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The Rank and Minimal Border Strip Decompositions of a Skew Partition

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The rank of an ordinary partition of a nonnegative integer n is the length of the main diagonal of its Ferrers or Young diagram. Nazarov and Tarasov gave a generalization of this definition for skew partitions and proved some basic properties. We show the close connection between the rank of a skew partition λ/μ and the minimal number of border strips whose union is λ/μ. A general theory of minimal border strip decompositions is developed and an application is given to the evaluation of certain values of irreducible characters of the symmetric group.

1. INTRODUCTION

Let $\lambda = (\lambda_1, \lambda_2, \ldots)$ be a partition of the integer n, i.e., $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant 0$ and $\sum \lambda_i = n$. The (Durfee or Frobenius) *rank* of λ, denoted $\operatorname{rank}(\lambda)$, is the length of the main diagonal of the diagram of λ, or equivalently, the largest integer i for which $\lambda_i \geqslant i$ [11, p. 289]. We will assume familiarity with the notation and terminology involving partitions and symmetric functions found in [7, 11]. Nazarov and Tarasov [9, Sect. 1], in connection with tensor products of Yangian modules $Y(\mathfrak{gl}_n)$, defined a generalization of rank to skew partitions (or skew diagrams) λ/μ. There are several simple equivalent definitions of $\operatorname{rank}(\lambda/\mu)$ which we summarize in Proposition 2.2. In particular, $\operatorname{rank}(\lambda/\mu)$ is the least integer r such that λ/μ is a disjoint union of r border strips (also called ribbons or rim hooks). In Section 4, we consider the structure of the decompositions of λ/μ into this minimal number r of border strips. For instance, we show that the number of ways to write λ/μ as a disjoint union of r border strips is a perfect square. A consequence of our results will be that if $\chi^{\lambda/\mu}$ is the skew character of the

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symmetric group $\mathfrak{S}_n$ indexed by λ/μ and if w is a permutation in $\mathfrak{S}_n$ with $\text{rank}(\lambda/\mu)$ cycles (in its disjoint cycle decomposition) for which exactly m_i cycles have length i, then $\chi^{\lambda/\mu}(w)$ is divisible by $m_1!\,m_2!\cdots$.

In addition to the various characterizations of $\text{rank}(\lambda/\mu)$ given by Proposition 2.2 we have a further possible characterization which we have been unable to prove or disprove. Namely, let $s_{\lambda/\mu}(1^t)$ denote the skew Schur function $s_{\lambda/\mu}$ evaluated at $x_1 = \cdots = x_t = 1$, $x_i = 0$ for $i > t$. For fixed λ/μ, $s_{\lambda/\mu}(1^t)$ is a polynomial in t. Let $\text{zrank}(\lambda/\mu)$ denote the exponent of the largest power of t dividing $s_{\lambda/\mu}(1^t)$ (as a polynomial in t). It is easy to see (Proposition 3.1) that $\text{zrank}(\lambda/\mu) \geqslant \text{rank}(\lambda/\mu)$, and we ask whether equality always holds. We know of two main cases where the answer is affirmative: (1) when λ/μ is an ordinary partition (i.e., $\mu = \emptyset$), a trivial consequence of known results on Schur functions (Theorem 3.2(a)), and (2) when every row of the Jacobi–Trudi matrix for λ/μ which contains an entry equal to 0 also contains an entry equal to 1 (Theorem 3.2(b)).

2. CHARACTERIZATIONS OF FROBENIUS RANK

Let λ/μ be a skew shape, which we identify with its Young diagram $\{(i,j)\colon \mu_i < j \leqslant \lambda_i\}$. While all our results are stated in terms of the partitions λ and μ, it should be mentioned that these results depend on λ and μ only up to translation of the skew shape λ/μ. We regard the points (i,j) of the Young diagram as squares. An *outside top corner* of λ/μ is a square $(i,j) \in \lambda/\mu$ such that $(i-1,j), (i,j-1) \notin \lambda/\mu$. An *outside diagonal* of λ/μ consists of all squares $(i+p, j+p) \in \lambda/\mu$ for which (i,j) is a fixed outside top corner. Similarly, an *inside top corner* of λ/μ is a square $(i,j) \in \lambda/\mu$ such that $(i-1,j), (i,j-1) \in \lambda/\mu$ but $(i-1,j-1) \notin \lambda/\mu$. An *inside diagonal* of λ/μ consists of all squares $(i+p, j+p) \in \lambda/\mu$ for which (i,j) is a fixed inside top corner. If $\mu = \emptyset$, then λ/μ has one outside diagonal (the main diagonal) and no inside diagonals. Figure 1 shows the skew shape 8874/411, with outside diagonal squares marked by $+$ and inside diagonal squares by $-$.

Let $d^+(\lambda/\mu)$ (respectively, $d^-(\lambda/\mu)$) denote the total number of outside diagonal squares (respectively, inside diagonal squares) of λ/μ. Following Nazarov and Tazarov [9, Sect. 1], we define the (Durfee or Frobenius) *rank* of λ/μ, denoted $\text{rank}(\lambda/\mu)$, to be $d^+(\lambda/\mu) - d^-(\lambda/\mu)$. Clearly, when $\mu = \emptyset$ this reduces to the usual definition of $\text{rank}(\lambda)$ mentioned in the Introduction. We see, for instance, from Fig. 1 that rank $(8874/411) = 4$.

We wish to give several equivalent definitions of $\text{rank}(\lambda/\mu)$. First, we discuss the necessary background. A skew shape λ/μ is *connected* if the interior of the Young diagram of λ/μ, regarded as a union of solid squares,

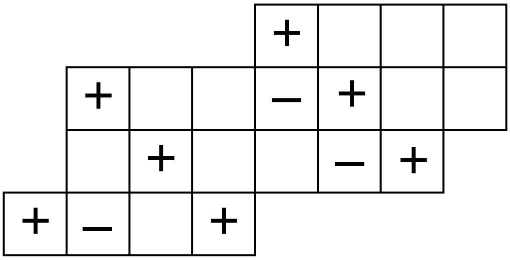

FIG. 1. Outside and inside diagonals of the skew shape 8874/411.

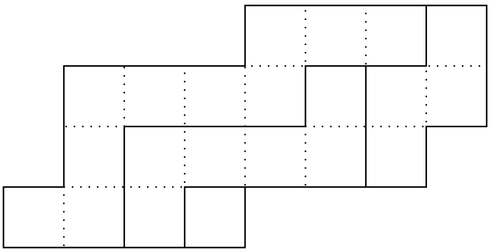

FIG. 2. A minimal border strip decomposition of the skew shape 8874/411.

is a connected (open) set. A *border strip* [11, p. 345] is a connected skew shape with no 2×2 square. (The empty diagram $\emptyset$ is *not* a border strip.) A border strip is uniquely determined, up to translation, by its row lengths; there are exactly 2^{n-1} border strips with n squares (up to translation). We say that a border strip $B \subseteq \lambda/\mu$ is a *border strip of* λ/μ if $\lambda/\mu - B$ is a skew shape ν/μ (so $B = \lambda/\nu$). Equivalently, we say that B can be *removed* from λ/μ. A border strip B of λ/μ is determined by its lower left-hand square $\text{init}(B)$ and upper right-hand square $\text{fin}(B)$. A *border strip decomposition* [11, p. 470] of λ/μ is a partitioning of the squares of λ/μ into (pairwise disjoint) border strips. Let $N = |\lambda/\mu| := \sum \lambda_i - \sum \mu_i$ and $\sigma = (\sigma_1, \ldots, \sigma_\ell) \vdash N$, where $\sigma_\ell > 0$. We say that a border strip decomposition $\boldsymbol{D}$ has *type* $\sigma \vdash N$ if the sizes (number of squares) of the border strips appearing in $\boldsymbol{D}$ are $\sigma_1, \ldots, \sigma_\ell$. A border strip decomposition of λ/μ is *minimal* if the number of border strips is minimized, i.e., there does not exist a border strip decomposition with fewer border strips. Figure 2 shows a minimal border strip decomposition of the skew shape 8874/411.

A concept closely related to border strip decompositions is that of border strip tableaux [11, p. 346]. Let $\lambda/\mu \vdash N$. Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_m)$ be a composition of N, i.e., $\alpha_i \in \mathbb{P} = \{1, 2, \ldots\}$ and $\sum \alpha_i = N$. A *border strip tableau* of (shape) λ/μ and type α is a sequence

$$\mu = \lambda^0 \subset \lambda^1 \subset \cdots \subset \lambda^r = \lambda \tag{1}$$

such that λ^i/λ^{i-1} is a border strip of size α_i. (Note that the type of a border strip decomposition is a partition but of a border strip tableau is a composition.) Often in the definition of a border strip tableau there is allowed $\lambda^i/\lambda^{i-1} = \emptyset$, but it will be convenient for us not to permit this. Every border strip tableau $\boldsymbol{T}$ of shape λ/μ defines a border strip decomposition $\boldsymbol{D}$ of λ/μ, viz., the border strips λ^i/λ^{i-1} of $\boldsymbol{T}$ are just the border strips of $\boldsymbol{D}$. We say that $\boldsymbol{D}$ *corresponds* to $\boldsymbol{T}$ and conversely that $\boldsymbol{T}$ *corresponds* to $\boldsymbol{D}$. Of course given $\boldsymbol{T}$, the corresponding $\boldsymbol{D}$ is unique, but not conversely. If $\boldsymbol{T}$ corresponds to a minimal border strip decomposition $\boldsymbol{D}$, then we call $\boldsymbol{T}$ a *minimal* border strip tableau.

