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Estimating Functions by Partial Sums of Their Fourier Series

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For classes of functions with convergent Fourier series, the problem of estimating the rate of convergence of the Fourier series has always been of interest. A classical theorem like that of Dirichlet and Jordan for functions of bounded variation (BV) assures the convergence of the Fourier series but gives no estimate of the rate of convergence. Such an estimate was recently provided by Bojanic [1]. For certain classes of functions of generalized bounded variation the conclusions of the Dirichlet–Jordan theorem also hold. Waterman [5, 7] has shown that the class of functions of harmonic bounded variation (HBV) is, in a certain sense, the largest such class. An estimate of the rate of convergence of the Fourier series has been made for certain classes which lie between BV and HBV [2]. Here we consider this problem in greater generality for ΛBV classes in that range to obtain a result which includes the previous estimates and allows us to make an estimate for a particular class which is closer to HBV than the classes previously considered.

If f is a real valued function on the interval $[a, b]$ and $\Lambda = \{\lambda_n\}$ is a nondecreasing sequence of positive numbers such that $\sum 1/\lambda_n$ diverges, we say that f is of Λ-bounded variation (ΛBV) if the sums

$$\sum_1^n |f(a_k) - f(b_k)|/\lambda_k$$

are bounded uniformly over the sequences of nonoverlapping intervals $[a_k, b_k]$ in $[a, b]$. If $f \in \Lambda$BV, the Λ-variation of f is defined to be the supremum of such sums. When $\Lambda = \{n\}$ the class is called the functions of harmonic bounded variation (HBV). We have shown that the conclusions of the Dirichlet–Jordan theorem can be extended to the class HBV, but if $\Lambda\text{BV} \supsetneq \text{HBV}$, then there is an $f \in \Lambda$BV whose Fourier series diverges at a point.

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Bojanic and Waterman considered classes between BV and HBV by setting $\Lambda = \{n^\alpha\}$, $0 < \alpha < 1$. These classes are of interest for two reasons: we have information on the summability of Fourier series of functions in these classes [6], and these classes mediate between two extremes, HBV and BV.

In Section 1 we state our main result and deduce from it a number of corollaries which demonstrate its applicability and scope. In Section 2 we prove the main theorem, discuss the restriction on Λ in its hypothesis, and show the applicability of our estimate to other classes of functions of generealized bounded variation.

1. Suppose that f is defined on $[-\pi, \pi]$ has period 2π, and is *regulated*, i.e., $f(x) = \frac{1}{2}[f(x+) + f(x-)]$ for each x. Functions of generalized bounded variation in our sense have only simple discontinuities, so the last requirement is not restrictive.

We will now prescribe certain notation and conventions to which we will adhere for the entire discussion.

We suppose that λ_k/k is *nonincreasing* and, for *fixed* n, $H(t)$ is a *continuous nonincreasing* function on $(0, \pi]$ such that

$$H(t) = \lambda_k/t \qquad \text{for} \quad t = k\pi/(n+1),\ k = 1, 2, \dots, n+1.$$

We let $g(t) = g_x(t) = f(x+t) + f(x-t) - 2f(x)$ and let $V(t)$ be the Λ-variation of g on $[0, t]$ for *whatever particular* Λ we are discussing. $S_n(x)$ will denote the nth partial sum of the Fourier series of f evaluated at x.

The following is our main result.

THEOREM. *Let $f \in \Lambda\mathrm{BV}$ and $\pi/(n+1) = a_n < a_{n-1} < \cdots < a_0 = \pi$. Then*

$$|S_n(x) - f(x)| \leqq (1 + 2/\pi) \left[\frac{\lambda n + 1}{n+1} V(\pi) + \frac{\pi}{n+1} \sum_{i=0}^{N-1} V(a_i)(H(a_{i+1}) - H(a_i)) \right].$$

The significance of this result is best seen by considering specific applications. It should also be noted that $V(t) \searrow 0$ as $t \searrow 0$. Our applications are listed as corollaries. The estimate of Bojanic [1] is

