ELSEVIER Journal of Computational and Applied Mathematics 91 (1998) 231–247

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Derivation and analytical investigation of three direct boundary integral equations for the fundamental biharmonic problem

Søren Christiansen *

Department of Mathematical Modelling, Building 321, The Technical University of Denmark, DK-2800 Lyngby, Denmark

Received 3 February 1997; received in revised form 17 February 1998

Abstract

We derive and investigate three families of direct boundary integral equations for the solution of the plane, fundamental biharmonic boundary value problem. These three families are fairly general so that they, as special cases, encompass various known and applied equations as demonstrated by giving many references to the literature. We investigate the families by analytical means for a circular boundary curve where the radius is a parameter. We find for all three combinations of equations (i) that the solution of the equations is non-unique for one or more critical radius/radii, and (ii) that this lack of uniqueness can always be removed by combining the integral equations with a suitable combination of one or more supplementary condition(s). We conjecture how the results obtained can, or cannot, be generalized to other boundary curves through the concept logarithmic capacity. A few published general results about uniqueness are compared with our findings. © 1998 Elsevier Science B.V. All rights reserved.

AMS classification: 45F99; 35J40

Keywords: Boundary integral equation; Fundamental biharmonic problem; Uniqueness of solution; Logarithmic capacity

1. Formulation of the problem

We derive and investigate three families of direct integral equations for the solution of the plane, fundamental biharmonic boundary value problem for the function $u = u(\mathbf{r})$, viz.

$$\Delta^2 u(\mathbf{r}) = 0, \quad \mathbf{r} \in D, \tag{1a}$$

$$u(\mathbf{r}) = F_0(\mathbf{r}), \quad \frac{\partial u}{\partial v}(\mathbf{r}) = F_1(\mathbf{r}), \quad \mathbf{r} \in \Gamma, \tag{1b}$$

* E-mail: sc@imm.dtu.dk

0377-0427/98/$19.00
PII S0377-0427(98)00041-7

where Δ means the Laplacian operator, D is a bounded, simply connected domain, and Γ its boundary which is assumed to be closed, smooth and simple; the outward normal derivative to Γ at the point $\boldsymbol{r} \in \Gamma$ is denoted $\partial/\partial\nu$; further $F_0(\boldsymbol{r})$ and $F_1(\boldsymbol{r})$ are prescribed continuous functions of the arc length s on Γ.

The plane biharmonic problem arises in connection with a variety of branches within applied mathematics, most notably within:

(1) the theory of plane elasticity: (i) elastic plates: the deflection of the plate is biharmonic [67, Ch. 4], (ii) elastic sheets: the Airy stress function (from which stresses are determined) is biharmonic [65, Ch. 5];

(2) the theory of plane "slow" viscous flow: a velocity potential (from which the velocity components are determined) is biharmonic [56].

The solution of the problem (1) is unique; in particular are Δu and $\partial\Delta u/\partial\nu$ on Γ uniquely determined. If Δu and $\partial\Delta u/\partial\nu$ on Γ were available, the value of u at a point $\boldsymbol{r}'$ inside Γ can be found from the identity

$$u(\boldsymbol{r}') \equiv \oint_\Gamma \left\{ \frac{\partial\Delta u}{\partial\nu}(\boldsymbol{r})G - \Delta u(\boldsymbol{r})\frac{\partial G}{\partial\nu} + \frac{\partial u}{\partial\nu}(\boldsymbol{r})\Delta G - u(\boldsymbol{r})\frac{\partial\Delta G}{\partial\nu} \right\} \mathrm{d}s, \quad \boldsymbol{r}' \in D, \quad \boldsymbol{r} \in \Gamma, \tag{2a}$$

with

$$G \equiv G(\boldsymbol{r}', \boldsymbol{r}; \mu) := -\frac{1}{8\pi}\rho^2(\ln\rho - \mu), \quad \rho = |\,\boldsymbol{r}' - \boldsymbol{r}\,|,\ \mu \in \mathbb{R}, \tag{2b}$$

where the point $\boldsymbol{r}'$ is a parameter which is kept fixed during the integration; the derivative $\partial/\partial\nu$ acts here and in what follows on $\boldsymbol{r} \in \Gamma$. For $\mu = 0$ the identity (2) is known [69, Section 36] and because $\Delta^2\rho^2 = 0$ it is still valid for $\mu \neq 0$. The introduction of the term with factor μ creates a whole family of identities which gives a great flexibility.

• Harmonic problems: The plane Dirichlet boundary value problem, for Laplace's equation, can be solved by means of Green's third identity [60, pp. 73f.], which leads to a *direct* first kind integral equation. This equation does not have a unique solution for certain boundary curves. We denote them *critical* curves (in [39, Section 4.2], called Γ-contours) and they are characterized by means of the concept logarithmic capacity, LC (discussed thoroughly in [36]) (or exterior/outer mapping radius, external conformal radius, transfinite capacity/diameter). By amending one supplementary integral condition which the unknown function must satisfy, it is possible to eliminate this non-uniqueness [8]. The analytical part of the investigation in [8] is carried out for a circular boundary curve; to generalize the analytical investigation to elliptical boundary curves should be tractable. By a suitable quadrature method the integrals of the equation and the supplementary condition are replaced by a matrix. The condition number of this matrix is used to detect critical geometries for the integral equation and the supplementary condition [9].

• Biharmonic problems: Here we consider the plane, fundamental boundary value problem for the biharmonic equation (1) and our purpose is

(i) to derive from (2), in a systematic manner, three families of *direct* boundary integral equations with Δu and $\partial\Delta u/\partial\nu$ as unknown functions,

(ii) to analyze the equations with respect to non-uniqueness,

(iii) to find how to eliminate the non-uniqueness by amending one or more supplementary condition(s) which the unknown functions must satisfy, and
(iv) to relate the present findings to some published results.

In the following we
–(Section 2) derive the equations; state the possible supplementary conditions; specify the boundary curve; give reference to the appearance of the equations,
–(Section 3) investigate, by analytical means for a circular boundary curve, the three families with respect to uniqueness and critical radius/radii; find how to combine the equations with the supplementary integral conditions; illustrate by examples, and
–(Section 4) conjecture how to generalize our findings to other curves (in some cases by means of the logarithmic capacity of the curves); relate our findings to a few published results; point out some unsolved problems about how to characterize the critical curves. To generalize the present analytical investigation to non-circular curves seems prohibitive, but we shall subsequently [12] treat the problem by numerical methods.

There is a great amount of literature on application of integral equations to the solution of boundary value problems within the theory of elasticity and the theory of viscous flow; see the textbooks [37, 39, 42, 50, 53, 58, 61] and the literature cited therein. Here we solely consider integral equations for the boundary value problem (1) derived from the identity (2). However, other classes of integral equations are available for solving the fundamental biharmonic problem (Section 4.3).

2. Three families of direct integral equations

2.1. Derivation of equations

The derivation of the three families of direct integral equations departs from the identity (2) which depends on the parameter point $\mathbf{r}'$. With the point $\mathbf{r}'$ as origo there is introduced a local, right-handed, orthogonal coordinate system $(\hat{v}', \hat{\tau}')$ where $\hat{}$ denotes a unit vector. In this system the identity (2) is differentiated at the point $\mathbf{r}'$ with the three operators: I, $\partial/\partial v'$ and $\Delta' := \partial^2/\partial v'^2 + \partial^2/\partial \tau'^2$ where I is the identity operator which reproduce the identity (2) unchanged. Hereby we get three identities which we for easy reference denote by the number of differentiations performed, namely $\underline{0}$, $\underline{1}$ and $\underline{2}$. Each of the three identities contain the quantity μ which however need not to be the same in the three identities; in fact, we choose three quantities which we denote μ_0, μ_1 and μ_2. Let $\mathbf{r}_0$ be a point on Γ. With $\mathbf{r}_0$ as origo there is similarly introduced a local, right handed, orthogonal coordinate system $(\hat{v}_0, \hat{\tau}_0)$ where $\hat{v}_0$ is the outward normal to Γ at $\mathbf{r}_0$.