Now suppose that $\ell(\lambda) \leqslant n$, where $\ell(\lambda)$ denotes the number of (nonzero) parts of λ. Recall that the Jacobi–Trudi identity for the skew Schur function $s_{\lambda/\mu}$ [11, Theorem 7.16.1] asserts that

$$s_{\lambda/\mu} = \det(h_{\lambda_i - \mu_j - i + j})_{i,j=1}^{n},$$

where h_k denotes the complete homogeneous symmetric function of degree k, with the convention $h_0 = 1$ and $h_k = 0$ for $k < 0$. Denote the matrix $(h_{\lambda_i - \mu_j - i + j})$ appearing in the Jacobi–Trudi identity by $\mathrm{JT}_{\lambda/\mu}$, called the *Jacobi–Trudi matrix* of the skew shape λ/μ. Let $\mathrm{jrank}(\lambda/\mu)$ denote the number of rows of $\mathrm{JT}_{\lambda/\mu}$ that do not contain a 1. Note that $\mathrm{JT}_{\lambda/\mu}$ implicitly depends on n, but $\mathrm{jrank}(\lambda/\mu)$ does not depend on the choice of n.

Our final piece of background material concerns the (Comét) code of a shape λ [11, Exercise 7.59], generalized to skew shapes λ/μ. Let λ/μ be a skew shape, with its left-hand edge and upper edge extended to infinity, as shown in Fig. 3 for $\lambda/\mu = 8874/411$. Put a 0 next to each vertical edge and a 1 next to each horizontal edge of the "lower envelope" and "upper envelope" of λ/μ (whose definition should be clear from Fig. 3). If we read these numbers as we move north and east along the lower envelope we obtain a binary sequence $C_{\lambda/\mu} = \cdots c_{-2}c_{-1}c_0c_1c_2\cdots$ beginning with infinitely many 0's and ending with infinitely many 1's. Similarly, if we read these numbers as we move north and east along the upper envelope we obtain another such binary sequence $D_{\lambda/\mu} = \cdots d_{-2}d_{-1}d_0d_1d_2\cdots$. The indexing of the terms of $C_{\lambda/\mu}$ and $D_{\lambda/\mu}$ is arbitrary (it does not affect the sequences themselves), but we require them to "line up" in the sense that common steps in the two envelopes should have common indices. We call the

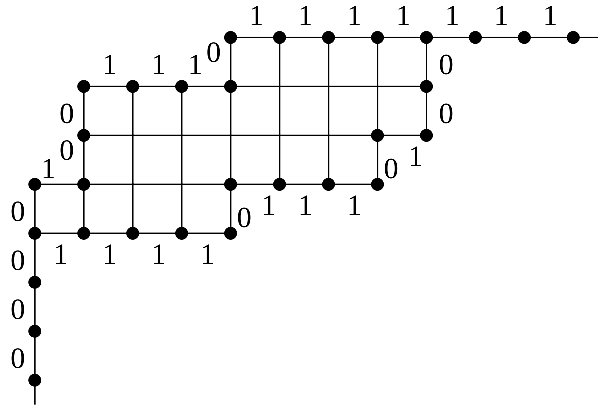

FIG. 3. Constructing the code of 8874/411.

resulting two-line array

$$\operatorname{code}(\lambda/\mu) = \begin{matrix} \cdots & c_{-2} & c_{-1} & c_0 & c_1 & c_2 & \cdots \\ \cdots & d_{-2} & d_{-1} & d_0 & d_1 & d_2 & \cdots \end{matrix}, \tag{2}$$

the (Comét) *code* of λ/μ (also known as the *partition sequence* of λ/μ [1, 2]). If we omit the infinitely many initial columns $\substack{0\\0}$ and final columns $\substack{1\\1}$ from code(λ/μ), then we call the resulting array the *reduced code* of λ/μ, denoted $\overline{\text{code}}(\lambda/\mu)$. Thus for instance from Fig. 3 we see that

$$\overline{\text{code}}(8874/411) = \begin{matrix} 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \end{matrix}.$$

A two-line array (2) with infinitely many initial columns $\substack{0\\0}$ and final columns $\substack{1\\1}$ is the code of some λ/μ if and only if for all i,

$$\#\{j \leqslant i : (c_j, d_j) = (1, 0)\} \geqslant \#\{j \leqslant i : (c_j, d_j) = (0, 1)\}, \tag{3}$$

and

$$\#\{j \in \mathbb{Z} : (c_j, d_j) = (1, 0)\} = \#\{j \in \mathbb{Z} : (c_j, d_j) = (0, 1)\}. \tag{4}$$

If $\mu = \emptyset$ then the second row of code(λ/μ) is redundant, so we define code(λ) to be the first row of code(λ/μ). If code(λ/μ) is given by (2) then we write $s(c_i)$ (respectively, $s(d_i)$) for the (unique) square of λ/μ that contains the edge of the lower envelope (respectively, upper envelope) of λ/μ corresponding to c_i (respectively, d_i). The following fundamental property of $\mathrm{code}(\lambda/\mu)$ appears e.g. in [11, Exercise 7.59(b)] for ordinary shapes and carries over directly to skew shapes.

PROPOSITION 2.1. *Let* $\mathrm{code}(\lambda/\mu)$ *be given by* (2). *Then removing a border strip of size p from* λ/μ *is equivalent to choosing i with* $c_i = 1$ *and* $c_{i+p} = 0$, *and then replacing* c_i *with* 0 *and* c_{i+p} *with* 1, *provided that* (3) *continues to hold. Specifically, such a pair* $(i, i+p)$ *corresponds to the border strip B of size p defined by*

$$\mathrm{init}(B) = s(c_i), \qquad \mathrm{fin}(B) = s(c_{i+p}).$$

Moreover, $\mathrm{code}(\lambda/\mu - B)$ *is obtained from* $\mathrm{code}(\lambda/\mu)$ *by setting* $c_i = 0$ *and* $c_{i+p} = 1$.

We can now state several characterizations of $\mathrm{rank}(\lambda/\mu)$.

PROPOSITION 2.2. *For any skew shape* λ/μ, *the following numbers are equal*:

(a) $\mathrm{rank}(\lambda/\mu)$,

(b) *the number of border strips in a minimal border strip decomposition of* λ/μ,

(c) $\mathrm{jrank}(\lambda/\mu)$,

(d) *the number of columns of* $\mathrm{code}(\lambda/\mu)$ *equal to* $\genfrac{}{}{0pt}{}{0}{1}$ (*or to* $\genfrac{}{}{0pt}{}{1}{0}$).

Proof. By Eqs. (3) and (4) there exists a bijection

$$\vartheta : \{i : (c_i, d_i) = (1, 0)\} \to \{i : (c_i, d_i) = (0, 1)\}$$

such that $\vartheta(i) > i$ for all i in the domain of ϑ. By Proposition 2.1, as we successively remove border strips from λ/μ the bottom line $\cdots d_{-1}d_0d_1 \cdots$ of $\mathrm{code}(\lambda/\mu)$ remains the same, while the top line $\cdots c_{-1}c_0c_1 \cdots$ interchanges a 0 and 1. We will exhaust all of λ/μ when the top line becomes equal to the bottom. Hence the number of border strips appearing in a border strip decomposition of λ/μ is at least the number of columns $\genfrac{}{}{0pt}{}{0}{1}$ of $\mathrm{code}(\lambda/\mu)$. On the other hand, we can achieve exactly this number by interchanging c_i with $c_{\vartheta(i)}$ for all i such that $(c_i, d_i) = (0, 1)$. It follows that (b) and (d) are equal.

Let B be the (unique) largest border strip of λ/μ such that $\mathrm{init}(B)$ is the bottom square of the leftmost column of λ/μ. B will intersect each diagonal (running from upper-left to lower-right) of its connected component σ of λ/μ exactly once. The number of outside diagonals of σ is one more than the number of inside diagonals. Hence $\mathrm{rank}(\lambda/\mu) = \mathrm{rank}(\lambda/\mu - B) + 1$. Continuing to remove the largest border strip results in a minimal border strip decomposition of λ/μ. (Minimality is an easy consequence of Proposition 2.1). Since each border strip removal reduces the rank by one, it follows that (a) and (b) are equal.

Finally, consider the Jacobi–Trudi matrix $\mathrm{JT}_{\lambda/\mu}$. We prove by induction on the number of rows of $\mathrm{JT}_{\lambda/\mu}$ that (b) and (c) are equal. The assertion is clear when $\mathrm{JT}_{\lambda/\mu}$ has one row, so assume that $\mathrm{JT}_{\lambda/\mu}$ has more than one row. We may assume that λ/μ has no empty rows, since "compressing" λ/μ by removing all empty rows does not change (c). Let $\mathrm{JT}'_{\lambda/\mu}$ denote $\mathrm{JT}_{\lambda/\mu}$ with the first row and last column removed. Let v/μ be the shape obtained by removing a maximal border strip from each connected component of λ/μ and deleting the bottom (empty) row. If λ/μ has c connected components, then $\mathrm{rank}(v/\mu) = \mathrm{rank}(\lambda/\mu) - c$. Now the (i,j)-entry $h_{\lambda_{i+1}-\mu_j-i+j-1}$ of the matrix $\mathrm{JT}'_{\lambda/\mu}$ satisfies

$$h_{\lambda_{i+1}-\mu_j-i+j-1} = \begin{cases} h_{v_i-\mu_j-i+j} & \text{if row } i \text{ of } \lambda/\mu \text{ is not the last row of a} \\ & \text{connected component of } \lambda/\mu, \\ h_{v_i-\mu_j-i+j+1} & \text{otherwise.} \end{cases}$$

Moreover, if row i is the last row of a connected component of λ/μ (other than the bottom row of λ/μ) then the (i,i)-entry of $\mathrm{JT}_{v/\mu}$ is 1, while the ith row of $\mathrm{JT}_{\lambda/\mu}$ does not contain a 1. It follows that $\mathrm{jrank}(v/\mu) = \mathrm{jrank}(\lambda/\mu) - c$, and the equality of (b) and (c) follows by induction. ∎

The equivalence of (a) and (c) in Proposition 2.2 is also an immediate consequence of [9, Proposition 1.32].

The following corollary was first proved by Nazarov and Tarasov [9, Theorem 1.4] using the definition $\mathrm{rank}(\lambda/\mu) = d^+(\lambda/\mu) - d^-(\lambda/\mu)$. The result is not obvious (even for nonskew shapes λ) using this definition, but it is an immediate consequence of parts (b) or (d) of Proposition 2.2.

