2 \) has an integral solution.

Conversely, suppose \( x^2 + 3y^2 = 6a - 2 \) has an integral solution \((x, y)\). Then \( x^2 \equiv 1 \) (mod 3). Since \((x, y)\) is also a solution, we may assume \( x \equiv 1 \) (mod 3). Also, \( a \equiv 3 \) (mod 8) implies \( 6a - 2 \equiv 0 \) (mod 8) and \( x \equiv y \equiv 0 \) or \( 2 \) (mod 4). Let \( y_1 = (x - 1)/3, \)

\[
y_2 = (y - 1 - y_1)/2, \text{ and } y_3 = -1 - y_1 - y_2 = (-y - 1 - y_1)/2.
\]

For (C1), we only need to check the case of \( n = 1. \) In this case, \( a = 3 \) and \((x, y) = (4, 0) \) or \((-2, \pm 2)\).
Thus each $|y_i| \leq 1$, hence (C1) holds. Next, $y_1 \equiv (x-1)/3 \equiv 1 - x$ (mod 4) and $2y_2 = y - 1 - y_1 \equiv x + y - 2 \equiv 2$ (mod 4), so both $y_1$ and $y_2$ are odd; consequently, so is $y_3$. (C3) and (C4) follow from a straightforward calculation. \hfill \Box

In order to use Theorem 4 effectively, we need a result from number theory.

**Theorem 5** (Bolker [2, p. 122]). The equation $x^2 + 3y^2 = n$ has an integral solution if and only if $n = 3^{a_0} p_1^{a_1} \cdots p_k^{a_k} q_1^{b_1} \cdots q_s^{b_s}$ for some nonnegative integers $a_i$ and $b_i$, where $p_i$ and $q_i$ are primes satisfying $p_i \equiv 1$ (mod 6) and $q_i \equiv 5$ (mod 6), respectively.

The cordiality of $3K_n$ is summarized in the next theorem.

**Theorem 6.** Define $b = b(n)$ as

$$b = \begin{cases} 
3n/2 & \text{if } n \equiv 0 \text{ (mod 8)}, \\
3n/2 + 1 & \text{if } n \equiv 2 \text{ (mod 8)}, \\
18n - 2 & \text{if } n \equiv 1 \text{ (mod 8)}, \\
18n + 10 & \text{if } n \equiv 3 \text{ (mod 8)}, \\
18n - 14 & \text{if } n \equiv 7 \text{ (mod 8)}, 
\end{cases}$$

and let $v_p(z)$ denote the highest power of the prime $p$ that divides $z$. Then $3K_n$ is cordial if and only if $n \equiv 0, 1, 2, 3, 7$ (mod 8) and $v_2(b)$ and $v_q(b)$ are even for all prime divisors $q$ of $b$ with $q \equiv 5$ (mod 6).

We now turn our attention to $4K_n$.

**Theorem 7** (Grosswald [7, p. 24]). The equation $x^2 + y^2 + z^2 = n$ has an integral solution $(x,y,z)$ if and only if $n$ is not of the form $4a(8b + 7)$ for any nonnegative integers $a$ and $b$.

**Theorem 8.** $4K_n$ is cordial if and only if $n$ is not of the form $4a(8b + 7)$ for any nonnegative integers $a$ and $b$.

**Proof.** Suppose $4K_n$ is cordial and $(y_1, y_2, y_3, y_4)$ is a solution to (C1)-(C4). By (C2), $y_1, y_2, y_3,$ and $y_4$ are of the same parity. (C3) then implies $y_1 + y_2 + y_3 + y_4 = 0$. (C4) implies $y_1^2 + y_2^2 + y_3^2 + y_4^2 = 4n$. Hence $4n = y_1^2 + y_2^2 + y_3^2 + (-y_1 - y_2 - y_3)^2 = (y_1 + y_2)^2 + (y_2 + y_3)^2 + (y_3 + y_1)^2$. By Theorem 7, $4n$, and hence $n$, are not of the form $4a(8b + 7)$ for any nonnegative integers $a$ and $b$.

