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Global topological properties of the Hopf bifurcation

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Abstract

We study the homotopical and homological properties of the attractors evolving from a generalized Hopf bifurcation. We consider the Lorenz equations for parameter values near the Hopf bifurcation and study a natural Morse decomposition of the global attractor, calculating the Čech homotopy type of the Lorenz attractor, the shape indexes of the Morse sets and the Morse equation of the decomposition.

Keywords: Hopf bifurcation; Lorenz equations; Lorenz attractor; Morse equations; Conley index; Cohomological index

1. Introduction

This paper is devoted to the study of some properties of the generalized Poincaré–Andronov–Hopf bifurcation for dynamical systems from a topological point of view. We consider a situation related to the transition from asymptotic stability to complete instability, not only in the case of equilibrium points or periodic orbits but also in the case of general attractors of flows in manifolds. We show that this kind of bifurcations produces families of attractors whose properties can be examined using those parts of topology designed to study spaces from a global point of view, in particular Borsuk's shape theory, Čech cohomology and the classical duality theory of manifolds.

We also study some properties of the Lorenz equations for parameter values near the Hopf bifurcation. We calculate the Čech homotopy type (or shape) of the Lorenz attractor, which gives information about its global topological structure, and consider some natural Morse decompo-

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sitions of the global attractor induced by the non-wandering set. We find the cohomological Conley index of the Morse sets (including the Lorenz attractor) and calculate the corresponding Morse equations. We also see how these equations change when the parameter crosses the value corresponding to the Hopf bifurcation.

This paper illustrates the fact, already noticed by Kennedy and Yorke in [21], that "bizarre Topology is natural in Dynamical Systems." In fact, the most natural formulation of the topological ideas related to dynamics is very often achieved through Čech theory, which provides deeper insight when applied to objects with complicated properties, like many attractors. Standard notions and techniques would produce unsatisfactory results in this context, since most of the theorems proved in the paper do not hold when classical homotopy theory or singular homology replace the corresponding Čech notions. We assume as known some elementary facts on shape theory, algebraic topology and the theory of the Conley index of isolated invariant compacta. Some familiarity with the most topological parts of the theory of dynamical systems would be a helpful prerequisite for the reading of this paper. We use, in particular, some well-known facts related to properties of stability and attraction of flows. For information about the topological and the dynamical aspects of these theories we refer the reader to the papers and books [2,4,6,7,14,24,27,28,32,34,35,40] (see also [3,11–13,20,30,31,36–38] for applications of the theory of shape to dynamical systems). The paper [44] by Viana explores the relations between the study of attractors and that of main bifurcation mechanisms. We also recommend the book [41] by Sparrow and the survey article [45] by Viana for general information about the Lorenz attractor.

2. Notation and terminology

We shall consider continuous dynamical systems (or flows) $\varphi : W \times \mathbb{R} \to W$, where W is a (metrizable) topological manifold. On some occasions W will be required to be differentiable and we use the term "smooth manifold" to refer to a C^r manifold with $r \geqslant 2$. We always consider a fixed metric d defined in W. Suppose K is an isolated invariant set of W and (N, L) is an index pair for K. If we take the quotient N/L then the point corresponding to the equivalence class of L will be denoted by $*$ (note that $N/\emptyset$ is obtained from N by adjoining the isolated point $*$, i.e. $N/\emptyset = N \cup \{*\}$). The (pointed) homotopy class of $(N/L, *)$ is the homotopy index (or Conley index) of K and will be denoted by $h(K)$ and the (pointed) shape of $(N/L, *)$ is the shape index and will be denoted by $s(K)$. The shape index has been defined by Robbin and Salamon [30].

We shall consider parametrized families of flows $\varphi_\lambda : W \times \mathbb{R} \to W$ with $\lambda \in I$ (the unit interval). We shall assume that the map $\hat{\varphi} : W \times \mathbb{R} \times I \to W$ defined by $\hat{\varphi}(x, t, \lambda) = \varphi_\lambda(x, t)$ is continuous. Our notation is the usual one in dynamical systems and the Conley index theory. In particular, the unstable manifold of K is the set $W^u(K) = \{x \in W \mid \emptyset \neq \omega^*(x) \subset K\}$ (the notation $\alpha(x)$ is often used in the literature for the α-limit of x, that we denote by $\omega^*(x)$).