COROLLARY 2.3. *Let* $(\lambda/\mu)^\natural$ *denote the skew shape obtained by rotating the diagram of* λ/μ 180°, *i.e., replacing* $(i,j) \in \lambda/\mu$ *with* $(h-i, k-i)$ *for some* h *and* k. *Then* $\mathrm{rank}(\lambda/\mu) = \mathrm{rank}((\lambda/\mu)^\natural)$.

3. AN OPEN CHARACTERIZATION OF RANK(λ/μ)

Recall that in Section 1 we defined $\mathrm{zrank}(\lambda/\mu)$ to be the largest power of t dividing the polynomial $s_{\lambda/\mu}(1^t)$.

Open problem. Is it true that

$$\operatorname{rank}(\lambda/\mu) = \operatorname{zrank}(\lambda/\mu) \tag{5}$$

for all λ/μ?

PROPOSITION 3.1. *For all* λ/μ *we have* $\operatorname{rank}(\lambda/\mu) \leqslant \operatorname{zrank}(\lambda/\mu)$.

Proof. We have (see [11, Proposition 7.8.3])

$$h_i(1^t) = \binom{t+i-1}{i} = \frac{t(t+1)\cdots(t+i-1)}{i!}.$$

Hence by the Jacobi–Trudi identity,

$$s_{\lambda/\mu}(1^t) = \det\left(\binom{t+\lambda_i-\mu_j-i+j-1}{\lambda_i-\mu_j-i+j}\right)_{i,j=1}^{n}. \tag{6}$$

By Proposition 2.2 exactly $\operatorname{rank}(\lambda/\mu)$ rows of this matrix have every entry equal either to 0 or a polynomial divisible by t. Hence $s_{\lambda/\mu}(1^t)$ is divisible by $t^{\operatorname{rank}(\lambda/\mu)}$, so $\operatorname{rank}(\lambda/\mu) \leqslant \operatorname{zrank}(\lambda/\mu)$ as desired.

Alternatively, we can expand $s_{\lambda/\mu}$ in terms of power sums p_ν instead of complete symmetric functions h_ν. If

$$s_{\lambda/\mu} = \sum_{\nu} z_\nu^{-1} \chi^{\lambda/\mu}(\nu) p_\nu, \tag{7}$$

then by the Murnaghan–Nakayama rule [11, Corollary 7.17.5] $\chi^{\lambda/\mu}(\nu) = 0$ unless there exists a border strip tableau of λ/μ of type ν. By Proposition 2.2 it follows that $\chi^{\lambda/\mu}(\nu) = 0$ unless $\ell(\nu) \geqslant \operatorname{rank}(\lambda/\mu)$. Since $p_\nu(1^t) = t^{\ell(\nu)}$, it again follows that $s_{\lambda/\mu}(1^t)$ is divisible by $t^{\operatorname{rank}(\lambda/\mu)}$. ∎

The next result establishes that $\operatorname{rank}(\lambda/\mu) = \operatorname{zrank}(\lambda/\mu)$ in two special cases.

THEOREM 3.2. (a) *If* $\mu = \emptyset$ (*so* $\lambda/\mu = \lambda$) *then* $\operatorname{rank}(\lambda) = \operatorname{zrank}(\lambda)$.

(b) *If every row of* $\mathrm{JT}_{\lambda/\mu}$ *that contains a* 0 *also contains a* 1, *then* rank $(\lambda/\mu) = \operatorname{zrank}(\lambda/\mu)$.

Proof. (a) A basic formula in the theory of symmetric functions [11, Corollary 7.21.4] asserts that

$$s_\lambda(1^t) = \prod_{(i,j)\in\lambda} \frac{t-i+j}{h(i,j)},$$

where $h(i,j) = \lambda_i + \lambda'_j - i - j + 1$, the hook length of λ at (i,j). Hence

$$\text{zrank}(\lambda) = \#\{i : (i,i) \in \lambda\} = \text{rank}(\lambda).$$

(b) Let

$$y(\lambda/\mu) = (t^{-\text{rank}(\lambda/\mu)} s_{\lambda/\mu}(1^t))_{t=0}.$$

By Proposition 3.1 $y(\lambda/\mu)$ is finite (and in fact is just the coefficient of $t^{\text{rank}(\lambda/\mu)}$ in $s_{\lambda/\mu}(1^t)$), and the assertion that $\text{rank}(\lambda/\mu) = \text{zrank}(\lambda/\mu)$ is equivalent to $y(\lambda/\mu) \neq 0$. Now factor out t from every row not containing a 1 of the matrix on the right-hand side of Eq. (6). By Proposition 2.2 the number of such rows is $\text{rank}(\lambda/\mu)$. Divide by $t^{\text{rank}(\lambda/\mu)}$ and set $t = 0$. Denote the resulting matrix by $R_{\lambda/\mu}$, so

$$y(\lambda/\mu) = \det R_{\lambda/\mu}|_{t=0}.$$

Note that

$$(t^{-1} h_i(1^t))_{t=0} = \frac{1}{i}, \qquad i \geqslant 1. \tag{8}$$

If row i of $\text{JT}_{\lambda/\mu}$ contains a 1, say in column j, then row i of $R_{\lambda/\mu}$ has all entries equal to 0 except for a 1 in column j. Hence we can remove row i and column j from $R_{\lambda/\mu}$ without changing the determinant $\det R_{\lambda/\mu}$, except possibly for the sign. When we do this for all rows i of $\text{JT}_{\lambda/\mu}$ containing a 1, then using (8) we obtain a matrix of the form

$$R'_{\lambda/\mu} = \left(\frac{1}{a_i + b_j} \right)_{i,j=1}^{r}, \tag{9}$$

where $a_1 > a_2 > \cdots > a_r > 0$ and $0 = b_1 < b_2 < \cdots < b_r$. In particular, the denominators $a_i + b_j$ are never 0. But it was shown by Cauchy (e.g., [8, Sect. 353]) that

$$\det R'_{\lambda/\mu} = \frac{\prod_{i<j} (a_i - a_j)(b_i - b_j)}{\prod_{i,j} (a_i + b_j)} \neq 0,$$

as was to be shown. ∎

4. MINIMAL BORDER STRIP DECOMPOSITIONS OF λ/μ

In the proof of Proposition 3.1 we mentioned the Murnaghan–Nakayama rule [11, Corollary 7.17.5] in connection with the expansion of $s_{\lambda/\mu}$ in terms of power sums. This rule asserts that if $\chi^{\lambda/\mu}(v)$ is defined by Eq. (7), then

$$\chi^{\lambda/\mu}(v) = \sum_{\boldsymbol{T}} (-1)^{\mathrm{ht}(\boldsymbol{T})}, \tag{10}$$

summed over all border-strip tableaux $\boldsymbol{T}$ of shape λ/μ and type v. Here

$$\mathrm{ht}(\boldsymbol{T}) = \sum_{B} \mathrm{ht}(B),$$

where B ranges over all border strips in $\boldsymbol{T}$ and $\mathrm{ht}(B)$ is one less than the number of rows of B. In fact, in Eq. (10) v can be a composition rather than just a partition. In other words, let $\alpha = (\alpha_1, \dots, \alpha_m)$ be a composition of $N = |\lambda/\mu|$ and let

$$\chi^{\lambda/\mu}(\alpha) = \sum_{\boldsymbol{T}} (-1)^{\mathrm{ht}(\boldsymbol{T})},$$

summed over all border strip tableaux $\boldsymbol{T}$ of shape λ/μ and type α. Then $\chi^{\lambda/\mu}(\alpha) = \chi^{\lambda/\mu}(v)$, where v is the decreasing rearrangement of α. The second proof of Proposition 3.1 showed that $s_{\lambda/\mu}$ has minimal degree $r = \mathrm{rank}(\lambda/\mu)$ as a polynomial in the p_i's (with deg $p_i = 1$ for $i \geqslant 1$). Since $p_\alpha(1^t) = t^{\ell(\alpha)}$ we see that the coefficient $y(\lambda/\mu)$ of $t^{\mathrm{rank}(\lambda/\mu)}$ in $s_{\lambda/\mu}(1^t)$ is given by

$$y(\lambda/\mu) = \sum_{\substack{v \vdash N \\ \ell(v) = r}} z_v^{-1} \chi^{\lambda/\mu}(v). \tag{11}$$

As mentioned above, an affirmative answer to (5) is equivalent to $y(\lambda/\mu) \neq 0$. Although we are unable to resolve this question here, we will show that there is some interesting combinatorics associated with minimal border strip decompositions and border strip tableaux of shape λ/μ. In particular, a more combinatorial version of Eq. (11) is given by (30).