The limiting process $\mathbf{r}' \to \mathbf{r}_0$ is now carried out in the three identities so that the direction $\hat{v}'$ becomes coincident with the direction $\hat{v}_0$. Thereby the directional derivatives $\partial/\partial v'$ and $\partial/\partial \tau'$ become $\partial/\partial v_0$ and $\partial/\partial \tau_0$, respectively; furthermore, the Laplacian Δ' becomes $\partial^2/\partial v_0^2 + \partial^2/\partial \tau_0^2$ which we, for mnemotechnical reasons, denote Δ_0. However, this limiting process can not immediately be carried out under the sign of integration; the following precautions are needed:

(i) The integrals with the kernels $\partial \Delta G/\partial v, \partial \Delta' G/\partial v$ and $\partial \Delta G/\partial v'$ correspond to the logarithmic potential of a double layer or to the normal derivative of the logarithmic potential of a single layer [60, p. 125ff.], so that the limiting value of the integrals are obtained using the jump relations with a factor $-\frac{1}{2}$ or $+\frac{1}{2}$, respectively.

(ii) The integral with the kernel $\partial^2 \Delta G/\partial\nu'\partial\nu$ becomes divergent. Because of the form of $\partial\Delta G/\partial\nu$ this difficulty can be overcome [52, p. 138], using Cauchy-Riemann's equations, integration by parts and an integration around Γ giving

$$\frac{\partial}{\partial\nu'}\oint_\Gamma u(\boldsymbol{r})\frac{\partial\Delta G}{\partial\nu}(\boldsymbol{r}',\boldsymbol{r};\mu)\,\mathrm{d}s = \oint_\Gamma \frac{\partial u}{\partial s}(\boldsymbol{r})\,\frac{\partial\Delta G}{\partial\tau'}(\boldsymbol{r}',\boldsymbol{r};\mu)\,\mathrm{d}s.$$

When $\boldsymbol{r}' \to \boldsymbol{r}_0$ the last integral is to be considered as Cauchy principal value, denoted by $\int^*$.

When the above-mentioned rewritings are performed in the three identities, and the prescribed boundary values (1b) are inserted, we arrive at the following three families of direct boundary integral equations with the unknowns $W(\boldsymbol{r})$ and $V(\boldsymbol{r})$

$$\underline{0}: \quad \oint_\Gamma \left\{W(\boldsymbol{r})G(\boldsymbol{r}_0,\boldsymbol{r};\mu_0) - V(\boldsymbol{r})\frac{\partial G}{\partial\nu}(\boldsymbol{r}_0,\boldsymbol{r};\mu_0)\right\}\mathrm{d}s$$
$$= \frac{1}{2}F_0(\boldsymbol{r}_0) - \oint_\Gamma\left\{F_1(\boldsymbol{r})\Delta G(\boldsymbol{r}_0,\boldsymbol{r};\mu_0) - F_0(\boldsymbol{r})\frac{\partial\Delta G}{\partial\nu}(\boldsymbol{r}_0,\boldsymbol{r};\mu_0)\right\}\mathrm{d}s, \qquad (3\text{-}\underline{0})$$

$$\underline{1}: \quad \oint_\Gamma \left\{W(\boldsymbol{r})\frac{\partial G}{\partial\nu_0}(\boldsymbol{r}_0,\boldsymbol{r};\mu_1) - V(\boldsymbol{r})\frac{\partial^2 G}{\partial\nu_0\partial\nu}(\boldsymbol{r}_0,\boldsymbol{r};\mu_1)\right\}\mathrm{d}s$$
$$= \frac{1}{2}F_1(\boldsymbol{r}_0) - \oint_\Gamma F_1(\boldsymbol{r})\frac{\partial\Delta G}{\partial\nu_0}(\boldsymbol{r}_0,\boldsymbol{r};\mu_1)\,\mathrm{d}s + \oint_\Gamma^* \frac{\partial F_0}{\partial s}(\boldsymbol{r})\frac{\partial\Delta G}{\partial\tau_0}(\boldsymbol{r}_0,\boldsymbol{r};\mu_1)\,\mathrm{d}s, \qquad (3\text{-}\underline{1})$$

$$\underline{2}: \quad \oint_\Gamma \left\{W(\boldsymbol{r})\Delta_0 G(\boldsymbol{r}_0,\boldsymbol{r};\mu_2) - V(\boldsymbol{r})\frac{\Delta_0\partial G}{\partial\nu}(\boldsymbol{r}_0,\boldsymbol{r};\mu_2)\right\}\mathrm{d}s - \frac{1}{2}V(\boldsymbol{r}_0) = 0. \qquad (3\text{-}\underline{2})$$

For easy reference the three families of equations are also denoted $\underline{0}$, $\underline{1}$ and $\underline{2}$. For a more specific reference the actual value of μ, viz., μ_0, μ_1 or μ_2, is added as a superscript in parentheses, e.g., $\underline{2}^{(1)}$. Because $\Delta_0 G(\boldsymbol{r}_0,\boldsymbol{r};\mu) = \Delta G(\boldsymbol{r}_0,\boldsymbol{r};\mu) = -(1/2\pi)(\ln\rho + 1 - \mu)$ equation $\underline{2}^{(1)}$ can be Green's third identity [60, p. 73f.], for $W(\boldsymbol{r}) = (\partial\Delta u/\partial\nu)(\boldsymbol{r})$ and $V(\boldsymbol{r}) = \Delta u(\boldsymbol{r})$ because $\Delta u(\boldsymbol{r})$ is harmonic.

By way of constructing the three families of integral equations, the solutions $W(\boldsymbol{r}) := (\partial\Delta u/\partial\nu)(\boldsymbol{r})$ and $V(\boldsymbol{r}) := \Delta u(\boldsymbol{r})$ must satisfy the equations. But the questions arise naturally: Are the solutions of the integral equations unique ? Are there other solutions ? These questions can not simply be answered just by referring to the method of deriving the equations. In Section 3 we shall investigate this question using analytical means.

2.2. Supplementary conditions

When u is biharmonic inside Γ the boundary values of u, $\partial u/\partial\nu$, Δu and $\partial\Delta u/\partial\nu$ must satisfy some conditions. They have the form of curve integrals along Γ where the point of integration is $\boldsymbol{r} = (x,y) \in \Gamma$ with $r = |\,\boldsymbol{r}\,|$, and the outward unit normal at $\boldsymbol{r}$ is $\hat{\nu} = (\nu_x,\nu_y)$. Below we quote from

[14] (Appendix A) those conditions which turn out to be useful; we have inserted the prescribed boundary values (1b),

$$\underline{\mathrm{A}}: \quad \oint_\Gamma \frac{\partial \Delta u}{\partial \nu}(\boldsymbol{r})\,\mathrm{d}s = 0, \tag{4-$\underline{\mathrm{A}}$}$$

$$\underline{\mathrm{B}}_x: \quad \oint_\Gamma \left\{ x\frac{\partial \Delta u}{\partial \nu}(\boldsymbol{r}) - \nu_x \Delta u(\boldsymbol{r}) \right\} \mathrm{d}s = 0, \tag{4-$\underline{\mathrm{B}}_x$}$$

$$\underline{\mathrm{B}}_y: \quad \oint_\Gamma \left\{ y\frac{\partial \Delta u}{\partial \nu}(\boldsymbol{r}) - \nu_y \Delta u(\boldsymbol{r}) \right\} \mathrm{d}s = 0, \tag{4-$\underline{\mathrm{B}}_y$}$$

$$\underline{\mathrm{C}}: \quad \oint_\Gamma \left\{ r^2\frac{\partial \Delta u}{\partial \nu}(\boldsymbol{r}) - 2\hat{\nu}\cdot\boldsymbol{r}\Delta u(\boldsymbol{r}) \right\} \mathrm{d}s = -\oint_\Gamma 4F_1(\boldsymbol{r})\,\mathrm{d}s. \tag{4-$\underline{\mathrm{C}}$}$$

For easy reference the conditions are also denoted $\underline{A}$, $\underline{\mathrm{B}}_x$, $\underline{\mathrm{B}}_y$ and $\underline{C}$.