COROLLARY 1. *Let $f \in BV$. Then*

$$|S_n(x) - f(x)| \leqq \frac{2(1 + 2/\pi)}{n+1} \sum_{k=1}^{n} V(\pi/k).$$

To see this, set $\lambda_k = 1$ for all k, $a_i = \pi/(i+1)$, $i = 0, \dots, n$, and $H(t) = 1/t$. Then

$$
\begin{aligned}
&|S_n(x) - f(x)| \\
&\leqq (1 + 2/\pi) \left[\frac{V(\pi)}{n+1} + \frac{\pi}{n+1} \sum_{k=1}^{n} V(\pi/k) \left(\frac{k+1}{\pi} - \frac{k}{\pi} \right) \right] \\
&= (1 + 2/\pi) \left[\frac{V(\pi)}{n+1} + \sum_{k=1}^{n})V(\pi/k) \right] < \frac{2(1+2/\pi)}{n+1} \sum_{k=1}^{n} V(\pi/k).
\end{aligned}
$$

The result of Bojanic and Waterman [2] is

COROLLARY 2. *Let $f \in \{n^\alpha\} - BV$ for $0 < \alpha < 1$. Then*

$$
|S_n(x) - f(x)| \leqq \frac{(2-\alpha)(1+2/\pi)}{(n+1)^{1-\alpha}} \sum_{k=1}^{n} \frac{1}{k^\alpha} V(\pi/k).
$$

Here we set $\lambda_k = k^\alpha$ and $H(t) = t^{\alpha-1}((n+1)/\pi)^\alpha$. With a_i as above we have

$$
\begin{aligned}
|S_n(x) - f(x)| \leqq (1 + 2/\pi) &\left[\frac{V(\pi)}{(n+1)^{1-\alpha}} + \frac{\pi}{n+1} \sum_{i=0}^{n-1} V\left(\frac{\pi}{i+1}\right) \left(\frac{n+1}{\pi}\right)^\alpha \right. \\
&\left. \times \left[\left(\frac{\pi}{i+2}\right)^{\alpha-1} - \left(\frac{\pi}{i+1}\right)^{\alpha-1} \right] \right] \\
&= \frac{(1+2/\pi)}{(n+1)^{1-\alpha}} \left[V(\pi) + \sum_{1}^{n} V(\pi/k)((k+1)^{1-\alpha} - k^{1-\alpha}) \right] \\
&< \frac{1+2/\pi}{(n+1)^{1-\alpha}} \left[V(\pi) + (1-\alpha) \sum_{1}^{n} V(\pi/k)\, k^{-\alpha} \right]
\end{aligned}
$$

by the mean value theorem. The result follows immediately.

In the next application we show how the flexibility in the choice of $\{a_i\}$ may be used. If $\lambda_n \sim n/\log n$, then it is easily seen that $\Lambda\mathrm{BV} \subsetneqq \mathrm{HBV}$, but $\Lambda\mathrm{BV} \supsetneqq \{n^\alpha\} - \mathrm{BV}$ for $0 < \alpha < 1$. This result shows how the estimated rate of convergence worsens as we approach HBV.

COROLLARY 3. *Let $f \in \Lambda\mathrm{BV}$ with $\lambda_1 = 1/\log 3$, $\lambda_2 = 2/\log 3$, and $\lambda_n = n/\log n$ for $n \geqq 3$. Let N be the least integer not less than $\log(n+1)/\log 3$ and let $r = (n+1)^{1/N}$. Then for $n \geqq 8$,*

$$
|S_n(x) - f(x)| \leqq \frac{1+2/\pi}{\log 3} \left(\frac{V(\pi)}{N-1} + \sum_{k=1}^{N-1} \frac{1}{(N-k)^2} V(\pi/r^{k-1}) \right).
$$

Let $a_i = \pi/r^i$, $i = 0, 1, \ldots, N$ and

$$\begin{aligned} H(t) &= (n+1)/\pi \log 3, & t &< 3\pi/(n+1) \\ &= (n+1)/\pi \log(n+1)t/\pi, & t &\geqq 3\pi/(n+1). \end{aligned}$$