Let e be an edge of the lower envelope of λ/μ, i.e., no square of λ/μ has e as its upper or left-hand edge. We will define a certain subset S_e of squares of λ/μ, called a *snake*. If e is also an edge of the upper envelope of λ/μ, then set $S_e = \emptyset$. Otherwise, if e is horizontal and (i,j) is the square of λ/μ having e as its lower edge, then define

$$\begin{aligned} S_e = (\lambda/\mu) \cap \{&(i,j), (i-1,j), (i-1,j-1), \\ &(i-2,j-1), (i-2,j-2), \dots\}. \end{aligned} \tag{12}$$

Finally if e is vertical and (i,j) is the square of λ/μ having e as its right-hand edge, then define

$$\begin{aligned} S_e = (\lambda/\mu) \cap \{&(i,j), (i,j-1), (i-1,j-1), \\ &(i-1,j-2), (i-2,j-2), \dots\}. \end{aligned} \tag{13}$$

In Fig. 4 the nonempty snakes of the skew shape 8744/411 are shown with dashed paths through their squares, with a single bullet in the two snakes with just one square. The *length* $\ell(S)$ of a snake S is one fewer than its number of squares; a snake of length $k-1$ (so with k squares) is called a *k-snake*. In particular, if $S_e = \emptyset$ then $\ell(S_e) = -1$. Call a snake of even length a *right snake* if it has form (12) and a *left snake* if it has form (13). (We could just as well make the same definitions for snakes of odd length, but we only need the definitions for those of even length.) It is clear that the snakes are linearly ordered from lower left to upper right. In this linear ordering replace a left snake of length $2k$ with the symbol L_k, a right snake of length $2k$ with R_k, and a snake of odd length with O. The resulting sequence (which does *not* determine λ/μ), with infinitely many initial and final O's removed, is called the *snake sequence* of λ/μ, denoted $\mathrm{SS}(\lambda/\mu)$. For instance, from Fig. 4 we see that

$$\mathrm{SS}(8874/411) = L_0 O L_1 L_2 R_2 O O L_2 R_2 O R_1 R_0.$$

Snakes (though not with that name) appear in the solution to [11, Exercise 7.66]. Call two consecutive squares of a snake S (i.e., two squares with a common edge) a *link* of S. Thus a k-snake has $k-1$ links. A link of a left snake is called a *left link*, and similarly a link of a right snake is called a *right link*. Two links l_1 and l_2 are said to be *consecutive* if they have a square in common. We say that a border strip B *uses* a link l of some snake if B contains the two squares of l. Similarly, a border strip decomposition $\boldsymbol{D}$ or border strip tableau $\boldsymbol{T}$ *uses* l if some border strip in $\boldsymbol{D}$ or $\boldsymbol{T}$ uses l. The exercise cited above shows the following.

LEMMA 4.1. *Let $\boldsymbol{D}$ be a border strip decomposition of λ/μ. Then no $B \in \boldsymbol{D}$ uses two consecutive links of a snake. Conversely, if we choose a set $\mathscr{L}$ of links*

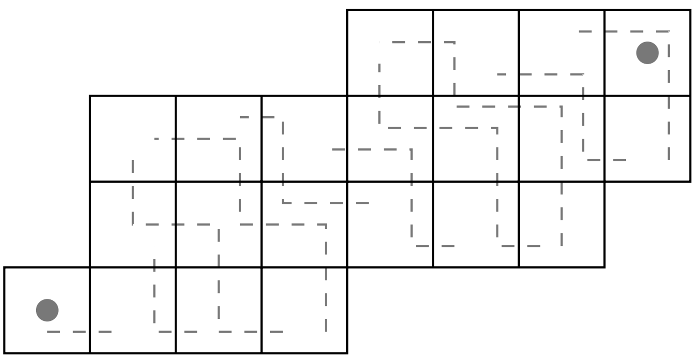

FIG. 4. Snakes for the skew shape 8874/411.

from the snakes of λ/μ such that no two of these links are consecutive, then there is a unique border strip decomposition $\boldsymbol{D}$ *of* λ/μ *that uses precisely the links in* $\mathscr{L}$ (*and no other links*).

Lemma 4.1 sets up a bijection between border strip decompositions of λ/μ and sets $\mathscr{L}$ of links of the snakes of λ/μ such that no two links are consecutive. In particular, if F_n denotes a Fibonacci number ($F_1 = F_2 = 1$, $F_{n+1} = F_n + F_{n-1}$ for $n > 1$), then there are F_{k+1} ways to choose a subset $\mathscr{L}$ of links of a k-snake such that no two links are consecutive. Hence if the snakes of λ/μ have sizes $a_1, \dots, a_r$, then the number of border strip decompositions of λ/μ is $F_{a_1+1} \cdots F_{a_r+1}$ (as is clear from the solution to [11, Exercise 7.66]). Moreover, the size (number of border strips) of the border strip decomposition $\boldsymbol{D}$ is given by

$$\#\boldsymbol{D} = |\lambda/\mu| - \#\mathscr{L}. \tag{14}$$

Consider now the *minimal* border strip decompositions $\boldsymbol{D}$ of λ/μ, i.e., $\#\boldsymbol{D}$ is minimized. Thus by Proposition 2.2 we have $\#\boldsymbol{D} = \text{rank}(\lambda/\mu)$. By Eq. (14) we wish to maximize the number of links, no two consecutive. For snakes with an odd number $2m - 1$ of links we have no choice—there is a unique way to choose m links, no two consecutive, and this is the maximum number possible. For snakes with an even number $2m$ of links there are $m + 1$ ways to choose the maximum number m of links. Thus if $\text{mbsd}(\lambda/\mu)$ denotes the number of minimal border strip decompositions of λ/μ, then we have proved the following result (which will be improved in Theorem 4.5).

Proposition 4.2. *We have*

$$\text{mbsd}(\lambda/\mu) = \prod_S \left(1 + \frac{\ell(S)}{2}\right),$$

where S ranges over all snakes of λ/μ of even length.

To proceed further with the structure of the minimal border strip decompositions of λ/μ, we will develop their connection with $\text{code}(\lambda/\mu)$. Let p be the bottom-leftmost point of (the diagram of) λ/μ, and let q be the top-rightmost point. We regard the boundary of λ/μ as consisting of two lattice paths from p to q with steps $(1,0)$ or $(0,1)$, or in other words, the restriction of the upper and lower envelopes of λ/μ between p and q. The top-left path (regarded as a sequence of edges $e_1, \dots, e_k$) is denoted $\Lambda_1(\lambda/\mu)$, and the bottom-right path $f_1, \dots, f_k$ by $\Lambda_2(\lambda/\mu)$. Note that if in the two-line array

$$\begin{matrix} f_1 & f_2 & \cdots & f_k \\ e_1 & e_2 & \cdots & e_k \end{matrix}$$

we replace each vertical edge by 1 and each horizontal edge by 0, then we obtain $\overline{\text{code}}(\lambda/\mu)$.

Continue the zigzag pattern of the links of each snake of λ/μ one further step in each direction, as illustrated in Fig. 5 for $\lambda/\mu = 8874/411$. These steps will cross an edge on the boundary of λ/μ. Denote the top-left boundary edge crossed by the extended link of the snake S by $\tau(S)$, called the *top edge* of S. Similarly, denote the bottom-right boundary edge crossed by the extended link of the snake S by $\beta(S)$, called the *bottom edge* of the snake S. (In fact, the snake S_e has $\beta(S_e) = e$.) When $S_e = \emptyset$ we have $\tau(S_e) = \beta(S_e) = e$. See Fig. 6 for the case $\lambda/\mu = 43111/2211$, which has three edges e for which $S_e = \emptyset$.

We thus have the following situation. Write S_i as short for S_{f_i}, so $\tau(S_i) = e_i$ and $\beta(S_i) = f_i$. Let

$$\overline{\text{code}}(\lambda/\mu) = \begin{pmatrix} c_1 & c_2 & \cdots & c_k \\ d_1 & d_2 & \cdots & d_k \end{pmatrix}. \tag{15}$$

It is easy to see that S_i is a left snake if and only if $(c_i, d_i) = (1, 0)$. In this case, if S_i has length $2m$ then

$$m + 1 = \#\{j > i : (c_j, d_j) = (0, 1)\} - \#\{j > i : (c_j, d_j) = (1, 0)\}. \tag{16}$$

Similarly S_i is a right snake if and only if $(c_i, d_i) = (0, 1)$; and if S_i has length $2m$ then

$$m + 1 = \#\{j < i : (c_j, d_j) = (1, 0)\} - \#\{j < i : (c_j, d_j) = (0, 1)\}. \tag{17}$$

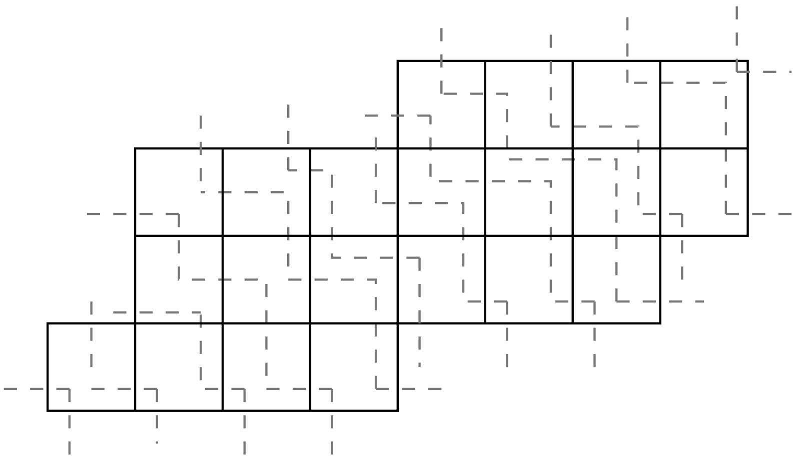

FIG. 5. Extended links for the skew shape 8874/411.

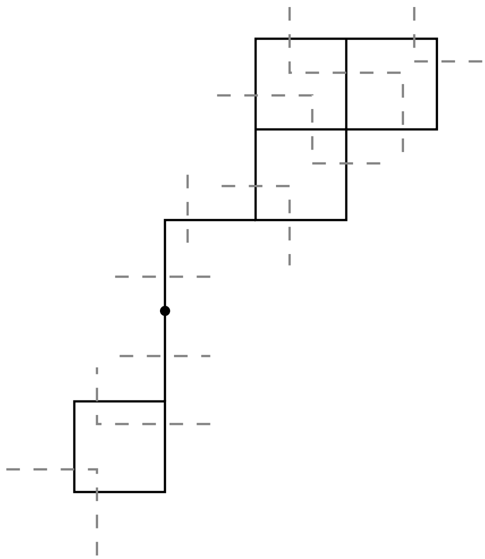

FIG. 6. Extended links for the skew shape 43111/2211.

PROPOSITION 4.3. *The snake sequence* $\mathrm{SS}(\lambda/\mu) = q_1 q_2 \cdots q_k$ *is "well-parenthesized" in the following sense. There exists a* (*unique*) *set* $\mathscr{P}(\lambda/\mu) = \{(u_1, v_1), \ldots, (u_r, v_r)\}$, *where* $r = \mathrm{rank}(\lambda/\mu)$, *such that*:

(a) *The* u_i*'s and* v_i*'s are all distinct integers*,

(b) $1 \leqslant u_i < v_i \leqslant k$,

(c) $q_{u_i} = L_t$ *and* $q_{v_i} = R_t$ *for some* t (*depending on* i),

(d) *for no* i *and* j *do we have* $u_i < u_j < v_i < v_j$.