We use in the paper some of the most elementary notions of Borsuk's theory of shape (also called Čech homotopy theory), as can be found in [4,7,10,24] and [27]. Shape can be understood as a generalized homotopy type which agrees with the usual one in spaces with good local topological behaviour, such as polyhedra or manifolds, but which provides more geometrical and topological insight in the case of spaces with complicated topological properties such as attractors or other kinds of isolated invariant compacta.

3. Bifurcations of flows in manifolds

The study of the Hopf bifurcation had its origins in the works of Poincaré at the end of the nineteenth century and was later continued by A.A. Andronov. E. Hopf started his research on this subject [15,18] in the decade of the forties of the twentieth century. Although the term "Hopf bifurcation" is commonly used, the name "Poincaré–Andronov–Hopf bifurcation" is more respectful towards the historical origins. We recommend the book [23] by Marsden and McCracken for information about this topic. The authors follow in the book the methods used by Ruelle and Takens in their fundamental paper [33]. A useful reference for the study of the foundations of bifurcation theory, in particular bifurcations from general attractors, is the paper [39] by Seibert and Florio.

We discuss in this section a generalized form of the Poincaré–Andronov–Hopf bifurcation for flows defined on manifolds. We consider the supercritical case, in which the bifurcation results from the transition of asymptotic stability to instability in such a way that the attractor expels a family of new attractors that are created in the bifurcation and whose global topological properties we study. The subcritical case can be studied in a similar way. The properties that we consider are of homotopical and homological nature. In the first case we express them in the language of Borsuk's shape theory and in the second one we use Čech cohomology. We distinguish three different situations. First we consider the case in which the bifurcation evolves from an equilibrium, in the second place we study bifurcations from periodic orbits and, finally, we consider the general case of bifurcations evolving from attractors.

Theorem 1 *(Bifurcations from equilibrium points). Let W be an n-dimensional manifold. Let $\varphi_\lambda : W \times \mathbb{R} \to W$ be a parametrized family of flows with $\lambda \in I$ (the unit interval) and such that the point $p \in W$ is an attractor of the flow φ_0. Then the two following statements hold:*

(1) *If p is a repeller of φ_λ for every $\lambda > 0$, then for every compact neighborhood V of p contained in the basin of attraction of p for the flow φ_0, there exists λ_0 such that for every λ, with $0 < \lambda \leqslant \lambda_0$, there exists an attractor K_λ of φ_λ with the shape (and hence with the Čech homology and cohomology) of S^{n-1}, the attractor K_λ is contained in $V - \{p\}$ and attracts all points in $V - \{p\}$. Moreover, the multivalued function $\Theta : [0, \lambda_0] \to W$ defined by $\Theta(0) = \{p\}$ and $\Theta(\lambda) = K_\lambda$ (when $\lambda \neq 0$) is upper semi-continuous.*

(2) *If the following conditions hold for $\lambda > 0$:*
   (a) *there exists a k-dimensional submanifold W_0 of W such that W_0 is invariant by φ_λ,*
   (b) *there exists a neighborhood U of p (the same for all λ) such that the maximal invariant set of φ_λ inside U is contained in W_0,*
   (c) *p is a repeller of the restriction flow $\varphi_\lambda | W_0 : W_0 \times \mathbb{R} \to W_0$,*

   then there is λ_0 such that for every λ, with $0 < \lambda \leqslant \lambda_0$, there is an attractor $K_\lambda \subset U$ of the unrestricted flow *$\varphi_\lambda : W \times \mathbb{R} \to W$ with the shape (and hence with the Čech homology and cohomology) of S^{k-1}. In particular, K_λ has the shape of S^1 when W_0 is of dimension 2. The attractors K_λ are contained in arbitrarily small neighborhoods of p for values of λ close to 0.*