Note that

$$\frac{\pi}{n+1} = a_N < a_{N-1} \leqq 3\pi/(n+1) < a_{N-2}.$$

Thus

$$\frac{\pi}{n+1} V(a_{N-1})(H(a_N) - H(a_{N-1})) = 0$$

and

$$\begin{aligned} &\frac{\pi}{n+1} V(a_{N-2})(H(a_{N-1}) - H(a_{N-2})) \\ &\quad \leqq \frac{\pi}{n+1} V(\pi/r^{n-2}) \left(\frac{n+1}{\pi \log 3} - \frac{n+1}{\log(n+1)^{2/N}}\right) \\ &\quad = V(\pi/r^{n-2}) \left(\frac{1}{\log 3} - \frac{N}{2\log(n+1)}\right) \leqq V(\pi/r^{N-2}) \left(\frac{1}{\log 3} - \frac{1}{2\log 3}\right) \\ &\quad = V(\pi/r^{n-2})/2 \log 3. \end{aligned}$$

For $k < N-2$,

$$\begin{aligned} (\pi/n+1)\, V(a_k)(H(a_{k+1}) - H(a_k)) &= V(\pi/r^k)N/(N-k)(N-k-1)\log(n+1) \\ &< V(\pi/r^k)/(N-k-1)^2 \log 3, \end{aligned}$$

since $N/(N-k)\log(n+1) < 1/(N-k-1)\log 3$. We have then

$$\begin{aligned} |S_n(x) - f(x)| &\leqq (1 + 2/\pi) \left[V(\pi)/\log(n+1) + \frac{1}{\log 3} \sum_{k=0}^{N-3} \frac{1}{(N-k-1)^2} V(\pi/r^k) \right. \\ &\qquad \left. + V(\pi/r^{n-2})/2 \log 3 \right] \\ &< (1 + 2/\pi) \left[V(\pi)/\log(n+1) + \frac{1}{\log 3} \sum_{k=1}^{N-1} \frac{1}{(N-k)^2} V(\pi/r^{k-1}) \right] \\ &< \frac{(1 + 2/\pi)}{\log 3} \left[\frac{V(\pi)}{N-1} + \sum_{k=1}^{N-1} \frac{1}{(N-k)^2} V(\pi/r^{k-1}) \right]. \end{aligned}$$

2. If I is an interval, let

$$\operatorname{osc}(g, I) = \sup\{|g(t) - g(t')| \mid t, t' \in I\}.$$

Let $I_{k,n} = [k\pi/(n+1), (k+1)\pi/(n+1)]$, $k = 0, \ldots, n$. The proof of our theorem rests on the following estimate of Bojanic and Waterman [2]:

LEMMA. *If f is a regulated function of period 2π, then*

$$|S_n(x) - f(x)| \leqq (1 + 2/\pi) \sum_{k=0}^{n} \frac{1}{k+1} \operatorname{osc}(g, I_{k,n}).$$

Now let

$$M_k = \sum_{j=0}^{k} \frac{1}{\lambda_{j+1}} \operatorname{osc}(g, I_{j,n})$$

and set

$$M(t) = M_k \qquad \text{for } \frac{(k+1)\pi}{n+1} \leqq t < \frac{(k+2)\pi}{n+1}, \ k = 0, \ldots, n-1.$$

Then

$$\sum_{k=0}^{n} \frac{1}{k+1} \operatorname{osc}(g, I_{k,n}) = \frac{\lambda_{n+1}}{n+1} M_n + \sum_{k=0}^{n-1} M_k \left(\frac{\lambda_{k+1}}{k+1} - \frac{\lambda_{k+2}}{k+2} \right)$$

and this last sum is

$$\frac{-\pi}{n+1} \sum_{k=0}^{n-1} \int_{I_{k+1,n}} M(t)\, dH(t) = \frac{-\pi}{n+1} \int_{\pi/n+1}^{\pi} M(t)\, dH(t)$$
$$\leqq \frac{\pi}{n+1} \sum_{i=0}^{N-1} M(a_i)(H(a_{i+1}) - H(a_i)).$$