Proof. Equations (3) and (4) assert that for any $1 \leqslant i \leqslant k$ we have

$$\#\{j : 1 \leqslant j \leqslant i,\ q_j = L_s \text{ for some } s\}$$
$$\geqslant \#\{j : 1 \leqslant j \leqslant i,\ q_j = R_s \text{ for some } s\}, \tag{18}$$

and that the total number of L's in $\mathrm{SS}(\lambda/\mu)$ equals the total number of R's. It now follows from a standard bijection (e.g., [11, solution to Exercise 6.19(n) and (o)]) that there is a unique set $\mathscr{P}(\lambda/\mu)$ satisfying (a), (b), and (d). But (c) is then a consequence of Eqs. (16) and (17). ∎

We can depict the set $\mathscr{P}(\lambda/\mu)$ by drawing arcs above the terms of $\mathrm{SS}(\lambda/\mu)$, such that the left and right endpoints of an arc are some L_t and R_t, and such that the arcs are noncrossing. For instance,

$$\mathscr{P}(8874/411) = \{(1,12),(3,11),(4,5),(8,9)\},$$

as illustrated in Fig. 7.

Let $\mathrm{SS}(\lambda/\mu) = q_1 q_2 \cdots q_k$ as in Proposition 4.3, and define an *interval set* of λ/μ to be a collection $\mathscr{I}$ of r ordered pairs,

$$\mathscr{I} = \{(u_1, v_1), \ldots, (u_r, v_r)\},$$

satisfying the following conditions:

- The u_i's and v_i's are all distinct integers,
- $1 \leqslant u_i < v_i \leqslant k$,
- $q_{u_i} = L_s$ and $q_{v_i} = R_t$ for some s and t (depending on i).

Thus $\mathscr{P}(\lambda/\mu)$ is itself an interval set. Figure 8 illustrates the interval set $\{(1,5),(3,12),(4,9),(8,11)\}$ of the skew shape 8874/411. Let $\mathrm{is}(\lambda/\mu)$ denote the number of interval sets of λ/μ.

THEOREM 4.4. *Let $T_1, \ldots, T_r$ be the left snakes* (*or right snakes*) *of λ/μ. Then*

$$\mathrm{is}(\lambda/\mu) = \prod_{i=1}^{r}\left(1 + \frac{\ell(T_i)}{2}\right).$$

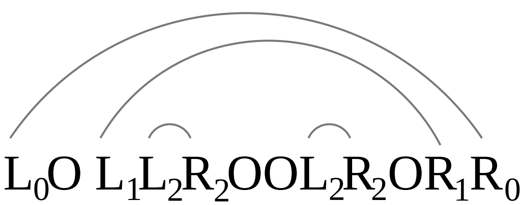

FIG. 7. Parenthesization of the snake sequence SS(8874/411).

FIG. 8. An interval set of the skew shape 8874/411.

Proof. Let $\mathrm{SS}(\lambda/\mu) = q_1 q_2 \cdots q_k$. Let $q_{u_1}, \ldots, q_{u_r}$ be the positions of the terms L_s, with $u_1 < \cdots < u_r$. Let $q_{u_i} = L_{m_i}$. We can obtain an interval set by pairing q_{u_r} with some R_s to the right of q_{u_r}, then pairing $q_{u_{r-1}}$ with some R_s to the right of $q_{u_{r-1}}$ not already paired, etc. By Eq. (16) the number of choices for pairing q_{u_i} is just $m_i + 1$, and the proof follows. ∎

We are now in a position to count the number of minimal border strip decompositions and minimal border strip tableaux of shape λ/μ. Let us denote this latter number by $\mathrm{mbst}(\lambda/\mu)$.

THEOREM 4.5. *Let* $\mathrm{rank}(\lambda/\mu) = r$. *Then*

$$\mathrm{mbsd}(\lambda/\mu) = \mathrm{is}(\lambda/\mu)^2, \tag{19}$$

$$\mathrm{mbst}(\lambda/\mu) = r!\,\mathrm{is}(\lambda/\mu). \tag{20}$$

Proof. Eq. (19) is an immediate consequence of Proposition 4.2 and Theorem 4.4 (using that in Theorem 4.4 we can take $T_1, \ldots, T_r$ to consist of either all left snakes or all right snakes).

To prove Eq. (20) we use Proposition 2.1. Let

$$\overline{\mathrm{code}}(\lambda/\mu) = \begin{matrix} c_1 & c_2 & \cdots & c_k \\ d_1 & d_2 & \cdots & d_k \end{matrix}$$

and let $r = \mathrm{rank}(\lambda/\mu)$. It follows from Proposition 2.1 that a minimal border strip tableau of shape λ/μ is equivalent to choosing a sequence $(u_1, v_1), \ldots, (u_r, v_r)$ where $1 \leqslant u_i < v_i \leqslant k$, $c_{u_i} = 1$, $c_{v_i} = 0$, the u_i's and v_i's are distinct, and then successively changing (u_i, v_i) from $(1,0)$ to $(0,1)$, so that at the end we obtain the sequence $d_1, \ldots, d_k$. Since there are exactly r pairs (c_i, d_i) equal to $(0,1)$ and r pairs equal to $(1,0)$, the condition that we end up with $d_1, \ldots, d_k$ is equivalent to $d_{u_i} = 0$ and $d_{v_i} = 1$. Hence the possible sets $\{(u_1, v_1), \ldots, (u_r, v_r)\}$ are just the interval sets of λ/μ. There are $\mathrm{is}(\lambda/\mu)$ ways to choose an interval set and $r!$ ways to linearly order its elements, so the proof follows. ∎

As discussed in the above proof, every interval set $\mathscr{I}$ of λ/μ gives rise to $r!$ minimal border strip tableaux $\boldsymbol{T}$ of shape λ/μ. The set of border strips appearing in such a tableau is a border strip decomposition $\boldsymbol{D}$ of λ/μ. Extending our terminology that $\boldsymbol{T}$ and $\boldsymbol{D}$ correspond to each other, we will say that $\mathscr{I}$, $\boldsymbol{D}$, and $\boldsymbol{T}$ all *correspond* to each other.

How many of the above $r!$ border strip decompositions corresponding to $\mathscr{I}$ are distinct? Rather remarkably, the number is $\mathrm{is}(\lambda/\mu)$, independent of the interval set $\mathscr{I}$. This is a consequence of Theorem 4.8. Our proof of this result is best understood in the context of posets. Let P be a finite poset with p elements $x_1, \ldots, x_p$. A bijection $f : P \to [p] = \{1, 2, \ldots, p\}$ is called a *dropless labeling* of P if we never have $f^{-1}(i+1) < f^{-1}(i)$. Let $\mathrm{inc}(P)$ denote the incomparability graph of P, i.e., the vertex set of $\mathrm{inc}(P)$ is $\{x_1, \ldots, x_p\}$, with an edge between x_i and x_j if and only if x_i and x_j are incomparable in P. The next result is implicit in [5, Theorem 2; 3, Theorem on p. 322] (namely, in [5, Theorem 2] put $x = -1$ and in [3, Theorem on p. 322] put $\lambda = -1$, and use (22)) and explicit in [12, Theorem 4.12]. For the sake of completeness we repeat the essence of the proof in [12].

LEMMA 4.6. *The number* $\mathrm{dl}(P)$ *of dropless labelings of* P *is equal to the number* $\mathrm{ao}(\mathrm{inc}(P))$ *of acyclic orientations of* $\mathrm{inc}(P)$.

Proof. Given the dropless labeling $f : P \to [p]$, define an acyclic orientation $\mathfrak{o} = \mathfrak{o}(f)$ as follows. If $x_i x_j$ is an edge of $\mathrm{inc}(P)$, then let $x_i \to x_j$ in $\mathfrak{o}$ if $f(x_i) < f(x_j)$, and let $x_j \to x_i$ otherwise. Clearly, $\mathfrak{o}$ is an acyclic orientation of $\mathrm{inc}(P)$. Conversely, let $\mathfrak{o}$ be an acyclic orientation of $\mathrm{inc}(P)$. The set of sources (i.e., vertices with no arrows into them) form a chain in P since otherwise two are incomparable, so there is an arrow between them that must point into one of them. Let x be the minimal element of this chain, i.e., the unique minimal source. If f is a dropless labeling of P with $\mathfrak{o} = \mathfrak{o}(f)$, then we claim $f(x) = 1$. Suppose to the contrary that $f(x) = i > 1$. Let j be the largest integer satisfying $j < i$ and $y := f^{-1}(j) \not< x$. Note that j exists since $f^{-1}(1) > x$. We must have $y > x$ since x is a source. But then $f^{-1}(j+1) \leqslant x < y = f^{-1}(j)$, contradicting the fact that f is dropless. Thus we can set $f(x) = 1$, remove x from $\mathrm{inc}(P)$, and proceed inductively to construct a unique f satisfying $\mathfrak{o} = \mathfrak{o}(f)$. ∎

Now given any set

$$\mathscr{I} = \{(u_1, v_1), \ldots, (u_r, v_r)\} \tag{21}$$

with $u_i < v_i$, define a partial order $P_{\mathscr{I}}$ on $\mathscr{I}$ by setting $(u_i, v_i) < (u_j, v_j)$ if $v_i < u_j$. If we regard the pairs (u_i, v_i) as closed intervals $[u_i, v_i]$ in $\mathbb{R}$, then $P_{\mathscr{I}}$ is just the *interval order* corresponding to these intervals (e.g., [4, 13]).