When u represents an Airy stress function, the conditions (4) can be given an elastostatic interpretation [11].

2.3. Equations in parametrized form

Assume that the boundary curve Γ has a 2π-periodic C^∞ parametrization $z(\theta) = x(\theta) + \mathrm{i}y(\theta) = \gamma\zeta(\theta) = \gamma(\xi(\theta) + \mathrm{i}\eta(\theta))$; $0 \leqslant \theta \leqslant 2\pi$, with $|\, z'(\theta)\,| > 0$ and where $\gamma > 0$ is a scaling parameter. Hereby $\boldsymbol{r}$ and $\boldsymbol{r}_0$ correspond to $z(\theta)$ and $z(\theta_0)$, respectively. Instead of using the functions $W(\boldsymbol{r}) = (\partial\Delta u/\partial\nu)(\boldsymbol{r})$ and $V(\boldsymbol{r}) = \Delta u(\boldsymbol{r})$ we express the integral equations and the supplementary conditions in terms of $w(\theta) := W(z(\theta))\ |\, z'(\theta)\,|$ and $v(\theta) := V(z(\theta))\ |\, z'(\theta)\,|$. Further are $G(\boldsymbol{r}_0, \boldsymbol{r}; \mu_\bullet)$ and ρ expressed as $g(\theta_0, \theta; \mu_\bullet) := G(z(\theta_0), z(\theta); \mu_\bullet)$ and $\rho :=|\, z(\theta_0) - z(\theta)\,|$ with similar expressions for the derivatives.

Subsequently, we shall only consider the integral equations and the supplementary conditions in the homogeneous case, i.e., with $F_0(\boldsymbol{r}) \equiv 0$ and $F_1(\boldsymbol{r}) \equiv 0$, in which case we get

- The homogeneous integral equations:

$$\underline{0}: \quad \int_0^{2\pi} g(\theta_0, \theta; \mu_0) w(\theta)\,\mathrm{d}\theta - \int_0^{2\pi} \frac{\partial g}{\partial \nu}(\theta_0, \theta; \mu_0) v(\theta)\,\mathrm{d}\theta = 0, \tag{5-$\underline{0}$}$$

$$\underline{1}: \quad \int_0^{2\pi} \frac{\partial g}{\partial \nu_0}(\theta_0, \theta; \mu_1) w(\theta)\,\mathrm{d}\theta - \int_0^{2\pi} \frac{\partial^2 g}{\partial \nu_0 \partial \nu}(\theta_0, \theta; \mu_1) v(\theta)\,\mathrm{d}\theta = 0, \tag{5-$\underline{1}$}$$

$$\underline{2}: \quad \int_0^{2\pi} \Delta_0 g(\theta_0, \theta; \mu_2) w(\theta)\,\mathrm{d}\theta - \frac{v(\theta_0)}{2\,|\, z'(\theta_0)\,|} - \int_0^{2\pi} \frac{\Delta_0 \partial g}{\partial \nu}(\theta_0, \theta; \mu_2) v(\theta)\,\mathrm{d}\theta = 0\,. \tag{5-$\underline{2}$}$$

For easy reference we still denote the equations by $\underline{0}$, $\underline{1}$ and $\underline{2}$. The detailed form of the six kernels in (5) is lengthy for a general curve $z(\theta)$. The formulas are not given here [12].

- The homogeneous supplementary conditions:

$$\underline{\mathrm{A}}: \quad \int_0^{2\pi} w\,\mathrm{d}\theta = 0 \tag{6-$\underline{\mathrm{A}}$}$$

$$\underline{\mathrm{B}}_x: \quad \int_0^{2\pi} \left(xw - \frac{y'}{|z'|} v \right) \mathrm{d}\theta = 0, \tag{6-$\underline{\mathrm{B}}_x$}$$

$$\underline{\mathrm{B}}_y: \quad \int_0^{2\pi} \left(yw + \frac{x'}{|z'|} v \right) \mathrm{d}\theta = 0, \tag{6-$\underline{\mathrm{B}}_y$}$$

$$\underline{\mathrm{C}}: \quad \int_0^{2\pi} \left(|z|^2 w - 2 \left(x\frac{y'}{|z'|} - y\frac{x'}{|z'|} \right) v \right) \mathrm{d}\theta = 0, \tag{6-$\underline{\mathrm{C}}$}$$

where $w, v, x, y, z, x', y', z'$ all depend on θ. For easy reference we still denote the conditions by $\underline{\mathrm{A}}$, $\underline{\mathrm{B}}_x$, $\underline{\mathrm{B}}_y$ and $\underline{\mathrm{C}}$.

2.4. Appearance of the equations

It is not our aim here to promote some of the equations, or to discriminate among the three families of equations. We just want to analyze the derived equations with respect to uniqueness. The reason for doing this is that all three (families of) equations appear in the literature (in the form here considered or in an analogous form):

The three integral equations (3) with $\mu_0 = \mu_1 = \mu_2 = 0$ were derived by Hougaard [59] (and the present author) for use in plane elasticity (stresses in elastic sheets). Some of these equations (or some closely related to them) have been used for formulating and solving problems within (i) stresses in elastic sheets [13, 44, 54, 59, 62] (ii) deflection of elastic plates [3, 5, 29, 63] and (iii) plane "slow" viscous flow [4, 25, 30–32, 45–49, 54]. For deflection of Kirchhoff's plates the identity (2) can be reformulated using physical quantities (bending moments, etc.) leading to integral equations involving physical quantities [1, 6, 26, 33, 43, 57, 64, 66, 68, 70–73] giving the possibility to treat further physical phenomena [2, 55, 72]; in these "plate" equations no quantities μ are introduced.

An analysis of the numerical solution of all three combinations of the complete system (3) has apparently not been carried out. But for the system $\underline{0}^{(0)}$ & $\underline{2}^{(1)}$ a thorough numerical analysis is available [21]. For the systems $\underline{0}^{(0)}$ & $\underline{1}^{(0)}$ and $\underline{0}^{(0)}$ & $\underline{2}^{(1)}$ a comparative study of the numerical feasibility has been performed [22].

