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On the Topological Regularity of the Solution Set of Differential Inclusions with Constraints

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In this paper we establish the topological regularity of the solution set of differential inclusions with constraints, defined in $\mathbb{R}^n$. The result is then extended to systems defined in Banach spaces. Our proof makes use of an approximation result by Lipschitz functions of Caratheodory functions, which we also prove in this paper.

1. INTRODUCTION

It is known that the solution set of the Cauchy problem $\dot{x}(t)=f(t, x(t))$ a.e., $x(0)=x_0$, with $t\in T=[0, r]$, $f(\cdot,\cdot)$ a bounded, continuous vector field on $T\times\mathbb{R}^n$, is an R_δ-set (see Yorke [15]). Recall that a subset of a metric space is called an R_δ-set if it is the intersection of a decreasing sequence of nonempty, compact absolute retracts. So every R_δ-set is acyclic and, in particular, nonempty, compact, and connected. This result was extended recently to multivalued differential equations (differential inclusions), by Himmelberg and Van Vleck [7] and DeBlasi and Myjak [2] for differential inclusions in $\mathbb{R}^n$ and by Papageorgiou [11] and Deimling and Rao [4] for differential inclusions in Banach spaces. However, none of the above works considered systems with constraints (viability problems). The purpose of this note is to derive such a topological regularity result for the solution set of differential inclusions with constraints.

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2. Preliminaries

Let (Ω, Σ) be a measurable space and X a separable Banach space. Throughout this note, we use the following notation: $P_{f(c)}(X) = \{A \subseteq X$: nonempty, closed, (and convex)$\}$. A multifunction $F: \Omega \to P_f(X)$ is said to be measurable if, for all $x \in X$, the $\mathbb{R}_+$-valued function $\omega \to d(x, F(\omega))$ is measurable (note that Himmelberg [6] calls such a multifunction weakly measurable).

Let Y, Z be Hausdorff topological spaces and $G: Y \to 2^Z \setminus \{\varnothing\}$. We say that $G(\cdot)$ is upper semicontinuous (u.s.c.) (resp. lower semicontinuous (l.s.c.)) if, for all $U \subseteq Z$ nonempty, open, $G^+(U) = \{y \in Y: G(y) \subseteq U\}$ (resp. $G^-(U) = \{y \in Y: G(y) \cap U \neq \varnothing\}$) is open in Y.

On $P_f(X)$ we can define a generalized metric, known in the literature as the Hausdorff metric, by

$$h(A, B) = \max[\sup_{a \in A} d(a, B), \sup_{b \in B} d(b, A)].$$

The metric space $(P_f(X), h)$ is complete and a multifunction $G: X \to P_f(X)$ is said to be Hausdorff continuous (h-continuous) if it is continuous from X into $(P_f(X), h)$.

If $K \in P_{fc}(X)$ and $x \in K$, the tangent cone to K at x is defined by

$$T_K(x) = \left\{ v \in X: \lim_{\lambda \downarrow 0} \frac{d(x + \lambda v, K)}{\lambda} = 0 \right\}.$$

This is a closed and convex cone. If $\operatorname{int} K$ is nonempty, then so is $\operatorname{int} T_K(x)$. The normal cone to K at x is defined by

$$N_K(x) = \{x^* \in X^*: (x^*, x) = \sigma(x^*, K) = \sup_{y \in K} (x^*, y)\}.$$

It is well known (see for example, Aubin and Cellina [1]) that the normal cone is the negative polar cone $T_K(x)^- = \{x^* \in X: (x^*, v) \leqslant 0$ for all $v \in T_K(x)\}$.

Recall that a set $A \subseteq X$ is said to be contractible if there exist a continuous $h: [0, 1] \times A \to A$ and $x_0 \in A$ s.t. $h(0, x) = x$ and $h(1, x) = x_0$ on A. A set $C \subseteq X$ is said to be an absolute retract if it can replace $\mathbb{R}$ in Tietze's theorem; i.e., for every metric space Y and closed $A \subseteq Y$, each continuous $f: A \to C$ has an extension $\hat{f}: Y \to C$. Evidently, an absolute retract C is contractible. Let $Y = [0, 1] \times C$, $A = \{0, 1\} \times C \subseteq Y$ and $f(0, x) = x$, $f(1, x) = x_0$ on C. Hence, an R_δ-set is the intersection of compact, contractible sets. The converse is also true and this is a result of Hyman [8] (see the equivalence of statements (j) and (k) in the theorem of Hyman).