Conversely, suppose $n$ is not of the form $4a(8b + 7)$ for any nonnegative integers $a$ and $b$. By Theorem 7, $x^2 + y^2 + z^2 = n$ has an integer solution $(x,y,z)$. Let

$y_1 = x + y + z$, \hspace{1cm} y_2 = x - y - z,

$y_3 = -x + y - z$, \hspace{1cm} y_4 = -x - y + z.
Note that for each \(i\), \(|y_i| \leq |x| + |y| + |z| \leq x^2 + y^2 + z^2 = n\). Thus (C1) holds. Also, each \(y_i \equiv x^2 + y^2 + z^2 \equiv n \pmod{2}\), so (C2) holds. (C3) and (C4) follow from a straightforward calculation. Hence \(4K_n\) is cordial. \(\square\)

3. Cordiality of \(mK_n\) for \(m \geq 5\)

The cordiality of \(mK_n\) is much more uniform when \(m \geq 5\). There are three cases: \(m \equiv 0 \pmod{4}\), \(m \equiv 2 \pmod{4}\), and \(m \equiv 1 \pmod{2}\). Theorems 12, 14 and 15 contain the corresponding results. Because of the next lemma and its corollary, in each of these three cases, we only have to check the cordiality of \(mK_n\) for only a few values of \(m\).

**Lemma 9.** If \(G\) is perfectly cordial and \(H\) is cordial, then \(G \cup H\) is cordial.

**Corollary 10.** If \(mK_n\) and \(m'K_n\) are cordial and \(m\) is even, then \((m+m')K_n\) is cordial.

**Proof.** The corollary follows from Lemma 9 and the fact that cordiality and perfect cordiality of \(mK_n\) are equivalent when \(m\) is even. \(\square\)

To determine the cordiality of \(mK_n\), we need the famous ‘four squares theorem’ due to Lagrange (see, for example, [7, p. 25]):

**Theorem 11.** The equation \(z_1^2 + z_2^2 + z_3^2 + z_4^2 = n\) has an integral solution for any nonnegative integer \(n\).

**Theorem 12.** If \(m \geq 5\) and \(m \equiv 0 \pmod{4}\), then \(mK_n\) is cordial.

**Proof.** Because of Corollary 10, we only have to prove the theorem for \(m = 8\) and \(m = 12\).

**Case 1:** \(m = 8\). By Theorem 11, \(z_1^2 + z_2^2 + z_3^2 + z_4^2 = n\) has an integral solution \((z_1, z_2, z_3, z_4)\). Let

\[
\begin{align*}
y_1 &= z_1 + z_2 + z_3 + z_4, & y_2 &= z_1 - z_2 + z_3 - z_4, \\
y_3 &= z_1 - z_2 - z_3 + z_4, & y_4 &= z_1 + z_2 - z_3 - z_4, \\
y_5 &= -y_1, & y_6 &= -y_2, & y_7 &= -y_3, & y_8 &= -y_4.
\end{align*}
\]

Then for each \(i\),

\[
|y_i| \leq |z_1| + |z_2| + |z_3| + |z_4| \leq z_1^2 + z_2^2 + z_3^2 + z_4^2 = n,
\]

that is, (C1) holds. Also,

\[
y_i \equiv z_1 + z_2 + z_3 + z_4 \equiv z_1^2 + z_2^2 + z_3^2 + z_4^2 \equiv n \pmod{2}
\]

for each \(i\), so (C2) holds. (C3) is clearly true. (C4) follows a straightforward calculation.
Case 2: \( m = 12 \) and \( n \) is even. By Theorem 11, \( z_1^2 + z_2^2 + z_3^2 + z_4^2 = 3n/2 \) has an integral solution \((z_1, z_2, z_3, z_4)\). Let

\[
y_i = 2z_i, \quad y_{i+4} = -2z_i \quad \text{and} \quad y_{i+8} = 0 \quad \text{for} \quad 1 \leq i \leq 4.
\]

Then (C2)–(C4) hold. For \( n = 2 \) or 4, \( z_1^2 + z_2^2 + z_3^2 + z_4^2 = 3 \) or 6 implies each \( |z_i| \leq 1 \) or 2, respectively, hence (C1) holds. For \( n \geq 6 \), each \( |y_i| \leq 2|z_j| \leq 2\sqrt{3n/2} = \sqrt{6n} \leq n \), so (C1) holds again. Thus \( 12K_n \) is cordial.