LEMMA 4.7. *Let* $\mathscr{I}$ *be as in Eq.* (21). *For* $1 \leqslant i \leqslant r$ *let*

$$\varphi(i) = \#\{j : v_j > v_i\} - \#\{j : u_j > v_i\}.$$

Then

$$\mathrm{dl}(P_{\mathscr{I}}) = (\varphi(1) + 1)(\varphi(2) + 1) \cdots (\varphi(r) + 1).$$

Proof. Let $\chi_{\mathscr{I}}(q)$ denote the chromatic polynomial of the graph $\mathrm{inc}(P_{\mathscr{I}})$. We may suppose that the elements of $\mathscr{I}$ are indexed so that $v_1 > v_2 > \cdots > v_r$. We can properly color the vertices of $\mathrm{inc}(P_{\mathscr{I}})$ (i.e., adjacent vertices have different colors) in q colors as follows. First, color vertex (u_1, v_1) in q ways. Suppose that vertices $(u_1, v_1), \ldots, (u_i, v_i)$ have been colored, where $i < r$. Now for $1 \leqslant j \leqslant i$, (u_{i+1}, v_{i+1}) is incomparable in $P_{\mathscr{I}}$ to (u_j, v_j) if and only $v_{i+1} > u_j$. These vertices (u_j, v_j) form an antichain in $P_{\mathscr{I}}$; else either some $v_j < v_{i+1}$ or some $u_j > v_{i+1}$. The number of these vertices is $\varphi(i+1)$. Since they form a clique in $\mathrm{inc}(P_{\mathscr{I}})$ there are exactly $q - \varphi(i+1)$ ways to color vertex (u_{i+1}, v_{i+1}), independent of the colors previously assigned. It follows that

$$\chi_{\mathscr{I}}(q) = \prod_{i=1}^{r} (q - \varphi(i+1)).$$

For any graph G with r vertices it is known [10] that

$$\mathrm{ao}(G) = (-1)^r \chi_G(-1). \tag{22}$$

Hence

$$\mathrm{ao}(\mathrm{inc}(P_{\mathscr{I}})) = \prod_{i=1}^{r} (\varphi(i) + 1).$$

The proof follows from Lemma 4.6. ∎

Note. The fact (shown in the above proof) that we can order the vertices of $\mathrm{inc}(P_{\mathscr{I}})$ so that each vertex is adjacent to a set of previous vertices forming a clique is equivalent to the statement that the incomparability graph of an interval order is *chordal*. Note that the above proof shows that for any interval order P coming from intervals $[u_1, v_1], \ldots, [u_r, v_r]$, the chromatic polynomial of $\mathrm{inc}(P)$ depends only on the sets $\{u_1, \ldots, u_r\}$ and $\{v_1, \ldots, v_r\}$.

We now come to the result mentioned in the paragraph before Lemma 4.6.

THEOREM 4.8. *Let $\mathscr{I}$ be an interval set of λ/μ, thus giving rise to $r!$ minimal border strip tableaux of shape λ/μ. Then the number of distinct border strip decompositions that correspond to these $r!$ border strip tableaux is* $\mathrm{is}(\lambda/\mu)$.

Proof. Let $(u_i, v_i), (u_j, v_j) \in \mathscr{I}$. We say that (u_i, v_i) and (u_j, v_j) *overlap* if $[u_i, v_i] \cap [u_j, v_j] \neq \emptyset$, where $[a, b] = \{u_i, u_i + 1, \ldots, v_i\}$. Two linear orderings π and σ of $\mathscr{I}$ correspond to the same border strip decomposition if and only if any two overlapping elements (u_i, v_i) and (u_j, v_j) appear in the same order in π and σ. Suppose that π is given by the linear ordering

$$\pi = ((u_{i_1}, v_{i_1}), \ldots, (u_{i_r}, v_{i_r})). \tag{23}$$

If (u_{i_m}, v_{i_m}) and $(u_{i_{m+1}}, v_{i_{m+1}})$ are consecutive terms of π which do not overlap and if $i_m > i_{m+1}$, then we can transpose the two terms without affecting the border strip decomposition defined by π. By a series of such transpositions we can put π in the "canonical form" where consecutive nonoverlapping pairs appear in increasing order of their subscripts. The number of distinct border strip decompositions that correspond to the $r!$ permutations π is the number of π that are in canonical form. Let π be given by (23), and define $f : P_{\mathscr{I}} \to [r]$ by $f(u_{i_m}, v_{i_m}) = m$. Then π is in canonical form if and only if f is dropless. Comparing Eq. (16), Theorem 4.4, and Lemma 4.7 completes the proof. ∎

Note that Theorem 4.8 gives a refinement of Eq. (19), since we have partitioned the $\mathrm{is}(\lambda/\mu)^2$ minimal border strip decompositions of λ/μ into $\mathrm{is}(\lambda/\mu)$ blocks, each of size $\mathrm{is}(\lambda/\mu)$.

Now let $\mathscr{I} = \{(u_1, v_1), \ldots, (u_i, v_i)\}$ be an interval set of λ/μ. Define the *type* of $\mathscr{I}$ to be the partition σ whose parts are the integers $v_1 - u_1, \ldots, v_r - u_r$. Hence by Proposition 2.1 σ is also the type of any of the border strip decompositions corresponding to $\mathscr{I}$. Let $\mathrm{is}_\sigma(\lambda/\mu)$ denote the number of interval sets of λ/μ of type σ, and let $\mathrm{mbsd}_\sigma(\lambda/\mu)$ denote the number of minimal border strip decompositions of λ/μ of type σ. The following result is a refinement of Eq. (19).

Corollary 4.9. *Let $N = |\lambda/\mu|$. For any partition $\sigma \vdash N$, we have*

$$\mathrm{mbsd}_\sigma(\lambda/\mu) = \mathrm{is}_\sigma(\lambda/\mu)\ \mathrm{is}(\lambda/\mu).$$

Proof. Immediate consequence of Theorem 4.8 and the observation above that $\mathrm{type}(\mathscr{I}) = \mathrm{type}(\boldsymbol{D})$ for any interval set $\mathscr{I}$ and border strip decomposition $\boldsymbol{D}$ corresponding to $\mathscr{I}$. ∎

We can improve the above corollary by explicitly partitioning the minimal border strip decompositions of λ/μ into $\mathrm{is}(\lambda/\mu)$ blocks, each of which contains exactly $\mathrm{mbsd}_\sigma(\lambda/\mu)$ border strip decompositions of type σ.

Theorem 4.10. *For each right snake S of λ/μ fix a set F_S of $\ell(S)/2$ links of S, no two consecutive, and let $F = \bigcup_S F_S$. Let Q_F be the set of all minimal border strip decompositions $\boldsymbol{D}$ of λ/μ which use the links in Q_F. Then for each $\sigma \vdash N = |\lambda/\mu|$, Q_F contains exactly* $\mathrm{is}_\sigma(\lambda/\mu)$ *minimal border strip decompositions of type σ.*

Figure 9 illustrates Theorem 4.10 for the case $\lambda/\mu = 332/1$. We are using dots rather than squares in the diagram of λ/μ. The first column shows the right snakes, with the choice of links as a solid line and the remaining links as dashed lines. The first row shows the same for the left snakes. The remaining 16 entries are the minimal border strip decompositions of λ/μ using the right snake links for that row and the left snake links for that column. Theorem 4.10 asserts that each row (and hence by symmetry each column) contains the same number of minimal border strip decompositions of each type, viz., one of type $(5,1,1)$, two of type $(4,2,1)$, and one of type $(3,2,2)$. For general λ/μ there will also be snakes of odd length $2m-1$ yielding m links that must be used in every minimal border strip decomposition.

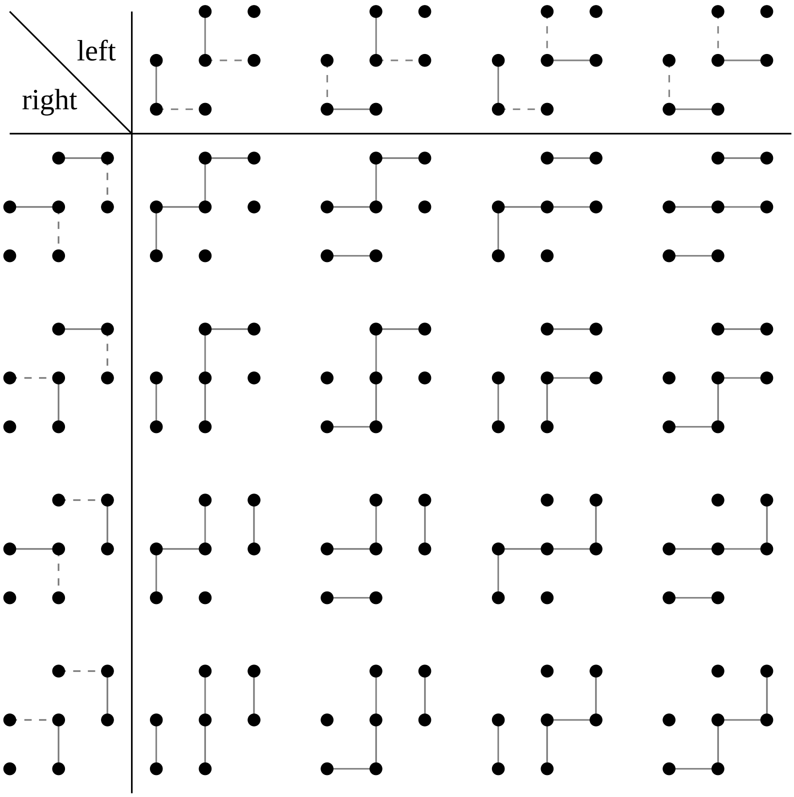

FIG. 9. Minimal border strip decompositions of the skew shape 332/1.

Proof of Theorem 4.10. Let $\mathscr{I}$ be an interval set of λ/μ of type σ. By Theorem 4.8 there are exactly $\mathrm{is}(\lambda/\mu)$ border strip decompositions (all of type σ) corresponding to $\mathscr{I}$.

Claim. Any two of the above $\mathrm{is}(\lambda/\mu)$ border strip decompositions $\boldsymbol{D}$ have a different set of left links and a different set of right links.