We have introduced the quantities μ in the equations (3) in order explicitly to encompass some published equations as special cases:

- The system $\underline{0}^{(0)}$ & $\underline{2}^{(1)}$ is treated in [14]: Introduced; investigated analytically for circular curves where critical geometries and relevant supplementary conditions are found (cf. Section 3.3.4). By numerical means it is conjectured that the supplementary conditions hold in general and that the critical boundary curves are characterized by means of their LC.
- The systems $\underline{0}^{(0)}$ & $\underline{2}^{(1)}$ and $\underline{0}^{(1)}$ & $\underline{2}^{(1)}$ are treated in [24]: The second system is introduced. Both systems are investigated thoroughly with respect to critical geometry and supplementary conditions whereby the above conjectures are proved. The system $\underline{0}^{(0)}$ & $\underline{2}^{(1)}$ has two critical geometries (cf. Section 3.3.4), but the system $\underline{0}^{(1)}$ & $\underline{2}^{(1)}$ has only one critical geometry (cf. Section 3.3.5).
- The systems $\underline{0}^{(\mu_0)}$ & $\underline{2}^{(\mu_2)}$ have recently been investigated [19], claiming that "uniqueness on any boundary curve" [19, p. 885], is guaranteed. Supplementary conditions are not taken into account.
- A pair of *indirect* boundary integral equations with the same four kernels as for $\underline{0}^{(0)}$ & $\underline{1}^{(0)}$ (but with other unknowns) has recently been investigated [20] with respect to invertibility using

functional analysis, proving that there is between one and four critical geometries (cf. Sections 3.3.1–3.3.2).

- Equation $\underline{0}^{(1)}$ is used in [4, 5, 23, 25, 43, 45–49, 54].
- Equation $\underline{1}^{(0)}$ is used (directly) in [44, 59] and (in the corresponding "plate" formulation) in [2, 29, 64].
- Equation $\underline{2}^{(1)}$ is used in [3–5, 23, 25, 32, 43, 45–49, 54, 63].

The abundant appearance of the equations in the literature should justify our analysis.

3. Analytical investigation of the uniqueness of solutions

The purpose of the present section is to investigate, by analytical means, whether the integral equations (3) have a unique solution. Such an investigation should precede any attempt to solve the equations; here, however, we do not carry out such an attempt. To investigate uniqueness we shall find out whether the homogeneous version of (3), which is formulated in (5) using the parameter θ, possesses other solutions than the trivial one, viz. $w(\theta) \equiv 0$ and $v(\theta) \equiv 0$. The analysis is carried out for a circular boundary curve (Section 3.1) leading to some results (Section 3.2) which are illustrated by seven examples (Section 3.3). The results obtained are based on a circular boundary curve, but to repeat the present analytical reasoning for a more general curve would be prohibitive, even for an elliptical boundary curve. For a general boundary curve numerical methods (cf. Section 4.2) can be used for obtaining conjectures about uniqueness, etc.

3.1. Analysis

We follow the analytical reasoning introduced in [14] for treating the system $\underline{0}^{(0)}$ & $\underline{2}^{(1)}$. We choose as curve Γ a special curve, viz. a circle Γ_c with radius c, where $\zeta(\theta) = \cos(\theta) + \mathrm{i}\sin(\theta)$ and $|z'(\theta)| = c\,|\zeta'(\theta)| = c$, so that c plays the rôle of the scaling parameter γ. In (5) the derivatives $\partial/\partial v$ and $\partial/\partial v_0$ become $\partial/\partial r$ and $\partial/\partial r_0$ evaluated at θ and θ_0, respectively; the various directional derivatives of g are expressed in terms of ρ, θ and θ_0. After some lengthy algebra the kernels of (5) can be expressed in a fairly tractable form, written below in { }, so that (5) is formulated as

$$\underline{0}: \quad \int_0^{2\pi} \left\{ -\frac{1}{8\pi}\rho^2(\tilde{L}_0 - 1) \right\} w(\theta)\,\mathrm{d}\theta - \int_0^{2\pi} \left\{ -\frac{1}{8\pi}\rho^2(2\tilde{L}_0 - 1)\frac{1}{2c} \right\} v(\theta)\,\mathrm{d}\theta = 0, \tag{7-$\underline{0}$}$$

$$\underline{1}: \quad \int_0^{2\pi} \left\{ -\frac{1}{8\pi}\rho^2(2\tilde{L}_1 - 1)\frac{1}{2c} \right\} w(\theta)\,\mathrm{d}\theta - \int_0^{2\pi} \left\{ -\frac{1}{8\pi}\left[1 - 2\tilde{L}_1\cos(\theta - \theta_0)\right] \right\} v(\theta)\,\mathrm{d}\theta = 0, \tag{7-$\underline{1}$}$$

$$\underline{2}: \quad \int_0^{2\pi} \left\{ -\frac{1}{8\pi} 4\tilde{L}_2 \right\} w(\theta)\, \mathrm{d}\theta - \frac{v(\theta_0)}{2c} - \int_0^{2\pi} \left\{ -\frac{1}{8\pi} \frac{2}{c} \right\} v(\theta)\, \mathrm{d}\theta = 0, \tag{7-$\underline{2}$}$$

where

$$\tilde{L}_m = \ln \rho + 1 - \mu_m, \quad m = 0, 1, 2 \tag{8a}$$

with

$$\rho^2 = 4c^2 \sin^2 \frac{\theta - \theta_0}{2}. \tag{8b}$$

For easy reference we still denote the equations by $\underline{0}$, $\underline{1}$ and $\underline{2}$.

Similarly, the conditions (6) can be reformulated as

$$\underline{\mathrm{A}}: \quad \int_0^{2\pi} w(\theta)\, \mathrm{d}\theta = 0, \tag{9-$\underline{\mathrm{A}}$}$$

$$\underline{\mathrm{B}}_x: \quad \int_0^{2\pi} \cos\theta (cw(\theta) - v(\theta))\, \mathrm{d}\theta = 0, \tag{9-$\underline{\mathrm{B}}_x$}$$

$$\underline{\mathrm{B}}_y: \quad \int_0^{2\pi} \sin\theta (cw(\theta) - v(\theta))\, \mathrm{d}\theta = 0, \tag{9-$\underline{\mathrm{B}}_y$}$$

$$\underline{\mathrm{C}}: \quad \int_0^{2\pi} (cw(\theta) - 2v(\theta))c\, \mathrm{d}\theta = 0. \tag{9-$\underline{\mathrm{C}}$}$$

For easy reference we still denote the conditions by $\underline{\mathrm{A}}$, $\underline{\mathrm{B}}_x$, $\underline{\mathrm{B}}_y$ and $\underline{\mathrm{C}}$.

Obviously, the homogeneous equations (7) have the solution $w(\theta) \equiv 0$ and $v(\theta) \equiv 0$. In order to find out if the equations have other solutions, the unknowns are written as the following Ansatz:

$$cw(\theta) = A_0 + \sum_{n=1}^{\infty} n(A_n \cos n\theta + B_n \sin n\theta), \tag{10a}$$

$$v(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta). \tag{10b}$$

The Ansatz (10) is inserted in the three integral equations (7) and a very lengthy algebra is carried out, including evaluation of complicated definite integrals using tables of integrals [28]. As a partial check a computer algebra system (CAS) (also called a symbolic computation system (SCS)) may be used. Here we have tried Maple V [7], which — with some help — could confirm the results given below, provided $\ln \rho$ in (8a) is expressed by means of [28, No. 1. 441-2],

$$\ln\left(2 \sin \frac{x}{2}\right) = -\sum_{k=1}^{\infty} \frac{\cos kx}{k}, \quad 0 < x < 2\pi.$$

For each integral equation there results an infinite sum of terms having the trigonometric factors $1, \cos\theta_0, \sin\theta_0, \cos 2\theta_0, \sin 2\theta_0, \ldots,$ Because the infinite sum must be zero, each expression multiplying the trigonometric factors must be zero. Below these expressions are presented as follows: For