Since $V(\pi) \geqq M_n$ and $V(t) \geqq M(t)$, we have

$$\sum_{k=0}^{n} \frac{1}{k+1} \operatorname{osc}(g, I_{k,n}) \leqq \frac{\lambda_{n+1}}{n+1} V(\pi)$$
$$+ \frac{\pi}{n+1} \sum_{i=0}^{N-1} V(a_i)(H(a_{i+1}) - H(a_i)),$$

which, with the lemma, yields our theorem.

In our result the hypothesis λ_k/k nonincreasing is essential. If Λ does not satisfy this hypothesis one can ask if there is a sequence $\Gamma = \{\gamma_n\}$ such that $\Gamma\mathrm{BV} = \Lambda\mathrm{BV}$ and λ_k/k is nonincreasing.

From known results on ΛBV spaces [3] we see that the hypothesis $\Delta\mathrm{BV} \subsetneqq \mathrm{HBV}$ is equivalent to

$$0 < c \leqq \left(\sum_1^n 1/\lambda_k\right) \Big/ \left(\sum_1^n 1/k\right) \neq O(1), \tag{1}$$

while $\Gamma\mathrm{BV} = \Lambda\mathrm{BV}$ is equivalent to

$$0 < c' \leqq \left(\sum_1^n 1/\gamma_k\right) \Big/ \left(\sum_1^n 1/\lambda_k\right) \leqq c'' < \infty. \tag{2}$$

Clearly (1), (2), and γ_k/k nonincreasing imply

$$\gamma_k/k \to 0$$

and, therefore,

$$\left(\sum_1^n 1/\lambda_k\right) \Big/ \left(\sum_1^n 1/k\right) \to \infty. \tag{3}$$

It is easy to construct an increasing sequence Λ such that (1) is true and (3) is false. Thus we see that we cannot omit the hypothesis λ_k/k nonincreasing.

James Smith has made the interesting observation [4] that there exists a sequence Λ satisfying (1) and (3) such that $\Gamma\mathrm{BV} = \Lambda\mathrm{BV}$ implies γ_k/k is *not* nonincreasing.

If a class of functions is contained in a ΛBV space to which our theorem is applicable, then our estimate may be used for the given class. We consider as an example certain classes of functions of Φ-bounded variation.

If Φ and Ψ are conjugate Young's functions, we say that $f \in \Phi\mathrm{BV}$ on $[a, b]$ if, for some $k > 0$, there is a $C < \infty$ such that

$$\sum \Phi(k \mid f(a_n) - f(b_n)|) < C$$

for every partition $\{[a_n, b_n]\}$ of $[a, b]$. For Λ as above and $\{[a_n, b_n]\}$ a collection of nonoverlapping intervals in $[a, b]$ we have

$$\sum k \, |f(a_n) - f(b_n)|/\lambda_n \leqq C + \sum_1^\infty \Psi(1/\lambda_n).$$

Thus $\sum_1^\infty \Psi(1/\lambda_n) < \infty$ and $f \in \Phi\mathrm{BV}$ implies $f \in \Lambda\mathrm{BV}$ and we can apply the estimate given above for $|S_n(x) - f(x)|$. For example, consider the Wiener class $W^p = \Phi\mathrm{BV}$ with $\Phi = x^p$, $p > 1$. Then $\Psi = cx^q$ with $q = p/(p-1)$. If

$\lambda_n = n^{\alpha}$, then $\sum_1^{\infty} \Psi(1/n^{\alpha}) = c$, $\sum_1^{\infty} 1/n^{\alpha q} < \infty$ if $\alpha > 1/q$. Thus for $f \in W^p$ we can estimate $|S_n(x) - f(x)|$ using the result of Corollary 2 for an $\alpha > (p-1)/p$.

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