Proof. Let $\mathcal{B}_0$ be the basin of attraction of p for the flow φ_0 and consider a compact neighborhood V of p contained in $\mathcal{B}_0$. Let N be a neighborhood of p contained in V and homeomorphic to a closed n-cell. Choose an $\epsilon > 0$ such that the closed ball $B[p, \epsilon] \subset \operatorname{int} N$. By using the stability of $\{p\}$, we select $\delta > 0$ such that $\varphi_0(x, t)$ is in the open ball $B(p, \epsilon/2)$ for every $x \in B[p, \delta]$

and every $t \geqslant 0$. By the definition of attractor there exists $T > 0$ such that $\varphi_0(V,t) \subset B(p, \delta/2)$ for $t \geqslant T$. We choose λ_0 such that $\varphi_\lambda(x,t)$ and $\varphi_0(x,t)$ are $\delta/2$-close for every $x \in V$, every t with $0 \leqslant t \leqslant T$ and every $\lambda \leqslant \lambda_0$. Define A_λ as the set of all points $y \in W$ such that there exist a sequence of points $x_n \in V$ and a sequence of times $t_n \to \infty$ with $y = \lim_{n\to\infty} \varphi_\lambda(x_n, t_n)$. Using standard arguments in topological dynamics, it is easy to see that A_λ is a compact invariant set contained in the interior of N. Moreover, V is uniformly attracted by A_λ and, hence, A_λ is an attractor of φ_λ and it is also the maximal invariant set in V. Since p is a repeller of φ_λ, p must be contained in A_λ and the basin of repulsion of p, $\mathcal{R}_\lambda$, is an open set of W contained in A_λ. In fact, the flow φ_λ restricted to A_λ defines a repeller–attractor decomposition of A_λ. We denote by K_λ the corresponding attractor

$$K_\lambda = A_\lambda - \mathcal{R}_\lambda.$$

We remark that K_λ is an attractor, not only for the restricted flow $\varphi_\lambda | A_\lambda$, but also for the unrestricted flow $\varphi_\lambda : W \times \mathbb{R} \to W$. Consider now an open neighborhood N_0 of p in $\mathcal{R}_\lambda$ contained in N and such that $N - N_0$ is homeomorphic to $S^{n-1} \times I$. Obviously, the compact set $N - N_0$ is contained in the region of attraction of K_λ for the flow φ_λ. We define a sequence of maps

$$r_k : N - N_0 \to W$$

by the expression

$$r_k(x) = \varphi_\lambda(x, k).$$

If U is a neighborhood of K_λ in W, there is an index k_0 such that the image of r_k is contained in U for every $k \geqslant k_0$. This is a consequence of the fact that $N - N_0$ is contained in the region of attraction of K_λ. Moreover, k_0 can be chosen in such a way that the homotopy between r_k and r_{k+1}

$$H_k(x,t) = \varphi_\lambda(x, k+t), \quad t \in [0,1],$$

induced by the flow takes place in U. This means that the sequence of maps r_k defines an approximative sequence

$$\mathbf{r} = \{r_k, N - N_0 \to K_\lambda\}$$

in the sense of Borsuk [4] and, hence, a shape morphism from $N - N_0$ to K_λ. This morphism is a left inverse of the shape morphism induced by the inclusion

$$i : K_\lambda \to N - N_0$$

since $r_k | K_\lambda$ is homotopic to the identity $i : K_\lambda \to K_\lambda$ for every k. In other words, $\mathbf{r}$ induces a shape domination and, since $N - N_0$ has the homotopy type of S^{n-1}, we deduce that K_λ is shape dominated by S^{n-1}.

On the other hand, K_λ separates the n-cell N into two disjoint non-empty open subsets, $N - A_\lambda$ and $\mathcal{R}_\lambda$. This implies that the Čech cohomology group $\check{H}^{n-1}(K_\lambda)$ is nontrivial. To conclude that K_λ has the shape of S^{n-1} we adopt an argument used by Borsuk and Holsztyński in [5]. The fact, already seen, that K_λ is shape dominated by S^{n-1} implies that there are two shape morphisms

$$\alpha : K_\lambda \to S^{n-1}$$

and

$$\beta : S^{n-1} \to K_\lambda$$

such that $\beta\alpha = \mathbf{i}_{K_\lambda}$. Suppose that $\alpha\beta \neq \mathbf{i}_{S^{n-1}}$. Then, if θ is a generator of $H^{n-1}(S^{n-1})$, θ is mapped by the endomorphism $(\alpha\beta)^*$ of $H^{n-1}(S^{n-1})$ induced by $\alpha\beta$ to $q\theta$, where $q \in \mathbb{Z}$. Hence $(\alpha\beta)^*(\alpha\beta)^*(\theta) = q^2\theta$. Since