By symmetry it suffices to show that any two, say $\boldsymbol{D}$ and $\boldsymbol{D}'$, have a different set of left links. Let $\overline{\mathrm{code}}(\lambda/\mu)$ be given by (15), and let $S_i = S_{f_i}$ as defined just before (15). Thus S_i is a left snake if and only if $(c_i, d_i) = (0, 1)$. Moreover, if S_i is a left snake and $\mathscr{I} = \{(u_1, v_1), \ldots, (u_r, v_r)\}$ is any interval set for λ/μ, then it follows from (16) that $\ell(S_i) = 2m$ where

$$m = \#\{j : u_j < i < v_j\}.$$

Let $j_1, \ldots, j_m$ be those j for which $u_j < i < v_j$. In a linear ordering π of $\mathscr{I}$ there are $m + 1$ choices for how many of the pairs (u_{j_s}, v_{j_s}) precede (u_i, v_i). The linear ordering π defines a border strip tableau with corresponding border strip decomposition $\boldsymbol{D}$. In turn $\boldsymbol{D}$ is defined by a choice of a maximum number of links, no two consecutive, from each left and right snake. The choices of links from the snake S_i are equivalent to choosing the number of pairs (u_{j_s}, v_{j_s}) preceding (u_i, v_i) in π, since S_i intersects precisely the border strips B_i and B_{j_s} corresponding to (u_i, v_i) and the (u_{j_s}, v_{j_s})'s, and the position of B_i within the snake determines the unique two consecutive unused links of the snake S_i extended by adding one square in each direction. Moreover, B_i will be the unique border strip whose initial square (reading from lower-left to upper-right) begins on S_i. As an example see Fig. 10, which shows the skew shape $\lambda/\mu = 66554/1$ with the left snake S_6 shaded. There are four border strips intersecting S_6, and the third one (reading from bottom-right to upper-left) begins on the square $(2, 3)$ of S_6. The two links of S_6 involving this square are not used in the border strip decomposition $\boldsymbol{D}$.

A dropless labeling of $\mathscr{I}$ is uniquely determined by specifying for each left snake S_i how many of the (u_{j_s}, v_{j_s})'s, as defined above, precede (u_i, v_i); for we can inductively determine, preceding from left-to-right in $\overline{\mathrm{code}}(\lambda/\mu)$, the relative order of any pair (u_i, v_i) and (u_j, v_j) of elements which cross, while all remaining ambiguities in the labeling are resolved by the dropless condition. Thus the $\mathrm{is}(\lambda/\mu)$ dropless labelings of $\mathscr{I}$ define border strip tableaux of shape λ/μ and type σ, no two of which have the same left links. Since these border strip tableaux correspond to different border strip decompositions (by the proof of Theorem 4.8), the proof of the claim follows.

By the claim, for each interval set $\mathscr{I}$ the $\mathrm{is}(\lambda/\mu)$ border strip decompositions corresponding to $\mathscr{I}$ all have the same type and belong to different Q_F's. Since there are $\mathrm{is}(\lambda/\mu)$ different Q_F's it follows that each Q_F

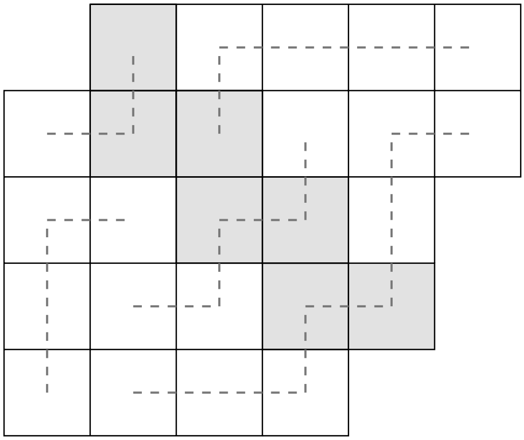

FIG. 10. Intersection of border strips with a left snake.

contains exactly $\mathrm{is}_\sigma(\lambda/\mu)$ minimal border strip decompositions of type σ, as was to be proved. ∎

Another way to state Theorem 4.10 is as follows. Let A be the square matrix whose columns (respectively, rows) are indexed by the maximum size sets G (respectively, F) of links, no two consecutive, of right snakes (respectively, left snakes) of λ/μ. The entry A_{FG} is defined to be the minimal border strip decomposition of λ/μ using the links F and G. Figure 9 shows this matrix for $\lambda/\mu = 332/1$. Let $t = \mathrm{is}(\lambda/\mu)$ and let $\mathscr{I}_1, \ldots, \mathscr{I}_t$ be the interval sets of λ/μ. If the border strip decomposition A_{FG} corresponds to $\mathscr{I}_j$, then let L be the matrix obtained by replacing A_{FG} with the integer j. Then the matrix L is a *Latin square*, i.e., every row and every column is a permutation of $1, 2, \ldots, t$. For instance, when $\lambda/\mu = 332/1$ the interval sets are

$$\mathscr{I}_1 = \{(1,6),(2,3),(4,5)\}, \qquad \mathscr{I}_2 = \{(1,3),(2,6),(4,5)\},$$

$$\mathscr{I}_3 = \{(1,5),(2,3),(4,6)\}, \qquad \mathscr{I}_4 = \{(1,3),(2,5),(4,6)\}.$$

The matrix A of Fig. 9 becomes the Latin square

$$L = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \\ 4 & 3 & 2 & 1 \end{bmatrix}.$$

5. AN APPLICATION TO THE CHARACTERS OF $\mathfrak{S}_n$

Expand the skew Schur function $s_{\lambda/\mu}$ in terms of power sums as in Eq. (7). Define $\deg(p_i) = 1$, so $\deg(p_\nu) = \ell(\nu)$. As mentioned after (7), the Murnaghan–Nakayama rule (10) implies that if p_ν appears in $s_{\lambda/\mu}$ then $\deg(p_\nu) \geqslant r = \text{rank}(\lambda/\mu)$. In fact, at least one such p_ν actually appears in $s_{\lambda/\mu}$, viz., let ν_1 be the length of the longest border strip B_1 of λ/μ, then ν_2 the length of the longest border strip B_2 of $\lambda/\mu - B_1$, etc. All border strip tableaux of λ/μ of type ν involve the same set of border strips, so there is no cancellation in the right-hand side of (10). Hence the coefficient of p_ν in $s_{\lambda/\mu}$ in nonzero. (See [11, Exercise 7.52] for the case $\mu = \emptyset$.) Let us write $\hat{s}_{\lambda/\mu}$ for the lowest degree part of $s_{\lambda/\mu}$, so

$$\hat{s}_{\lambda/\mu} = \sum_{\nu\,:\,\ell(\nu)=r} z_\nu^{-1} \chi^{\lambda/\mu}(\nu) p_\nu, \tag{24}$$

where $r = \text{rank}(\lambda/\mu)$. Also write $\tilde{p}_i = p_i/i$. For instance,

$$s_{332/1} = \tfrac{1}{120}\, p_1^7 - \tfrac{1}{12}\, p_1^4 p_3 + \tfrac{1}{24}\, p_1^3 p_2^2 + \tfrac{1}{5}\, p_1^2 p_5 - \tfrac{1}{4}\, p_1 p_2 p_4 + \tfrac{1}{12}\, p_2^2 p_3.$$

Hence

$$\begin{aligned}\hat{s}_{332/1} &= \tfrac{1}{5}\, p_1^2 p_5 - \tfrac{1}{4}\, p_1 p_2 p_4 + \tfrac{1}{12}\, p_2^2 p_3 \\ &= \tilde{p}_1^2 \tilde{p}_5 - 2\tilde{p}_1 \tilde{p}_2 \tilde{p}_4 + \tilde{p}_2^2 \tilde{p}_3.\end{aligned}$$

If $\mathscr{I} = \{(w_1, y_1), \ldots, (w_r, y_r)\}$ is an interval set, then let $c(\mathscr{I})$ denote the number of *crossings* of $\mathscr{I}$, i.e., the number of pairs (i,j) for which $w_i < w_j < y_i < y_j$. Moreover, let $\mathscr{P}(\lambda/\mu) = \{(u_1, v_1), \ldots, (u_r, v_r)\}$ be as in Proposition 4.3, and let

$$\overline{\text{code}}(\lambda/\mu) = \begin{matrix} c_1 & c_2 & \cdots & c_k \\ d_1 & d_2 & \cdots & d_k \end{matrix}.$$

For $1 \leqslant i \leqslant r$ define

$$z(i) = \#\{j : u_i < j < v_i,\ c_j = 0\},$$

$$z(\lambda/\mu) = z(1) + z(2) + \cdots + z(r).$$

It is easy to see (see the proof of Theorem 5.2 for more details) that $z(\lambda/\mu)$ is just the height $\text{ht}(\boldsymbol{T})$ of a "greedy border strip tableau" $\boldsymbol{T}$ of shape λ/μ obtained by starting with λ/μ and successively removing the largest possible border strip. (Although $\boldsymbol{T}$ may not be unique, the set of border strips appearing in $\boldsymbol{T}$ is unique, so $\text{ht}(\boldsymbol{T})$ is well-defined.)

LEMMA 5.1. *Let $\mathscr{I}$ be an interval set of λ/μ. If $\boldsymbol{T}$ and $\boldsymbol{T}'$ are two border strip tableaux corresponding to $\mathscr{I}$, then* $\mathrm{ht}(\boldsymbol{T}) \equiv \mathrm{ht}(\boldsymbol{T}') \pmod 2$.