Let $T=[0,r]$ and K be a nonempty, closed, and convex subset of X. We say that $f\colon T\times K\to X$ is a Caratheodory function if $t\to f(t,x)$ is measurable, $x\to f(t,x)$ is continuous, and there exists $\varphi(\cdot)\in L^1_+$ s.t. $|f(t,x)|\leqslant\varphi(t)$ a.e. for all $x\in X$. In the sequel we need the following approximation result concerning Caratheodory functions.

LEMMA α. If $f\colon T\times K\to X$ *is a Caratheodory function*, then *given any* $\varepsilon>0$, *there exists a jointly locally Lipschitz function*

$$f_\varepsilon\colon T\times K\to X \text{ s.t.} \int_0^r \sup_{x\in K}\|f(t,x)-f_\varepsilon(t,x)\|\,dt\leqslant\varepsilon.$$

Proof. From the absolute continuity of the Lebesgue integral, we know that given $\theta>0$ we can find $\delta>0$ s.t. $\int_A\varphi(t)\,dt<\theta$ for all $A\subseteq T$ with $\lambda(A)<\delta$ (here $\lambda(\cdot)$ denotes the Lebesgue measure on T). Also, from the Scorza–Dragoni theorem, we know that there exists $B\subseteq T$ closed with $\lambda(T\backslash B)<\delta$ s.t. $f|_{B\times K}$ and $\varphi|_B$ are both continuous. Let $m=\max_{t\in B}\varphi(t)$ and choose a closed set $C\subseteq T\backslash B$ s.t. $\lambda(T\backslash(B\cup C))\leqslant\theta/m$. Then define $f_1\colon(B\cup C)\times K\to X$ by

$$f_1(t,x)=\begin{cases} f(t,x) & t\in B\\ 0 & t\in C.\end{cases}$$

Thus $f_1(\cdot,\cdot)$ is continuous. Invoking Dugundji's extension theorem, we know that $f_1(\cdot,\cdot)$ has a continuous extension $f_2\colon T\times X\to X$ s.t. $\|f_2(t,x)\|\leqslant m$ for all $(t,x)\in T\times X$. Now from Lasota and Yorke [10], we know that there exists a locally Lipschitz function $f_3\colon T\times X\to X$ s.t. $\|f_2(t,x)-f_3(t,x)\|\leqslant\theta/r$ for all $(t,x)\in T\times X$. Hence we have

$$\begin{aligned}
&\int_0^r\sup_{x\in K}\|f(t,x)-f_3(t,x)\|\,dt\\
&\quad\leqslant\int_0^r\sup_{x\in K}\|f(t,x)-f_2(t,x)\|\,dt\\
&\qquad+\int_0^r\sup_{x\in K}\|f_2(t,x)-f_3(t,x)\|\,dt\\
&\quad\leqslant\int_B\sup_{x\in K}\|f(t,x)-f_2(t,x)\|\,dt\\
&\qquad+\int_C\sup_{x\in K}\|f(t,x)-f_2(t,x)\|\,dt\\
&\qquad+\int_{T\backslash(B\cup C)}\|f(t,x)-f_2(t,x)\|\,dt+\theta.
\end{aligned}$$

Recall that on $B\times K$ $f=f_2$ and on $C\times K$ $f_2=0$. So we get

$$\begin{aligned}\int_0^r \sup_{x\in K} \|f(t,x)-f_3(t,x)\|\,dt &\leqslant \int_C \varphi(t)\,dt+\int_{T\setminus(B\cup C)} \|f(t,x)\|\,dt \\ &\quad +\int_{T\setminus(B\cup C)} \|f_2(t,x)\|\,dt+\theta \\ &< \theta+\theta+m\frac{\theta}{m}+\theta=4\theta.\end{aligned}$$

Take $\theta=\varepsilon/4$ and let $f_\varepsilon=f_3|_{T\times K}$. Then the proof is finished. Q.E.D.

Remark. This lemma is an improvement and generalization of Theorem 2.3 of Kisielewicz *et al.* [9], since their approximation is Lipschitz only in x and they had to assume that $K\subseteq X$ is compact. Also, it generalizes Lemma 4 of DeBlasi and Myjak [2], where $X=\mathbb{R}^n$.