$$(\alpha\beta)^*(\alpha\beta)^* = (\alpha\beta\alpha\beta)^* = (\alpha\beta)^*$$

we have that $q = q^2$. Now, Hopf's classification theorem implies that $q \neq 1$ and from this we deduce that $q = 0$ and $\beta^*\alpha^* = 0$. Then

$$\mathbf{i}^*_{K_\lambda} = \alpha^*\beta^* = \alpha^*\beta^*\alpha^*\beta^* = \alpha^* 0 \beta^* = 0$$

and we deduce that $\check{H}^{n-1}(K_\lambda)$ is trivial in contradiction to our previous remark. We conclude that $\alpha\beta = \mathbf{i}_{S^{n-1}}$ and, hence, the shape of K_λ agrees with that of S^{n-1}.

Clearly, K_λ is the only attractor with the properties mentioned in the statement of the theorem and, hence, the multivalued function Θ from $[0, \lambda_0]$ to W given by $\Theta(0) = \{p\}$ and $\Theta(\lambda) = K_\lambda$ (when $\lambda \neq 0$) is well defined. The argument at the beginning of the proof shows that for a given δ the attractors A_λ (and hence the K_λ as well) are contained in $B[p, \delta]$ for λ close to 0. This shows the upper semicontinuity at 0 of the multivalued function. On the other hand for a given $\lambda \in (0, \lambda_0]$ there is an open neighborhood B of p such that B is contained in $\mathcal{R}_{\lambda'}$ for all λ' close to λ. Indeed, if B is such that $\bar{B} \subset \mathcal{R}_\lambda$ and N is an arbitrarily small neighborhood of p, take $\epsilon > 0$ such that the open ϵ-ball centered at p is contained in N and consider a smaller neighborhood of p, N_0, with diameter less than $\epsilon/2$ and such that N_0 is contained in B and negatively invariant by the flow φ_λ. Let $\delta < \epsilon/2$ be such that the open δ-ball centered at p is contained in N_0. Take now $T < 0$ such that $\varphi_\lambda(\bar{B}, t)$ is contained in the open $\delta/2$-ball centered at p for every $t \in (-\infty, T]$ and select $\eta > 0$ such that if λ' is η-close to λ we have that $\varphi_\lambda(x, t)$ and $\varphi_{\lambda'}(x, t)$ are $\delta/2$-close for every $x \in \bar{B}$ and every $t \in [T, 0]$. It is easy to see that $\varphi_{\lambda'}(\bar{B}, t) \subset N$ for every $t \in (-\infty, T]$, which implies that $\bar{B}$ (and hence B) is contained in $\mathcal{R}_{\lambda'}$. Consider now a neighborhood U of K_λ. Since $V - B$ is contained in the region of attraction of $K_{\lambda'}$ and it is also a compact neighborhood of $K_{\lambda'}$, we have that $K_{\lambda'}$ can be described as the set of all points $y \in W$ such that there exist a sequence of points x_n in $V - B$ and a sequence of times $t_n \to \infty$ such that $\varphi_{\lambda'}(x_n, t_n) \to y$. Now a standard argument shows that taking λ' still closer to λ, if necessary, we can guarantee that $\varphi_{\lambda'}(x_n, t_n)$ is in a closed neighborhood $U_0 \subset U$ of K_λ for all t_n larger than a certain T and for every λ' contained in a neighborhood of λ. This implies that $K_{\lambda'}$ is also in U and, hence, we have upper-semicontinuity at λ. This finishes the proof of the first statement in the theorem.

Concerning the second statement, consider a compact neighborhood V of p contained in U and in the basin of attraction of p for φ_0. The arguments in the first part of the proof show that there exists λ_0 such that the attractors A_λ of φ_λ are contained in V for $\lambda \leqslant \lambda_0$, which, by condition (a), implies that A_λ is contained in W_0. Moreover V is attracted by A_λ for $\lambda \leqslant \lambda_0$. Obviously A_λ is also an attractor for the restriction flow $\varphi_\lambda|W_0 : W_0 \times \mathbb{R} \to W_0$. Since p is a repeller for $\varphi_\lambda|W_0$, we have that $\mathcal{R}_\lambda \subset A_\lambda$ and

$$K_\lambda = A_\lambda - \mathcal{R}_\lambda$$

(where $\mathcal{R}_\lambda$ is the basin of repulsion of p with respect to $\varphi_\lambda|W_0$) is an attractor for the flow $\varphi_\lambda|W_0$. Now the repetition of the argument used in the first part of the theorem for this particular situation shows that the shape of K_λ is that of S^{k-1}. Moreover, since K_λ is an attractor inside the attractor A_λ it is also an attractor of the nonrestricted flow $\varphi_\lambda : W \times \mathbb{R} \to W$. The last statement in the theorem is clear from the previous discussion. □