Proof. When we remove a border strip B of size p from a skew shape α/β with $\mathrm{code}(\alpha) = \cdots c_0 c_1 c_2 \cdots$, then by Proposition 2.1 we replace some $(c_i, c_{i+p}) = (1, 0)$ with $(0, 1)$. It is easy to check (and is also equivalent to the discussion in [1, top of p. 3]) that

$$\mathrm{ht}(B) = \#\{h : i < h < i + p,\ \ c_h = 0\}. \tag{25}$$

Suppose we have $(c_i, c_{i+p}) = (c_j, c_{j+q}) = (1, 0)$, where the four numbers c_i, c_{i+p}, c_j, c_{j+q} are all distinct. Let B_1 be the border strip corresponding to $(i, i+p)$ and B_2 the border strip corresponding to $(j, j+q)$ after B_1 has been removed. Similarly, let B_1' correspond to $(j, j+q)$ and B_2' to $(i, i+p)$ after B_1' has been removed. If $i+p<j$ or $j+q<i$ then $B_1 = B_2'$ and $B_2 = B_1'$, so $\mathrm{ht}(B_1) + \mathrm{ht}(B_2) = \mathrm{ht}(B_1') + \mathrm{ht}(B_2')$. In particular,

$$\mathrm{ht}(B_1) + \mathrm{ht}(B_2) \equiv \mathrm{ht}(B_1') + \mathrm{ht}(B_2') \pmod 2. \tag{26}$$

If $c_i < c_j < c_{i+p} < c_{j+q}$, then using (25) we see that $\mathrm{ht}(B_1) = \mathrm{ht}(B_2') - 1$ and $\mathrm{ht}(B_2) = \mathrm{ht}(B_1') - 1$ so again (26) holds. Similarly, it is easy to check (26) in all remaining cases.

Iterating the above argument and using the fact that every permutation is a product of adjacent transpositions completes the proof. ∎

THEOREM 5.2. *For any skew shape λ/μ of rank r we have*

$$\hat{s}_{\lambda/\mu} = (-1)^{z(\lambda/\mu)} \sum_{\mathscr{I}=\{(u_1,v_1),\ldots,(u_r,v_r)\}} (-1)^{c(\mathscr{I})} \prod_{i=1}^{r} \tilde{p}_{v_i-u_i}, \tag{27}$$

where $\mathscr{I}$ ranges over all interval sets of λ/μ.

Proof. Let $\mathscr{I}$ be an interval set of λ/μ, and let $\boldsymbol{T}$ be a border strip tableau corresponding to $\mathscr{I}$. We claim that

$$\mathrm{ht}(\boldsymbol{T}) \equiv z(\lambda/\mu) + c(\mathscr{I}) \pmod 2. \tag{28}$$

The proof of the claim is by induction on $c(\mathscr{I})$.

First, note that by Lemma 5.1, it suffices to prove the claim for *some* $\boldsymbol{T}$ corresponding to each $\mathscr{I}$. Suppose that $c(\mathscr{I}) = 0$, so $\mathscr{I} = \mathscr{P}$. Let $\boldsymbol{T}$ be a greedy border strip tableau as defined before Lemma 5.1. The corresponding interval set is just $\mathscr{P}$, the unique interval set without crossings, since if $u_i < u_j < v_i < v_j$ we would pick the border strip corresponding to (u_i, v_j) rather than (u_i, v_i) or (u_j, v_j). Since by (25) we have $z(\lambda/\mu) = \mathrm{ht}(\boldsymbol{T})$, Eq. (28) holds when $c(\mathcal{I}) = 0$.

Now let $c(\mathcal{I}) > 0$. Suppose that (u_i, v_i) and (u_j, v_j) define a crossing in $\mathcal{I}$, say $u_i < u_j < v_i < v_j$. Let $\mathcal{I}'$ be obtained from $\mathcal{I}$ by replacing (u_i, v_i) and (u_j, v_j) with (u_i, v_j) and (u_j, v_i). It is easy to see that $c(\mathcal{I}) - c(\mathcal{I}')$ is an odd positive integer. By the induction hypothesis we may assume that (28) holds for $\mathcal{I}'$. Let $\boldsymbol{T}'$ be a border strip tableau corresponding to $\mathcal{I}'$ such that the border strips B_1 and B_2 indexed by (u_1, v_1) and (u_2, v_2) are removed first (say in the order B_1, B_2). Let $\boldsymbol{T}$ be the border strip tableau that differs from $\boldsymbol{T}'$ by replacing B_1, B_2 with the border strips indexed by (u_j, v_i) and (u_i, v_j). It is straightforward to verify, using (25) or a direct argument, that $\mathrm{ht}(\boldsymbol{T})$ and $\mathrm{ht}(\boldsymbol{T}')$ differ by an odd integer. Hence (28) holds for $\mathcal{I}$, and the proof of the claim follows by induction.

Now let $\ell(v) = r$ and $m_i(v) = \#\{j : v_j = i\}$, the number of parts of v equal to i. Since $z_v = 1^{v_1} v_1!\, 2^{v_2} v_2! \cdots$, we have

$$\begin{aligned} \hat{s}_{\lambda/\mu} &= \sum_{\ell(v)=r} z_v^{-1} \chi^{\lambda/\mu}(v) p_v \\ &= \sum_{\ell(v)=r} \frac{1}{m_1(v)!\, m_2(v)! \cdots} \chi^{\lambda/\mu}(v) \tilde{p}_v. \end{aligned}$$

Now by the Murnaghan–Nakayama rule we have

$$\chi^{\lambda/\mu}(v) = \sum_{\boldsymbol{T}} (-1)^{\mathrm{ht}(\boldsymbol{T})},$$

where $\boldsymbol{T}$ ranges over all border strip tableaux of shape λ/μ and some fixed type $\alpha = (\alpha_1, \ldots, \alpha_r)$ whose decreasing rearrangement is v. Since there are $r!/m_1(v)!m_2(v)! \cdots$ different permutations α of the entries of v, we have

$$\chi^{\lambda/\mu}(v) = \frac{m_1(v)!\, m_2(v)! \cdots}{r!} \sum_{\boldsymbol{T}} (-1)^{\mathrm{ht}(\boldsymbol{T})},$$

where $\boldsymbol{T}$ now ranges over all border strip tableaux of shape λ/μ whose type is some permutation α of v. By Theorem 4.8, Proposition 2.1, and Eq. (28) we then have

$$\chi^{\lambda/\mu}(v) = \frac{m_1(v)!\, m_2(v)! \cdots}{r!} \left(r! \sum_{\mathcal{I}\,:\,\mathrm{type}(\mathcal{I})=v} (-1)^{z(\lambda/\mu)+c(\mathcal{I})} \right), \tag{29}$$

where $\mathcal{I}$ ranges over all interval sets of λ/μ of type v, and the proof follows. ∎

Let us remark that just as in the Murnaghan–Nakayama rule, cancellation can occur in the sum on the right-hand side of (27). For instance, if $\lambda/\mu = 4442/11$ then there is one interval set of type $(6,3,2,1)$ with one crossing and one with two crossings.

The following corollary follows immediately from Eq. (29).

COROLLARY 5.3. *Let λ/μ be a skew shape of rank r and let $\ell(v) = r$. Then $\chi^{\lambda/\mu}(v)$ is divisible by $m_1(v)!\, m_2(v)! \cdots$.*

Let $A = (a_{ij})$ be an array of real numbers with $1 \leqslant i < j \leqslant 2r$. Recall that the *Pfaffian* $\mathrm{Pf}(A)$ may be defined by (e.g. [6, p. 616])

$$\mathrm{Pf}(A) = \sum_{\pi} (-1)^{c(\pi)} a_{i_1 j_1} \cdots a_{i_r j_r},$$

where the sum is over all partitions π of $\{1, 2, \ldots, 2r\}$ into two element blocks $i_k < j_k$, and where $c(\pi)$ is the number of crossings of π, i.e., the number of pairs $h < k$ for which $i_h < i_k < j_h < j_k$. Comparing with Theorem 5.2 gives the following alternative way of writing (27). Let $\mathrm{SS}(\lambda/\mu) = q_1 q_2 \cdots q_k$; let $u_1 < u_2 < \cdots < u_r$ be those indices for which $q_{u_i} = L_s$ for some s; and let $v_1 < v_2 < \cdots < v_r$ be those indices for which $q_{v_i} = R_s$ for some s. Let $w_1 < w_2 < \cdots < w_{2r}$ consist of the u_i's and v_i's arranged in increasing order. Then

$$\hat{s}_{\lambda/\mu} = (-1)^{z(\lambda/\mu)} \mathrm{Pf}(a_{ij}),$$

where

$$a_{ij} = \begin{cases} \tilde{p}_{w_j - w_i} & \text{if } w_i = u_s \text{ and } w_j = v_t \text{ for some } s < t, \\ 0 & \text{otherwise.} \end{cases}$$

For instance, $\mathrm{SS}(443/2) = L_0 L_1 O R_1 L_1 R_1 R_0$ and $z(443/2) = 2$, whence

$$\hat{s}_{443/2} = \mathrm{Pf} \begin{pmatrix} 0 & \tilde{p}_3 & 0 & \tilde{p}_5 & \tilde{p}_6 \\ & \tilde{p}_2 & 0 & \tilde{p}_4 & \tilde{p}_5 \\ & & 0 & 0 & 0 \\ & & & \tilde{p}_1 & \tilde{p}_2 \\ & & & & 0 \end{pmatrix}.$$

Note that from (11) or (24) we get the following Pfaffianic formula for the coefficient $y(\lambda/\mu)$ of $t^{\mathrm{rank}(\lambda/\mu)}$ in $s_{\lambda/\mu}(1^t)$:

$$y(\lambda/\mu) = (-1)^{z(\lambda/\mu)} \mathrm{Pf}(b_{ij}),$$

where

$$b_{ij} = \begin{cases} 1/(w_j - w_i) & \text{if } w_i = u_s \text{ and } w_j = v_t \text{ for some } s < t, \\ 0 & \text{otherwise.} \end{cases}$$

Similarly from Theorem 5.2 there follows

$$y(\lambda/\mu) = (-1)^{z(\lambda/\mu)} \sum_{\mathscr{I}=\{(u_1,v_1),\ldots,(u_r,v_r)\}} \frac{(-1)^{c(\mathscr{I})}}{\prod_{i=1}^{r} (v_i - u_i)}, \qquad (30)$$

summed over all interval sets $\mathscr{I}$ of λ/μ.

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