We also need the following extension of Michael's selection theorem due to Rybinski [13, Theorem 2].

LEMMA β. If $G\colon T\times K\to P_{fc}(X)$ *is a measurable multifunction and for every* $t\in T$, $G(t,\cdot)$ *is l.s.c.*, then *there exists* $g\colon T\times K\to X$ *s.t.* $t\to g(t,x)$ *is measurable,* $x\to g(t,x)$ *is continuous, and for all* $(t,x)\in T\times K$, $g(t,x)\in G(t,x)$.

3. Main Results

Let $T=[0,r]$ and $X=\mathbb{R}^N$. We consider the following Cauchy problem:

$$\left\{\begin{array}{l}\dot{x}(t)\in F(t,x(t)) \text{ a.e.}\\ x(0)=x_0\in K\end{array}\right\}. \tag{$*$}$$

Let $S(x_0)\subseteq C(T,\mathbb{R}^N)$ denote the solution set of $(*)$. Note that by a solution of $(*)$ we mean an absolutely continuous function $x\colon T\to\mathbb{R}^N$ such that $\dot{x}(t)\in F(t,x(t))$ a.e. and $x(0)=x_0$ (i.e., solution in the Caratheodory sense). We need the following hypotheses on the data of $(*)$:

H(F) $F\colon T\times K\to P_{fc}(\mathbb{R}^N)$ is a multifunction s.t.

(1) $t \to F(t, x)$ is measurable,

(2) $x \to F(t, x)$ is u.s.c.

(3) $|F(t, x)| = \sup\{\|y\| : y \in F(t, x)\} \leqslant k(t)(1 + \|x\|)$ a.e. with $k(\cdot) \in L^1_+$.

H(K) $K \subseteq \mathbb{R}^N$ is nonempty, compact, convex.

H_τ $F(t, x) \cap T_K(x) \neq \varnothing$ for all $(t, x) \in T \times K$.

Then we have the following theorem on the topological regularity of the solution set $S(x_0)$.

THEOREM 1. If *hypotheses* H(F), H(K), *and* H_τ *hold*, then $S(x_0)$ *is an* R_δ*-set in* $C(T, \mathbb{R}^N)$.

Proof. Without any loss of generality, we may assume that $|F(t, x)| = \sup\{\|y\| : y \in F(t, x)\} \leqslant 1$ (see Deimling [3] for the reduction). In addition, since K is bounded, we can assume that $|K| = \sup\{\|z\| : z \in K\} \leqslant 1$. We also claim that we have $\operatorname{int} K \neq \varnothing$. Indeed, if this is not the case, let $X_0 = \operatorname{span} K$. This is a subspace of X and clearly K has a nonempty interior in X_0. Furthermore, it is easy to see that $T_K(x) \subseteq X_0$ for all $x \in K$ and the orientor field $F(t, x) \cap X_0$ satisfies hypothesis H(F). Hence we can consider the following Cauchy problem, which is equivalent to $(*)$:

$$\left\{\begin{array}{ll} \dot{x}(t) \in F(t, x(t)) \cap X_0 & \text{a.e.} \\ x(0) = x_0 \in K & \end{array}\right\}. \tag{$*$'}$$

Thus there is no loss of generality in assuming that $\operatorname{int} K \neq \varnothing$. Finally, by translation, we can always have that $0 \in \operatorname{int} K$. This then means that there exists $\delta > 0$ s.t. for all $x^* \in \mathbb{R}^N$ we have

$$\delta \cdot \|x^*\| \leqslant \sigma(x^*, K) = \sup\{(x^*, x) : x \in K\}. \tag{1}$$