$n = 0$ they are shown for A_0 and a_0; for $n = 1$ and $n \geqslant 2$ they are shown separately, but only for $A_\bullet$ and $a_\bullet$; analogous expressions for $B_\bullet$ and $b_\bullet$ could have been shown, because the expressions for $n = 1$ and $n \geqslant 2$ have multiplicity *two*. The numbering $\underline{0}$, $\underline{1}$ and $\underline{2}$ indicate the integral equation from which the expression originate. From $\tilde{L}_m$ (8a) we are led to introduce

$$L_m = \ln c + 1 - \mu_m, \quad m = 0, 1, 2 \tag{11}$$

which are used in the following:
$n = 0$:

$$\underline{0}: \quad (2L_0 - 1)A_0 - 2L_0 a_0 = 0, \tag{12-00}$$

$$\underline{1}: \quad L_1 A_0 - a_0 = 0, \tag{12-01}$$

$$\underline{2}: \quad L_2 A_0 = 0. \tag{12-02}$$

$n = 1$:

$$\underline{0}: \quad (4L_0 - 1)A_1 - (4L_0 + 1)a_1 = 0, \tag{12-10}$$

$$\underline{1}: \quad (4L_1 + 1)A_1 - (4L_1 - 1)a_1 = 0, \tag{12-11}$$

$$\underline{2}: \quad A_1 - a_1 = 0. \tag{12-12}$$

$n \geqslant 2$:

$$\underline{0}: nA_n - a_n = 0, \tag{12-20}$$

$$\underline{1}: A_n - na_n = 0, \tag{12-21}$$

$$\underline{2}: A_n - a_n = 0\,. \tag{12-22}$$

The homogeneous equations (12) are to be understood as follows: For each value of n we have three linear algebraic equations with two unknown coefficients, $A_\bullet$ and $a_\bullet$; from these three equations three combinations of two equations are formed; for each combination the corresponding determinant of the 2×2 matrix is determined. The determinants are given in Table 1. We see that they are nonzero for $n \geqslant 2$ and that they may become zero for $n = 0$ and $n = 1$. For $n = 1$ the multiplicity is *two*. When a determinant is zero, the corresponding unknown coefficients are not all necessarily equal to zero. In Section 3.2 we find in which cases the various determinants are zero for certain *critical* values of c and indicate the coefficients not yet determined. For the determination of these

Table 1
The determinants of the three combinations (indicated in column C) of the three equations (12), for different values of n, expressed using the quantities L_m; $m = 0, 1, 2$; cf. (11).

C	$n = 0$	$n = 1$	$n \geqslant 2$
$\underline{0}$ & $\underline{1}$	$2L_0L_1 - 2L_0 + 1$	$8(L_0 + L_1)$	$1 - n^2$
$\underline{0}$ & $\underline{2}$	$2L_0L_2$	2	$1 - n$
$\underline{1}$ & $\underline{2}$	L_2	-2	$n - 1$

undetermined coefficients we invoke the supplementary conditions (9) and insert the Ansatz (10) in (9) which results in the following relations for some of the coefficients:

$$\underline{\mathrm{A}} : A_0 = 0, \tag{13-$\underline{\mathrm{A}}$}$$

$$\underline{\mathrm{B}}_x : A_1 - a_1 = 0, \tag{13-$\underline{\mathrm{B}}_x$}$$

$$\underline{\mathrm{B}}_y : B_1 - b_1 = 0, \tag{13-$\underline{\mathrm{B}}_y$}$$

$$\underline{\mathrm{C}} : A_0 - 2a_0 = 0\,, \tag{13-$\underline{\mathrm{C}}$}$$

which in fact furnish the lacking information about the undetermined coefficients. The two relations (13-$\underline{\mathrm{B}}_x$) and (13-$\underline{\mathrm{B}}_y$) reflect the multiplicity *two* for $n = 1$.

3.2. Results

By combining the determinants given in Table 1 with (12) and (13) we find (i) the *critical* radius/radii which result(s) in a zero for the determinant of the corresponding system taken from (12), (ii) the corresponding coefficient(s) which is/are not necessarily equal to zero according to (12), and (iii) the conditions(s) from (13) which is/are needed to force the above coefficient(s) to be zero. The results obtained are presented below for the three combinations of two equations in five main cases.

3.2.1. Equation $\underline{0}$ and Equation $\underline{1}$

Case I: $\underline{0}$ & $\underline{1}$; $n = 0$; $\mu_1 - \mu_0 > \sqrt{2} - 1$ or $\mu_1 - \mu_0 < -\sqrt{2} - 1$:

Case I(=): $\mu_1 - \mu_0 = \frac{1}{2}$:

$$c = \exp(\mu_0 - \tfrac{1}{2}) \tag{14-I(=)a}$$

Non-zero coefficient: A_0

Supplementary conditions: $\underline{\mathrm{A}}$ or $\underline{\mathrm{C}}$

$$c = \exp(\mu_0) \tag{14-I(=)b}$$

Non-zero coefficients: A_0 and a_0

Supplementary condition: $\underline{\mathrm{A}}$

Case I ($\neq$): $\mu_1 - \mu_0 \neq \frac{1}{2}$:

$$c = \exp\left(\frac{1}{2}\left[\mu_1 + \mu_0 - 1 \pm \sqrt{(\mu_1 - \mu_0)^2 + 2(\mu_1 - \mu_0) - 1}\right]\right) \tag{14-I($\neq$)^a_b}$$

Non-zero coefficients: A_0 and a_0

Supplementary conditions: $\underline{\mathrm{A}}$ or $\underline{\mathrm{C}}$

Case II: $\underline{0}$ & $\underline{1}$; $n = 1$: One value of c, but with multiplicity *two*, viz.,

$$c = \exp\left(\frac{\mu_0 + \mu_1}{2} - 1\right) \tag{14-II}$$

Supplementary conditions: $\underline{B}_x$ and $\underline{B}_y$
Case II(+): $\mu_1 - \mu_0 = +\frac{1}{2}$:
Non-zero coefficients: A_1 and B_1
Case II(−): $\mu_1 - \mu_0 = -\frac{1}{2}$:
Non-zero coefficients: a_1 and b_1
Case II(≠): $\mu_1 - \mu_0 \neq \pm\frac{1}{2}$:
Non-zero coefficients: A_1, a_1, B_1 and b_1

3.2.2. Equation $\underline{0}$ and Equation $\underline{2}$

Case III: $\underline{0}$ & $\underline{2}$; $n = 0$; $\mu_0 = \mu_2$:

$$c = \exp(\mu_0 - 1) \tag{14-III}$$

Non-zero coefficient: a_0
Supplementary condition: $\underline{C}$
Case IV: $\underline{0}$ & $\underline{2}$; $n = 0$; $\mu_0 \neq \mu_2$:
Case IV(0):

$$c = \exp(\mu_0 - 1) \tag{14-IV(0)}$$

Non-zero coefficient: a_0
Supplementary condition: $\underline{C}$
Case IV(2):

$$c = \exp(\mu_2 - 1) \tag{14-IV(2)}$$

Case IV(2)$\frac{1}{2}$: $\mu_2 - \mu_0 = \frac{1}{2}$:
Non-zero coefficient: A_0
Supplementary conditions: $\underline{A}$ or $\underline{C}$
Case IV(2)1: $\mu_2 - \mu_0 = 1$:
Non-zero coefficients: A_0 and a_0
Supplementary condition: $\underline{A}$
Case IV(2)≠: $\mu_2 - \mu_0 \neq \frac{1}{2}$ and 1 :
Non-zero coefficients: A_0 and a_0
Supplementary conditions: $\underline{A}$ or $\underline{C}$

3.2.3. Equation $\underline{1}$ and Equation $\underline{2}$

Case V: $\underline{1}$ & $\underline{2}$; $n = 0$:

$$c = \exp(\mu_2 - 1) \tag{14-V}$$

Case V(=): $\mu_1 = \mu_2$:
Non-zero coefficient: A_0
Supplementary conditions: $\underline{A}$ or $\underline{C}$
Case V(≠): $\mu_1 \neq \mu_2$:
Non-zero coefficients: A_0 and a_0
Case V(≠=): $\mu_2 - \mu_1 = \frac{1}{2}$:

Supplementary condition: $\underline{A}$
Case V($\neq\neq$): $\mu_2 - \mu_1 \neq \frac{1}{2}$:
Supplementary conditions: $\underline{A}$ or $\underline{C}$

3.3. Examples

Below we illustrate the use of the various Cases by seven Examples. We point to the relevant Case(s), give the critical value(s) of c, and state the supplementary conditions. All the Examples will subsequently be treated numerically [12].