Case 3: \( m = 12 \) and \( n \) is odd. By Theorem 11, \( z_1^2 + z_2^2 + z_3^2 + z_4^2 = 3(n - 1)/2 \) has an integral solution. Let

\[
y_i = \begin{cases} 1 + 2z_i & \text{if} \quad 1 \leq i \leq 4, \\ 1 - 2z_{i-6} & \text{if} \quad 7 \leq i \leq 10, \\ y_5 = y_6 = 1, \quad y_{11} = y_{12} = -1. \end{cases}
\]

Then (C2)–(C4) holds. For \( n = 1, 3 \) or 5, \( z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0, \) or 6 implies \( |z_i| \leq 0, 1, \) or 2, respectively, so (C1) holds. For \( n \geq 7 \), each \( |y_i| \leq 2|z_j| + 1 \leq 2\sqrt{3(n - 1)/2 + 1} = \sqrt{6(n - 1)} + 1 \leq (n - 1) + 1 = n \), so (C1) still holds. Thus \( 12K_n \) is cordial.

Lemma 13. If (i) \( n \) is even and \( mn \equiv 4 \pmod{8} \), or (ii) \( n \) is odd and \( m(n - 1) \equiv 4 \pmod{8} \), then \( mK_n \) is not cordial.

Proof. Suppose \( mK_n \) is cordial and \((y_1, y_2, \ldots, y_n)\) is an integral solution to (C1)–(C4).

Suppose \( n \) is even and \( mn \equiv 4 \pmod{8} \). By (C2), all \( y_i \) are even. By (C3), \( y_1 + y_2 + \cdots + y_n = 0 \). By (C4), \( y_1^2 + y_2^2 + \cdots + y_n^2 = mn \). Let \( z_i = y_i/2 \) for \( 1 \leq i \leq m \). Then \( z_1 + z_2 + \cdots + z_m = 0 \) and \( z_1^2 + z_2^2 + \cdots + z_m^2 = mn/4 \). Since \( mn \equiv 4 \pmod{8} \), \( mn/4 \) is odd. Consequently,

\[
0 \equiv z_1 + z_2 + \cdots + z_m \equiv z_1^2 + z_2^2 + \cdots + z_m^2 \equiv mn/4 \equiv 1 \pmod{2},
\]

a contradiction.

Suppose \( n \) is odd and \( m(n - 1) \equiv 4 \pmod{8} \). By (C2), each \( y_i \) is odd, say \( y_i = 2z_i + 1 \). By (C4), \( 4 \sum_{i=1}^{m} z_i(z_i + 1) - m(n - 1) \mid 2 \). This is impossible, since \( 4 \sum_{i=1}^{m} z_i(z_i + 1) \equiv 0 \pmod{8} \) and \( m(n - 1) \equiv 4 \pmod{8} \).

Theorem 14. For \( m \geq 5 \) and \( m \equiv 2 \pmod{4} \), \( mK_n \) is cordial if and only if \( n \equiv 0 \) or 1 (mod 4).

Proof. If \( mK_n \) is cordial, then \( n \equiv 0 \) or 1 (mod 4) by Lemma 13. Conversely, suppose \( n \equiv 0 \) or 1 (mod 4). By Corollary 10 and Theorem 12, we only have to prove that \( mK_n \) is cordial for \( m = 6 \) and \( m = 10 \). Solutions to (C1)–(C4) for \( n < m \) are
listed below.

\[ m = 6 \text{ and } n = 1: \ (1, 1, 1, -1, -1, -1). \]
\[ m = 6 \text{ and } n = 4: \ (2, 2, 2, -2, -2, -2). \]
\[ m = 6 \text{ and } n = 5: \ (3, 3, -3, -1, -1, -1). \]
\[ m = 10 \text{ and } n = 1: \ (1, 1, 1, 1, -1, -1, -1, -1, -1). \]
\[ m = 10 \text{ and } n = 4: \ (2, 2, 2, 2, -2, -2, -2, -2, -2). \]
\[ m = 10 \text{ and } n = 5: \ (5, 3, 1, -3, -1, -1, -1, -1, -1). \]
\[ m = 10 \text{ and } n = 8: \ (8, -2, -2, -2, 0, 0, 0, 0, 0). \]
\[ m = 10 \text{ and } n = 9: \ (9, -1, -1, -1, -1, -1, -1, -1, -1). \]

Now we may assume \( n \geq m \), and we only need to check (C2)–(C4) for a solution.

**Case 1:** \( n \equiv 0 \pmod{4} \) and \( mn/4 \) is not of the form \( 4a(8b + 7) \). By Theorem 7, \( x^2 + y^2 + z^2 = mn/4 \) has an integral solution \((x, y, z)\). Let

\[
\begin{align*}
y_1 &= x + y + z, & y_2 &= x - y - z, \\
y_3 &= -x + y - z, & y_4 &= -x - y + z, \\
y_i &= 0 \quad \text{for } i \geq 5.
\end{align*}
\]

For instance, when \( m = 10, n = 12 \), we could let \((x, y, z) = (5, 2, 1)\), from which we obtain the solution \((8, 2, -4, -6, 0, 0, 0, 0, 0, 0)\).