Invoking Lemma 3 of DeBlasi and Myjak [2], we know that there exist a multifunction $F_0 : T \times K \to P_{fc}(\mathbb{R}^N)$ and a sequence $\{G_n\}_{n \geqslant 1}$ of multifunctions $G_n : T \times K \to P_{fc}(\mathbb{R}^N)$ s.t. (i) $F_0(t, x) \subseteq F(t, x)$ for all $(t, x) \in T \times K$ and if $\Delta \subseteq T$ is measurable and $x, y : \Delta \to \mathbb{R}^N$ are measurable function s.t. $y(t) \in F(t, x(t))$ a.e., then $y(t) \in F_0(t, x(t))$ a.e. on Δ; (ii) $G_n(\cdot, x)$ is measurable for each $x \in K$ and $G_n(t, \cdot)$ is continuous for almost all $t \in T$; (iii) $G_1(t, x) \supseteq G_2(t, x) \supseteq \cdots \supseteq G_n(t, x) \supseteq G_{n+1}(t, x) \supseteq \cdots$ and $G_n(t, x) \supseteq F_0(t, x)$ for all $n \geqslant 1$ and all $(t, x) \in T \times K$; (iv) $h(G_n(t, x), F_0(t, x)) \to 0$ as $n \to \infty$ for each $(t, x) \in T \times K$; and (v) $|G_n(t, x)| = \sup\{\|z\| : z \in G_n(t, x)\} \leqslant 2$ for all $(t, x) \in T \times K$.

Note that because of hypothesis H(F)(1) and Proposition 2.4 of Himmelberg [6], $t \to F(t, x) \cap T_K(x)$ is measurable, so by the Kuratowski–Ryll–Nardzewski selection theorem, we can find $u: T \times X$ measurable s.t. $u(t) \in F(t, x) \cap T_K(z)$ for all $t \in T$ and so from the properties of multifunction $F_0(t, x)$, we have that $u(t) \in F_0(t, x) \cap T_K(x)$ for all $t \in T$. Thus we deduce that $F_0(t, x) \cap T_K(x) \neq \varnothing$ for all $(t, x) \in T \times K$. Therefore the following Cauchy problem is equivalent (i.e., has the same solution set) to $(*)$:

$$\left\{\begin{matrix} \dot{x}(t) \in F_0(t, x(t)) & \text{a.e.} \\ x(0) = x_0 \in K & \end{matrix}\right\}. \tag{$(*)_1$}$$

For $\varepsilon > 0$, we define

$$G_n^{\varepsilon}(t, x) = G_n(t, x) + B(\varepsilon),$$

where $B(\varepsilon) = \{x \in \mathbb{R}^N: \|x\| \leqslant \varepsilon\}$. It is then obvious that for all $(t, x) \in T \times K$, we have $G_n^{\varepsilon}(t, x) \cap \operatorname{int} T_K(x) \neq \varnothing$. Without loss of generality, we assume that $0 < \varepsilon \leqslant \delta \leqslant 1$, where $\delta > 0$ is from (1) above. Since $G_n(t, x)$ is measurable in t and h-continuous in x, from Theorem 3.3 of [12], we have that $(t, x) \to G_n(t, x)$ is measurable. Hence Theorem 4.1 of Himmelberg [6] tells us that $(t, x) \to G_n^{\varepsilon/3}(t, x) \cap T_K(x)$ is measurable (here $G_n^{\varepsilon/3}(t, x) = G_n(t, x) + B(\varepsilon/3)$). Also from Proposition 4, p. 221 of Aubin and Cellina [1], we know that $x \to \operatorname{int} T_K(x)$ has an open graph. Thus from Lemma β of Flytzanis and Papageorgiou [5], we get that $x \to G_n^{\varepsilon}(t, x) \cap \operatorname{int} T_K(x)$ is l.s.c. $\Rightarrow x \to \overline{G_n^{\varepsilon/3}(t, x) \cap \operatorname{int} T_K(x)} = G_n^{\varepsilon/3}(t, x) \cap T_K(x)$ is l.s.c. Therefore Lemma β of this paper is applicable and so we can find $g_n^{1,\varepsilon}: T \times K \to X$ measurable in t, continuous in x s.t. for all $(t, x) \in T \times K$

$$g_n^{1,\varepsilon}(t, x) \in G_n^{\varepsilon/3}(t, x) \cap T_K(x). \tag{2}$$

Invoking Lemma α of this paper, given any $\theta > 0$, we can find $g_n^{2,\varepsilon}: T \times K \to X$ which is locally Lipschitz in both (t, x) and such that

$$\int_0^r \sup_{x \in K} \|g_n^{1,\varepsilon}(t, x) - g_n^{2,\varepsilon}(t, x)\| \, dt < \theta.$$

It is obvious that for each fixed $n \geqslant 1$, by choosing θ sufficiently small, we can get $A_n \subseteq T$ measurable, with $\lambda(A_n) < 1/n^2$ s.t.