Example a. The three systems $\underline{0}^{(0)}$ & $\underline{1}^{(0)}$, $\underline{0}^{(0)}$ & $\underline{2}^{(0)}$ and $\underline{1}^{(0)}$ & $\underline{2}^{(0)}$, i.e., $\mu_0 = 0, \mu_1 = 0$ and $\mu_2 = 0$. They relate to the Cases II($\neq$), III and V($=$), respectively. All three systems have the critical radius $c = e^{-1}$ ($\simeq$ 0.3679) with multiplicity *two*, one and one, respectively. Supplementary conditions $\underline{B}$, $\underline{C}$, and $\underline{A}$ or $\underline{C}$, respectively. (These systems were derived and considered in [59].)

Example b. The system $\underline{0}^{(0)}$ & $\underline{1}^{(1/2)}$, i.e., $\mu_0 = 0$ and $\mu_1 = 1/2$. Case I($=$): $c = e^{-1/2}$ ($\simeq$ 0.6065), conditions $\underline{A}$ or $\underline{C}$, and $c = 1$, condition $\underline{A}$. Case II($+$): $c = e^{-3/4}$ ($\simeq$ 0.4724) with multiplicity *two*, and with *two* conditions $\underline{B}_x$ and $\underline{B}_y$. Counted with multiplicity there are *four* radii. (This observation has some relation to [20].)

Example c. The system $\underline{0}^{(1)}$ & $\underline{1}^{(0)}$, i.e., $\mu_0 = 1$ and $\mu_1 = 0$. Case II($\neq$) gives only one critical radius $c = e^{-1/2}$ ($\simeq$ 0.6065) with multiplicity *two*, and with *two* conditions $\underline{B}_x$ and $\underline{B}_y$. (This system relates to Section 4.2.)

Example d. The system $\underline{0}^{(0)}$ & $\underline{2}^{(1)}$, i.e., $\mu_0 = 0$ and $\mu_2 = 1$. Case IV(0) with $c = e^{-1}$ ($\simeq$ 0.3679), only condition $\underline{C}$. Case IV(2)1 with $c = 1$, only condition $\underline{A}$. (Introduced and considered in [14] and analysed in detail in [24].)

Example e. The system $\underline{0}^{(1)}$ & $\underline{2}^{(1)}$, i.e., $\mu_0 = 1$ and $\mu_2 = 1$. Case III with only one critical radius $c = 1$, only condition $\underline{C}$. (Introduced and analysed in detail in [24].)

Example f. The system $\underline{0}^{(0)}$ & $\underline{2}^{(1/2)}$, i.e., $\mu_0 = 0$ and $\mu_2 = 1/2$. Case IV(0): $c = e^{-1}$ ($\simeq$ 0.3679), condition $\underline{C}$. Case IV(0)$\frac{1}{2}$: $c = e^{-1/2}$ ($\simeq$ 0.6065), conditions $\underline{A}$ or $\underline{C}$. (This system has relation to [19].)

Example g. The system $\underline{1}^{(0)}$ & $\underline{2}^{(1/2)}$, i.e., $\mu_0 = 0$ and $\mu_2 = 1/2$. Case V($\neq=$): $c = e^{-1/2}$ ($\simeq$ 0.6065), condition $\underline{A}$. (Also this system will be treated numerically in [12].)

4. Conclusions, conjectures, comments

4.1. Conclusions

For the solution of the plane, fundamental biharmonic boundary value problem (1) we have presented three families of *direct* boundary integral equations (3) which depend on three parameters;

the three equations are combined to form three pairs of equations. For each pair (irrespective of how the parameters are chosen) there will always exist at least one *critical* geometry where the solution is not unique. The critical geometry depends on the parameters, and by choosing the parameters suitably, the critical geometry can be moved away from the actual geometry. But a complete elimination of the critical geometry can only be obtained by the amendment of one or more of the supplementary conditions (4). The actual combinations of pairs of equations, critical geometries and supplementary conditions are presented as a collection of Cases (Section 3.2). These results have been obtained for a circular boundary curve. For a circle the logarithmic capacity, *LC*, is equal to the radius [51, p. 172], Hereby we can conclude:

- For the systems $\underline{0}^{(0)}$ & $\underline{2}^{(1)}$ and $\underline{0}^{(1)}$ & $\underline{2}^{(1)}$ our results are in accordance with [24], cf. Sections 3.3.4–3.3.5.
- For the systems $\underline{0}^{(\mu_0)}$ & $\underline{2}^{(\mu_2)}$ we find always a critical geometry, cf. Section 3.2.2, in contrast to what has been claimed in [19]. But our result is in accordance with the proof [19, p. 888], which turns out to be valid only when LC is different from the values given in (14-IV(0)) and (14-IV(2)), cf., Sections 3.3.1, and 3.3.4–3.3.6.

4.2. Conjectures

For non-circular boundary curves it is necessary to apply other methods, e.g., numerical ones. Naturally, numerical methods cannot replace mathematical proofs, but they can provide conjectures to be proved afterwards, just as [14] was followed by [24]. A preliminary diagnostic investigation of the integral equations with respect to the critical geometries and the supplementary conditions has been carried out using numerical methods [10, Section 4], where the integrals are replaced by a corresponding matrix, for which the condition number is found. A more thorough investigation is in preparation [12] using the singular value decomposition of the matrix [27].

When the preliminary numerical method is applied to circles, cf. Section 3.3, it confirms (i) the critical radius/radii in accordance with (14), and (ii) the supplementary condition(s) stated. When the method is applied to other curves, primarily ellipses, it leads to the following conjectures:

- The critical circles with multiplicity *one*, i.e., all the Cases, except Case II, generalize through the concept LC to other curves having LC equal to the value given in (14). The supplementary condition(s) derived for circles is/are still to be used.
- The critical circles with multiplicity *two*, i.e., Case II, do not directly generalize through the concept LC, because one critical size (with multiplicity two) normally split into two critical sizes (each with multiplicity one). (Example b, Section 3.3.2, does split, but Example c, Section 3.3.3, does not.) The supplementary condition(s) derived for circles is/are still to be used. It is an open question how to characterize the critical geometries in Case II for non-circular geometries. (The numerical determination of multiplicity and non-zero coefficients is carried out in [12].)

These conjectures are compared with the literature:

- The systems $\underline{0}^{(0)}$ & $\underline{2}^{(1)}$ and $\underline{0}^{(1)}$ & $\underline{2}^{(1)}$: Our conjectures are proved in [24].
- The systems $\underline{0}^{(\mu_0)}$ & $\underline{2}^{(\mu_2)}$: The proof in [19] is not valid for curves having the values of LC here conjectured, viz. (14-IV(0)) and (14-IV(2)).
- The system $\underline{0}^{(0)}$ & $\underline{1}^{(0)}$: According to our conjecture, the critical geometry is not simply expressed by means of LC of the curve, as it has preciously been conjectured [22, p. 21], An investigation [20] of a pair of *indirect* boundary integral equations with the same four kernels (but other

unknowns) has proved that there is between one and four critical geometries. This general proof corroborates our conjectures for the system $\underline{0}$ & $\underline{1}$, cf. Sections 3.2.1, 3.3.1 and 3.3.2: For $\mu_0 = \mu_1 = 0$ we get, for a circle, only one radius with multiplicity *two*, cf., [20, p. 61], while for general choices of μ_0 and μ_1, we get one, three or four critical geometries, counted with multiplicity.