In general, each \( y_i \equiv x^2 + y^2 + z^2 \equiv mn/4 \equiv 0 \pmod{2} \), so (C2) holds. (C3) and (C4) follow from a straightforward calculation.

**Case 2:** \( n \equiv 0 \pmod{4} \) and \( mn/4 \) is of the form \( 4a(8b + 7) \). Then \( mn/2 \) is not of the form \( 4a(8b + 7) \). By Theorem 7, \( x^2 + y^2 + z^2 = mn/2 \) has an integral solution \((x, y, z)\). Since \( mn/2 \equiv 0 \pmod{4} \), \( x, y, z \) are all even. Let

\[
\begin{align*}
y_1 &= x, & y_2 &= y, & y_3 &= z, \\
y_4 &= -x, & y_5 &= -y, & y_6 &= -z;
\end{align*}
\]

and for \( m = 10 \), set

\[
y_7 = y_8 = y_9 = y_{10} = 0.
\]

Then (C2)–(C4) clearly hold.

For example, when \( m = 10, n = 24 \), we could pick \((x, y, z) = (10, 4, 2)\), which leads to the solution \((10, 4, 2, -10, -4, -2, 0, 0, 0, 0)\).

**Case 3:** \( n \equiv 1 \pmod{4} \). Note that \( (mn - m + 2)/2 \equiv 1 \pmod{4} \), so it is not of the form \( 4a(8b + 7) \). By Theorem 7, \( x^2 + y^2 + z^2 = (mn - m + 2)/2 \) has an integral solution \((x, y, z)\), two of them, say \( x \) and \( y \), are even and the other one is odd. Let

\[
\begin{align*}
y_1 &= 1 + x, & y_2 &= -1 + y, & y_3 &= z, \\
y_4 &= 1 - x, & y_5 &= -1 - y, & y_6 &= -z;
\end{align*}
\]

and for \( m = 10 \), set

\[
y_7 = y_8 = 1 \quad \text{and} \quad y_9 = y_{10} = -1.
\]

Then (C2)–(C4) hold.
An example: when \( m = 10, n = 13 \), selecting \((x, y, z) = (6, 4, 3)\) gives the solution \((7, 3, 3, -5, -5, -3, 1, 1, -1, -1)\).

We have shown that, in all cases, \( mK_n \) is cordial. □

**Theorem 15.** For odd \( m \) satisfying \( m \geq 5 \), \( mK_n \) is cordial if and only if \( n \not\equiv 4, 5 \pmod{8} \).

**Proof.** If \( mK_n \) is cordial, then \( n \not\equiv 4, 5 \pmod{8} \) by Lemma 13. Conversely, suppose \( n \not\equiv 4, 5 \pmod{8} \). By Corollary 10 and Theorem 12, we only need to prove that \( mK_n \) is cordial for \( m \in \{5, 7, 9, 11\} \).

For \( n < m \), the following are solutions to \((C1)-(C4)\).