$$\|g_n^{1,\varepsilon}(t, x) - g_n^{2,\varepsilon}(t, x)\| < \frac{\varepsilon\delta}{3} \tag{3}$$

for all $(t, x) \in (T \backslash A_n) \times K$. Set $B_n = \bigcup_{k=n}^{\infty} A_k$, let $\varepsilon = 1/n$ and define

$$G_n^*(t, x) = G_n(t, x) + B(1/n) + \chi_{B_n}(t)\, B(\beta/\delta),$$

for $(t, x) \in T \times K$, $\beta = \hat{m} + 1$, $\hat{m} = \sup\{\|g_n^{2, 1/n}(t, x)\| : (t, x) \in T \times K\}$.

It is clear that $G_n^*(\cdot, \cdot)$ is $P_{fc}(\mathbb{R}^N)$-valued, $t \to G_n^*(t, x)$ is measurable, and $x \to G_n^*(t, x)$ is h-continuous. Consider the following multivalued Cauchy problem:

$$\left\{\begin{array}{c} \dot{x}(t) \in G_n^*(t, x(t)) \text{ a.e.} \\ x(0) = x_0 \in K \end{array}\right\}. \tag{$*$)$_2$}$$

Denote the solution set of $(*)_2$ by $S_n^*(x_0)$. We know that $S_n^*(x_0) \subseteq C(T, \mathbb{R}^N)$ is nonempty and compact (see for example Aubin and Cellina [1] or Deimling [3]). Also since $\lambda(B_n) \leqslant \sum_{k=n}^{\infty} 1/k^2 \to 0$ as $n \to \infty$ and since by construction $h(G_n(t, x(t)), F_0(t, x(t))) \to 0$ as $n \to \infty$ for every $t \in T$ and every $x(\cdot) \in C(T, \mathbb{R}^N)$, we have that $h(G_n^*(t, x(t)), F_0(t, x(t))) \to 0$ in measure and thus by passing to a subsequence if necessary, we may assume that $h(G_n^*(t, x(t)), F_0(t, x(t))) \to 0$ a.e. Having that fact, it is easy to see that $S(x_0) = \bigcap_{n \geqslant 1} S_n^*(x_0)$. Therefore in order to complete the proof, we only need to show that $S_n^*(x_0)$ is contractible. To this end, for each $n \geqslant 1$, let $g_n^*(t, x) = g_n^{2, 1/n}(t, x) - (1/3n)\, x - (\beta/\delta)\, \chi_{B_n}(t)\, x$, where $g_n^{2, 1/n}(t, x)$ is from (3), and recall that $\beta = \hat{m} + 1$, $\hat{m} = \sup\{\|g_n^{2, 1/n}(t, x)\| : (t, x) \in T \times K\}$. Consider the Cauchy problem

$$\left\{\begin{array}{ll} \dot{x}(t) = g_n^*(t, x(t)) & \text{a.e. on } [s, r] \\ x(s) = y \in K & \end{array}\right\} \tag{$*$)$_2^n$}$$

for each $s \in [0, r)$ and each $y \in K$. We claim that for all $(t, x) \in T \times K$

$$g_n^*(t, x) \in T_K(x).$$

Indeed let $x^* \in N_K(x)$ and first assume that $t \notin B_n$. Then we have

$$\begin{aligned} (x^*, g_n^*(t, x)) &= (x^*, g_n^{2, 1/n}(t, x)) - \frac{1}{3n}(x^*, x) \\ &= (x^*, g_n^{2, 1/n}(t, x) - g_n^{1, 1/n}(t, x)) + (x^*, g_n^{1, 1/n}(t, x)) - \frac{1}{3n}(x^*, x) \\ &\leqslant \frac{\delta}{3n}\|x^*\| - \frac{\delta}{3n}\|x^*\| = 0 \end{aligned}$$

(recall that since $x^* \in N_K(x)$, $(x^*, x) = \sigma(x^*, K) = \sup_{y\in K}(x^*, y)$ and from (1) we have that $\delta\|x^*\| \leqslant \sigma(x^*, K)$).