4.3. Comments

As mentioned above, plane elastostatic problems can be formulated (in terms of biharmonic functions) as integral equations with critical geometries. Such problems can also be formulated as integral equations in terms of the inherent elastical quantities [15, 16, 18, 34, 35]. Also for such equations critical geometries exist [16, 34, 35]; in particular for an integral equation with a circular boundary curve the critical radius is e^{-1} [16, p. 268],

As seen above the integral equations (3) for a *simply* connected region do not have a unique solution for *certain* critical geometries. This result is not to be confused with the fact that for some integral equations for a *multiply* connected region [13, 41, 54] it is necessary for *all* geometries to amend some supplementary conditions in order to ensure uniqueness, e.g., the three conditions $\underline{A}$ and $\underline{B}$.

We have derived the integral equations (3) for the fundamental biharmonic problem (1) by using the representation (2) as the point of departure. Representations, other than (2), are available for the construction of biharmonic functions. This leads to other integral equations:

- (a) The Goursat representation: $u = \Re[\bar{z}\phi(z) + \chi(z)]$, where ϕ and χ are analytic functions of the complex variable z [53] (5.2.5), leads to integral equations of Muskhelishvili [53] (Ch. 5, § 3) and Lauricella-Sherman [53] (Ch. 5, § 6). For these integral equations no critical geometry exists.
- (b) The Almansi representation: $u = r^2\phi + \psi$, where ϕ and ψ are harmonic functions of the polar coordinates (r, θ) [38] (23). This approach may lead to (b1) a pair of coupled integral equations for the boundary values of ϕ and $\partial\phi/\partial v$ [38] (26), or (b2) a pair of coupled integral equations for two single layer densities used to generate ϕ and ψ, respectively [40] (p. 311). To try to create a connection between the pair (b1) and the system $\underline{0}^{(0)}$ & $\underline{2}^{(1)}$ is not easy [59] (§ 3.3). Therefore it is not known whether the present analysis concerning critical geometries for the system $\underline{0}^{(0)}$ & $\underline{2}^{(1)}$ can be carried over to the pair (b1).

Acknowledgements

For the inspiration to carry out the present investigation and for many stimulating discussions I want to thank Vice President, Ph.D., Peter Hougaard, Burmeister & Wain Energi A/S, Virum, Denmark. Professor, Dr. techn. Erik B. Hansen, Department of Mathematics, The Technical University of Denmark, is thanked for reading a previous version of the manuscript. Two referees are thanked for valuable comments which led to a thorough restructuralization of the paper. A referee is thanked for providing a reference [45].

References

[1] G. Bézine, Boundary integral formulation for plate flexure with arbitrary boundary conditions, Mech. Res. Commun. 5 (1978) 197–206.

[2] G. Bezine, A new boundary element method for bending of plates on elastic foundations, Int. J. Solids Struct. 24 (1988) 557–565.
[3] D.G.A. Brickell, Integral equation method for influence surfaces of a corner plate, Inst. Civil Eng. Proc. 46 (1970) 185–194.
[4] C.V. Camp, G.S. Gipson, A boundary element method for viscous flows at low Reynolds numbers, Eng. Anal. with Boundary Elements 6 (1989) 144–151.
[5] C.V. Camp, G.S. Gipson, Biharmonic analysis of rectilinear plates by the boundary element method, Int. J. Numer. Methods Eng. 30 (1990) 517–539.
[6] C. Chang-jun, W. Rong, Boundary integral equations and the boundary element method for buckling analysis of perforated plates, Eng. Anal. with Boundary Elements 17 (1996) 57–68.
[7] B.W. Char, K.O. Geddes, G.H. Gonnet, B.L. Leong, M.B. Monagan, S.M. Watt, Maple V, Library Reference Manual and Language Reference Manual, Springer, New York, 1991.
[8] S. Christiansen, Integral equations without a unique solution can be made useful for solving some plane harmonic problems, J. Inst. Math. Appl. 16 (1975) 143–159.
[9] S. Christiansen, On two methods for elimination of non-unique solutions of an integral equation with logarithmic kernel, Applicable Anal. 13 (1982) 1–18.
[10] S. Christiansen, Derivation and investigation of three direct boundary integral equations for the fundamental biharmonic problem, Department of Mathematical Modelling, The Technical University of Denmark, DK-2800 Lyngby, Denmark, ISSN 0909 6264. Technical Report IMM-REP-1996-22, 1997-01-30, 26 pp.
[11] S. Christiansen, On the elastostatic significance of four boundary integrals involving biharmonic functions, Acta Mechanica 126 (1998) 37–43.
[12] S. Christiansen, In preparation.
[13] S. Christiansen, E. Hansen, A direct integral equation method for computing the hoop stress at holes in plane isotropic sheets, J. Elasticity 5 (1975) 1–14.
[14] S. Christiansen, P. Hougaard, An investigation of a pair of integral equations for the biharmonic problem, J. Inst. Math. Appl. 22 (1978) 15–27.
[15] C. Constanda, Sur le problème de Dirichlet dans déformation plane, Comptes Rendus de l'Académie des Sciences Paris, Série I, Mathematique 316 (1993) 1107–1109.
[16] C. Constanda, On integral solutions of the equations of thin plates, Proc. Math. and Physical Sciences, Proc. Roy. Soc. London, Series A 444 (1994) 317–323.
[17] C. Constanda, On non-unique solutions of weakly singular integral equations in plane elasticity, Quart. J. Mech. and Appl. Math. 47 Part. 2 (1994) 261–268.
[18] C. Constanda, The boundary integral equation method in plane elasticity, Proc. Am. Math. Soc. 123 (1995) 3385–3396.
[19] C. Constanda, On the dirichlet problem for the two-dimensional biharmonic equation, Math. Methods Appl. Sci. 20 (1997) 885–890.
[20] M. Costabel, M. Dauge, Invertibility of the Biharmonic Single Layer Potential Operator, Integral Equations and Operator Theory 24 (1996) 46–67.
[21] M. Costabel, J. Saranen, Boundary element analysis of a direct method for the biharmonic dirichlet problem, Operator Theory: Advances and Applications 41 (1989) 77–95.
[22] M. Costabel, I. Lusikka, J. Saranen, Comparison of Three Boundary Element Approaches for the Solution of the Clamped Plate Problem, in: C.A. Brebbia, W.L. Wendland, G. Kuhn (Eds), Boundary Elements IX, Vol. 2: Stress Analysis, Applications Computational Mechanics Publications, Southampton and Springer, Berlin 1987, pp. 19–34.
[23] G. Fairweather, F.J. Rizzo, D.J. Shippy, Y.S. Wu, On the numerical solution of two-dimensional potential problems by an improved boundary integral equation method, J. Comput. Phys. 31 (1979) 96–112.
[24] B. Fuglede, On a direct method of integral equations for solving the biharmonic dirichlet problem, Z. Angew. Math. Mech. 61 (1981) 449–459.
[25] R. Goodwin, G.M. Homsy, Viscous flow down a slope in the vicinity of a contact line, Phys. Fluids A 3 (1991) 515–528.
[26] J. Giroire, J.-C. Nédélec, A new system of boundary integral equations for plates with free edges, Math. Methods Appl. Sci. 18 (1995) 755–772.
[27] G.H. Golub, C.F. Van Loan, Matrix Computations, 2nd ed., Johns Hopkins University Press, Baltimore, MD 1989.