\[
\begin{align*}
m = 5 \text{ and } n = 1: & \quad (1, 1, 1, -1, -1, -1). \\
m = 5 \text{ and } n = 2: & \quad (-2, 0, 0, 0, 0). \\
m = 5 \text{ and } n = 3: & \quad (3, 1, -1, -1, -1). \\
m = 7 \text{ and } n = 1: & \quad (1, 1, 1, -1, -1, -1, -1). \\
m = 7 \text{ and } n = 2: & \quad (2, 2, -2, -2, 0, 0, 0, 0). \\
m = 7 \text{ and } n = 3: & \quad (3, 1, 1, -3, -1, -1). \\
m = 7 \text{ and } n = 6: & \quad (4, -4, -2, 0, 0, 0, 0, 0). \\
m = 9 \text{ and } n = 1: & \quad (1, 1, 1, 1, -1, -1, -1, -1, -1). \\
m = 9 \text{ and } n = 2: & \quad (2, 2, -2, 0, 0, 0, 0, 0). \\
m = 9 \text{ and } n = 3: & \quad (3, 3, -1, -1, -1, -1, -1, -1). \\
m = 9 \text{ and } n = 6: & \quad (6, -4, -2, 0, 0, 0, 0, 0). \\
m = 9 \text{ and } n = 7: & \quad (5, 3, 1, -5, -1, -1, -1, -1, -1). \\
m = 9 \text{ and } n = 8: & \quad (6, -6, 0, 0, 0, 0, 0, 0). \\
m = 11 \text{ and } n = 1: & \quad (1, 1, 1, 1, 1, -1, -1, -1, -1, -1). \\
m = 11 \text{ and } n = 2: & \quad (2, 2, 2, -2, -2, 0, 0, 0, 0). \\
m = 11 \text{ and } n = 3: & \quad (3, 3, 1, 1, -3, -1, -1, -1, -1). \\
m = 11 \text{ and } n = 6: & \quad (6, 2, 2, -2, -2, -2, -2, 0, 0, 0). \\
m = 11 \text{ and } n = 7: & \quad (7, 3, 1, -3, -1, -1, -1, -1, -1, -1). \\
m = 11 \text{ and } n = 8: & \quad (8, -4, -2, 0, 0, 0, 0, 0, 0, 0). \\
m = 11 \text{ and } n = 9: & \quad (5, 5, 3, 1, -5, -1, -1, -1, -1, -1, -1). \\
m = 11 \text{ and } n = 10: & \quad (8, -6, -2, -2, 0, 0, 0, 0, 0, 0).
\end{align*}
\]

So we may assume \( n \geq m \). All we need now is to find solutions to \((C1)-(C4)\).

Let \( n' = (n \mod 2) \). Since \( n - n' \not\equiv 4 \pmod{8} \), exactly one of the three numbers \( mn - mn' - 2 \), \( mn - mn' \), \( mn - mn' + 2 \) is of the form \( 8t \).

**Case 1:** \( 2t + n' \) is not of the form \( 4a(8b + 7) \) for any nonnegative integers \( a \) and \( b \). By Theorem 7, \( x^2 + y^2 + z^2 = 2t + n' \) has an integral solution \((x, y, z)\). Let

\[
\begin{align*}
y_1 &= x + y + z, \\
y_2 &= x - y - z, \\
y_3 &= -x + y - z, \\
y_4 &= -x - y + z,
\end{align*}
\]

and

\[
y_i = \begin{cases} 
    n' & \text{if } 5 \leq i \leq (m + 5)/2, \\
    -n' & \text{if } (m + 7)/2 \leq i \leq m.
\end{cases}
\]
Note that $y_i = x^2 + y^2 + z^2 \equiv 2t + n' \equiv n \mod 2$ for each $i$, $1 \leq i \leq 4$. Also, for each $i \geq 5$, $y_i \equiv n' \equiv n \mod 2$. Then (C2) holds. Next, $\sum_{i=1}^{m} y_i = n' = 0$ or 1, thus (C3) holds. Finally, $\sum_{i=1}^{m} y_i^2 = 4(x^2 + y^2 + z^2) + (m-4)n' = 4(2t + n') + (m-4)n' = 8t + mn' = mn - 2$, $mn$, or $mn + 2$; therefore (C4) also holds.

For example, when $m = 9$, $n = 15$, we have $t = 16$; by letting $(x, y, z) = (4, 4, 1)$, we obtain the solution $(9, -1, -1, -7, 1, 1, -1, -1, -1)$. When $m = 9$, $n = 22$, we have $t = 25$, and $(x, y, z) = (5, 4, 3)$ gives rise to the solution $(12, -2, -4, -6, 0, 0, 0, 0, 0)$.

Case 2: $2t + n'$ is of the form $4^a(8b + 7)$ for some nonnegative integers $a$ and $b$.

Since $128b + 107 - 31n' \equiv 11 + n' \mod 32$, it is not of the form $4^a(8b + 7)$ for any nonnegative integers $a'$ and $b'$. By Theorem 7, $x^2 + y^2 + z^2 = 128b + 107 - 31n'$ has an integral solutions $(x, y, z)$.

For $n' = 0$, $x$, $y$, $z$ are all odd, for otherwise $x^2 + y^2 + z^2 \equiv 0$ or 4 $\mod 8$, whereas $128b + 107 - 31n' \equiv 3 \mod 8$. We can further assume $x \equiv y \equiv z \equiv 1 \mod 2$, for if necessary we could replace $x$ or $y$ or $z$ by $-x$ or $-y$ or $-z$, respectively.