Therefore, if $(t, x) \in (T\setminus B_n) \times K$, then $g_n^*(t, x) \in T_K(x)$.
Next let $t \in B_n$. Then for $x^* \in N_K(x)$, we have

$$(x^*, g_n^*(t, x)) = \left(x^*, g_n^{2,1/n}(t, x) - \frac{1}{3n}x\right) - \frac{\beta}{\delta}(x^*, x)$$

$$\leqslant \|x^*\| \cdot \left\| g_n^{2,1/n}(t, x) - \frac{1}{3n}x \right\| - \frac{\beta}{\delta}\sigma(x^*, K)$$

$$\leqslant \|x\|^* \left[\hat{m} + \frac{1}{3n}\right] - \frac{\beta}{\delta}\delta\|x^*\|$$

(since we have assumed that $|K| \leqslant 1$ and using inequality (1)). Thus we get

$$(x^*, g_n^*(t, x)) \leqslant \|x^*\| \, [\hat{m} + 1] - \beta\|x^*\| = \beta\|x^*\| - \beta\|x^*\| = 0$$

$$\Rightarrow g_n^*(t, x) \in T_K(x) \qquad \text{for all} \quad (t, x) \in B_n \times K.$$

Thus we have proved that for all $(t, x) \in T \times K$, $g_n^*(t, x) \in T_K(x)$. Also $t \to g_n^*(t, x)$ is measurable and $x \to g_n^*(t, x)$ is Lipschitz on K. Thus $(*)_2^n$ has a unique solution $u(t; s, y)$ and $(s, y) \to u(t; s, y)$ is continuous (continuous dependence result for o.d.e.'s). Also from the definitions of $G_n^*(t, x)$ and $g_n^*(t, x)$, we have that $g_n^*(t, x) \in G_n^*(t, x)$ for all $(t, x) \in T \times K$ and all $n \geqslant 1$. Consequently we may define a homotopy $h: [0, 1] \times S_n^*(x_0) \to S_n^*(x_0)$ as follows:

$$h(\lambda, x)(t) = \begin{cases} x(t) & \text{if} \quad t \in [0, \lambda r] \\ u(t; \lambda r, x(\lambda r)) & \text{if} \quad t \in (\lambda r, r]. \end{cases}$$

Hence $S_n^*(x_0)$ is contractible and so since $S(x_0) = \bigcap_{n\geqslant 1} S_n^*(x_0)$, we conclude that $S(x_0)$ is an R_δ-set. Q.E.D.

For X being a general separable Banach space (not necessarily finite dimensional), we need the following hypothesis on the orientor field and the set K:

H(F)′ $F: T \times K \to P_{fc}(X)$ is a multifunction s.t.

(1) $t \to F(t, x)$ is measurable,

(2) $x \to F(t, x)$ is u.s.c.,

(3) $|F(t, x)| = \sup\{\|y\| : y \in F(t, x)\} \leqslant k(t)(1 + \|x\|)$ a.e. with $k(\cdot) \in L_+^1$,

(4) $\lim_{\tau \to 0^+} \alpha(F(T_{t,\tau} \times B)) \leqslant l(t)\,\alpha(B)$, for $t \in (0, r)$, $B \subseteq K$ and $l(\cdot)$ in L^1_+ (here $T_{t,\tau} = [t-\tau, t+\tau] \cap T$ and $\alpha(\cdot)$ is the Kuratowski measure of noncompactness; see for example Deimling [3]).

H(K)′ $K \subseteq X$ is nonempty, closed, convex, bounded, and int $K \neq \varnothing$.

THEOREM 2. *If hypotheses* H(F)′, H(K)′, *and* H_τ *hold,* then $S(x_0)$ *is nonempty, compact and connected.*

Proof. Nonemptiness and compactness of $S(x_0)$ follows from Deimling [3]. Also, all the arguments in the proof of Theorem 1 are also valid in this more general situation except that $S_n^*(x_0)$ need not be closed or compact. Also it is clear from the proof of Lemma α that we can find $\hat{m}_1 > 0$ s.t. for all $(t, x) \in T \times K$, $\|g_n^{2,1/n}(t, x)\| \leqslant \hat{m}_1$, $n \geqslant 1$. Q.E.D.

Remark. If $X = \mathbb{R}^n$ and H_τ is replaced by the stronger condition

$$F(t, x) \subseteq T_K(x)$$

for all $(t, x) \in T \times K$, we recover from Theorem 1 a result due to Deimling [3]. Also for an analogous result for single-valued differential equation without constraints, we refer to the work of Szufla [14].

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