[28] I.S. Gradshteyn, I.M. Ryzhik, Table of Integrals, Series, and Products, 4th ed., Academic Press, New York, 1965.
[29] E.B. Hansen, Numerical solution of integro-differential and singular integral equations for plate bending problems, J. Elasticity 6 (1976) 39–56.
[30] E.B. Hansen, Stokes flow down a wall into an infinite pool, J. Fluid Mech. 178 (1987) 243–256.
[31] E.B. Hansen, M.A. Kelmanson, Steady, viscous, free-surface flow on a rotating cylinder, J. Fluid Mech. 272 1994 91–107.
[32] E.B. Hansen, M.A. Kelmanson, An integral equation justification of the boundary conditions of the driven-cavity problem, Comput. Fluids 23 (1994) 225–240.
[33] F. Hartmann, R. Zotemantel, The Direct Boundary Element Method in Plate Bending, Int. J. Numer. Methods Eng. 23 (1986) 2049–2069.
[34] U. Heise, The spectra of some integral operators for plane elastostatical boundary value problems, J. Elasticity 8 (1978) 47–79.
[35] U. Heise, Dependence of the round-off error in the solution of boundary integral equations on a geometrical scale factor. Comput. Methods Appl. Mech. Eng. 62 (1987) 115–126.
[36] G.C. Hsiao, R.E. Kleinman, On a Unified Characterization of Capacity, in: J. Kral, J.L.I. Netuka, J. Vesely (Eds.), Potential Theory, Plenum Press, New York, 1988 pp. 105–119.
[37] D.B. Ingham, M.A. Kelmanson, Boundary Integral Equation Analyses of Singular, Potential, and Biharmonic Problems, Lecture Notes in Eng. 7 Springer, Berlin, 1984.
[38] M.A. Jaswon, Integral equation methods in potential theory, I, Proc. Roy. Soc. London A 275 (1963) 23–32.
[39] M.A. Jaswon, G.T. Symm, Integral Equation Methods in Potential Theory and Elastostatics, Academic Press, London, 1977.
[40] M.A. Jaswon, M. Maity, G.T. Symm, Numerical biharmonic analysis and some applications, Int. J. Solids Struct. 3 (1967) 309–332.
[41] A.I. Kalandiia, Plane problem of elasticity theory in a multiconnected domain, J. Appl. Math. Mech. 43 (1979) 910–924.
[42] A.I. Kalandiya, Mathematical Methods of Two-dimensional Elasticity, Revised from the 1973 Russian edition, Mir Publishers, Moscow, 1975
[43] G. Karami, J. Zarrinchang, B. Foroughi, Analytical treatment of boundary integrals in direct boundary element analysis of plate bending problems, Int. J. Numer. Methods Eng. 37 (1994) 2409–2427.
[44] J.T. Katsikadelis, C.V. Massalas, G.J. Tzivanidis, An integral equation solution of the plane problem of the theory of elasticity, Mech. Res. Commun. 4 (1977) 199–208.
[45] M.A. Kelmanson, Modified integral equation solution of viscous flows near sharp corners, Comput. Fluids 11 (1983) 307–324.
[46] M.A. Kelmanson, Boundary integral solution of viscous flows with free surfaces, J. Eng. Math. 17 (1983) 329–343.
[47] M.A. Kelmanson, An integral equation method for the solution of singular slow flow problems, J. Comput. Phys. 51 (1983) 139–158.
[48] M.A. Kelmanson, A boundary integral equation method for the study of slow flow in bearings with arbitrary geometries, J. Tribology (Trans. ASME) 106 (1984) 260–264.
[49] W.Q. Lu, H.C. Chang, An extension of the biharmonic integral method to free surface flow in channels, J. Comput. Phys. 77 (1988) 340–360.
[50] K. Kitagawa, Boundary Element Analysis of Viscous Flow, Lecture Notes in Engineering 55, Springer, Berlin, 1990.
[51] N.S. Landkof, Foundations of Modern Potential Theory, Springer, Berlin, 1972.
[52] S.G. Mikhlin, Integral Equations and their applications (to certain problems in mechanics, mathematical physics and technology) 2nd rev. ed., Pergamon Press, Oxford, 1964.
[53] S.G. Mikhlin, N.F. Morozov, M.V. Paukshto, The Integral Equations of the Theory of Elasticity, Teubner-Texte zur Mathematik 135, Teubner, Stuttgart, 1995.
[54] A.K. Mitra, S. Das, Nonuniqueness in the integral equation formulation of the biharmonic equation in multiply connected domains, Computer Methods Appl. Mech. Eng. 69 (1988) 205–214.
[55] M.S. Nerantzaki, J.T. Katsikadelis, A Green's function method for large deflection analysis of plates, Acta Mechanica 75 (1988) 211–225.
[56] H. Ockendon, J.R. Ockendon, Viscous Flow, Cambridge Texts in Appl. Math., Cambridge University Press, Cambridge, 1995.

[57] F. París, S. de León, Simply supported plates by the boundary integral equation method, Int. J. Numer. Methods Eng. 23 (1986) 173–191.
[58] V.Z. Parton, P.I. Perlin, Integral Equations in Elasticity, Revised from the 1977 Russian edition, Mir Publishers, Moscow, 1982.
[59] P. Hougård Pedersen, M.Sc. Thesis (in Danish), Laboratory of Applied Mathematical Physics, The Technical University of Denmark, DK-2800 Lyngby, Denmark, 1971, 120 pp.
[60] G.N. Polozhiy, Equations of Mathematical Physics, Hayden Book Company, New York, 1967.
[61] C. Pozrikidis, Boundary Integral and Singularity Methods for Linearized Viscous Flow, Cambridge texts in appl. math. Cambridge University Press, Cambridge, 1972.
[62] W. Rzasnicki, A. Mendelson, L.U. Albers, Application of boundary integral method to elastic analysis of V-notched beams, NASA Technical Note, NASA TN D-7424, October 1973, 28 pp.
[63] C.M. Segedin, D.G.A. Brickell, Integral equation method for a corner plate, J. Struct. Div. Proc. Am. Soc. Civil Eng. 94 (1968) 41–52.
[64] V. Sládek, J. Sládek, On the boundary integral equation method in the classical plate theory, Mech. Res. Commun. 10 (1983) 111–120.
[65] I.S. Sokolnikoff, Mathematical Theory of Elasticity, 2nd ed., McGraw-Hill, New York, 1956.
[66] M. Stern, A general boundary integral formulation for the numerical solution of plate bending problems, Int. J. Solids Struct. 15 (1979) 769–782.
[67] S. Timoshenko, S.Woinowsky-Krieger, Theory of Plates and Shells 2nd ed., McGraw-Hill, New York, 1959.
[68] M. Vable, Y. Zhang, A boundary element method for plate bending problems, Int. J. Solids Struct. 29 (1992) 345–361.
[69] I.N. Vekua, New Methods for Solving Elliptic Equations, North-Holland, Amsterdam, 1967.
[70] W.S. Venturini, J.B. Paiva, Boundary element for plate bending analysis, Eng. Anal. Boundary Elements 11 1993 1–8.
[71] J.L. Wearing, O. Bettahar, The analysis of plate bending problems using the regular direct boundary element method, Eng. Anal. Boundary Elements 16 (1995) 261–271.
[72] J. Ye, An Alternative approach for large deflection analysis of axisymmetric plates on elastic foundation, Commun. Numer. Methods Eng. 10 (1994) 623–632.
[73] T.Q. Ye, Y. Liu, Finite deflection analysis of elastic plate by the boundary element method, Appl. Math. Modelling 9 (1985) 183–188.