For $n' = 1$, we must have $x = -y = z - 2 \mod 4$ in order for $128b + 107 - 31n' \equiv 12 \mod 16$. We can further assume $x = y = z - 4 = 2 \mod 8$, otherwise we could replace $x$ or $y$ or $z$ by $-x$ or $-y$ or $-z$, respectively.

In either case,

$$x - 1 - n' \equiv y - 1 - n' \equiv z - 1 - n' - 4 \equiv 0 \mod 4(1 + n').$$

Let

$$y_1 = 2^a(-x + y + z - 1 - n')/4, \quad y_2 = 2^a(x - y + z - 1 - n')/4,$$
$$y_3 = 2^a(x + y - z - 1 - n')/4, \quad y_4 = 2^a(-x - y - z - 1 - n')/4,$$
$$y_5 = 2^a(1 + 2n');$$

and for $m > 5$, set

$$y_i = \begin{cases} n' & \text{if } 6 \leq i \leq (m + 5)/2, \\
-n' & \text{if } (m + 7)/2 \leq i \leq m. \end{cases}$$

Note that $a \geq 1$ when $n' = 0$, and $a = 0$ when $n' = 1$, so $(4^a - 1)n' = 0$.

For the case of $n' = 0$, all $y_i \equiv 0 \equiv n' \equiv n \mod 2$, so (C2) holds. Also, for the case of $n' = 1$, all $y_i \equiv 1 \equiv n' \equiv n \mod 2$, so (C2) holds again. Next, $\sum_{i=1}^{m} y_i = 2^a(-1 - n') + 2^a(1 + 2n') = 2^a n' = 0$ or 1, so we also have (C3). Finally,

$$\sum_{i=1}^{m} y_i^2 = 4^a \left( \frac{x^2 + y^2 + z^2 + 1 + 3n'}{4} \right) + 4^a(1 + 8n') + (m - 5)n'$$
$$= 4^a \left( \frac{128b + 107 - 31n' + 1 + 3n'}{4} \right) + 4^a(1 + 8n') + (m - 5)n'$$
$$= 4^a(32b + 28) + (4^a + m - 5)n'$$
$$= 4(2t + n) + (4^a + m - 5)n'$$
$$= 8t + mn' + (4^a - 1)n'$$
$$= 8t + mn'$$
Since $8t + mn' \in \{mn - 2, mn, mn + 2\}$, (C4) holds.

Take $m = 11, n = 22$ as an illustration. We have $t = 30, a = 1, b = 1$. We could take $(x, y, z) = (-15, -3, 1)$, which leads to the solution $(6, -6, -10, 8, 2, 0, 0, 0, 0, 0, 0)$.

Consider $m = 9, n = 7$ as another example. We have $t = 7, a = 0, b = 1$. Consequently, we could let $(x, y, z) = (10, 10, -2)$. The solution thus obtained is $(-1, -1, 5, -5, 3, 1, 1, -1, -1, 1)$. □

4. Conclusion

This paper completely determines the cordiality of $mK_n$ as follows:
• $K_n$ is cordial if and only if $n \in \{1, 2, 3\}$.
• $2K_n$ is cordial if and only if $n = k^2$ for some integer $k$.
• Define $b = b(n)$ as

$$b = \begin{cases} 
3n/2 & \text{if } n \equiv 0 \pmod{8}, \\
3n/2 + 1 & \text{if } n \equiv 2 \pmod{8}, \\
18n - 2 & \text{if } n \equiv 1 \pmod{8}, \\
18n + 10 & \text{if } n \equiv 3 \pmod{8}, \\
18n - 14 & \text{if } n \equiv 7 \pmod{8},
\end{cases}$$

and let $v_p(z)$ denote the highest power of the prime $p$ that divides $z$. Then $3K_n$ is cordial if and only if $n \equiv 0, 1, 2, 3, 7 \pmod{8}$ and $v_2(b)$ and $v_q(b)$ are even for all prime divisors $q$ of $b$ with $q \equiv 5 \pmod{6}$.
• $4K_n$ is cordial if and only if $n$ is not of the form $4a(8b + 7)$ for any nonnegative integers $a$ and $b$.
• For $m \geq 5$, the following hold.
  - If $m \equiv 0 \pmod{4}$, $mK_n$ is cordial for all $n$.
  - If $m \equiv 2 \pmod{4}$, $mK_n$ is cordial if and only if $n \equiv 0$ or $1 \pmod{4}$.
  - If $m$ is odd, $mK_n$ is cordial if and only if $n \not\equiv 4, 5 \pmod{8}$.

References