Remark 2. It is not, in general, true that the attractors K_λ produced in the bifurcation described in Theorem 1 have the homotopy type of S^{n-1}. Using ideas similar to those developed by Hastings in [16] and [17] it is easy to define parametrized families of flows in surfaces such that the attractors K_λ evolving from equilibria are homeomorphic to Warsaw circles (which have the shape, but not the homotopy type, of S^1).

We consider now bifurcations evolving from attracting periodic orbits or, more generally, from attractors which are diffeomorphic to the circle.

Theorem 3 *(Bifurcations from periodic orbits). Let W be an orientable n-dimensional smooth manifold and $\varphi_\lambda : W \times \mathbb{R} \to W$ a parametrized family of smooth flows with $\lambda \in I$. Let S be a 1-dimensional submanifold of W such that S is a periodic orbit of φ_0 or, more generally, such that S is diffeomorphic to S^1. Suppose that S is an attractor of the flow φ_0. Then, if S is a repeller of φ_λ for every $\lambda > 0$, we have that for every compact neighborhood V of S contained in the basin of attraction of S for the flow φ_0, there exists λ_0 such that for every λ, with $0 < \lambda \leqslant \lambda_0$, there exists an attractor K_λ of φ_λ with the shape (and hence with the Čech homology and cohomology) of $\mathcal{T} = S^{n-2} \times S^1$. Moreover, the attractors K_λ are contained in $V - S$ and attract all points in $V - S$.*

Proof. Suppose that we have a submanifold $S \subset W$ diffeomorphic to S^1 and such that S is an attractor of φ_0 and a repeller of φ_λ for $\lambda \in (0, 1]$. If the compact neighborhood V of S is contained in the basin of attraction, $\mathcal{B}_0$, of S for the flow φ_0, we can argue exactly as in Theorem 1 to show that φ_λ has an attractor A_λ in V for all λ close to 0 and A_λ attracts V. A_λ can be described as the set of all points $y \in W$ such that there are a sequence of points $x_n \in V$ and a sequence of times $t_n \to \infty$ with $\varphi_\lambda(x_n, t_n) \to y$. Moreover, S is contained in A_λ. The basin of repulsion of S, $\mathcal{R}_\lambda$, is also contained in A_λ and we have, as before, that

$$K_\lambda = A_\lambda - \mathcal{R}_\lambda$$

is an attractor of φ_λ attracting $V - S$. We use the tubular neighborhood theorem on manifolds to choose a neighborhood $\mathcal{T}$ of S in V such that $\mathcal{T}$ is homeomorphic to $B^{n-1} \times S$ (where B^{n-1} is the unit $(n-1)$-dimensional closed ball). Then $\mathcal{T}$ is a neighborhood of A_λ for all λ close to 0 (for the same reasons as in Theorem 1 again). Now we claim that there is $\lambda_0 > 0$ and a time $t_0 \geqslant 0$ such that the following condition holds:

$\varphi_\lambda(\mathcal{T}, [0, t_0])$ is a positively invariant neighborhood of A_λ and it is contained in $\mathcal{B}_0$ for every $\lambda \leqslant \lambda_0$.

Indeed, take $\epsilon > 0$ such that the closed ϵ-neighborhood of S is contained in the interior of $\mathcal{T}$. By the stability of S, there is $\delta > 0$ such that $\varphi_0(x, t)$ belongs to the $\epsilon/2$-neighborhood of S for every x belonging to the δ-neighborhood of S and every $t \geqslant 0$. Moreover, since S is an attractor of φ_0 there exists $t_0 \geqslant 0$ such that $\varphi_0(\mathcal{T}, t)$ is contained in the $\delta/2$-neighborhood of S for $t \geqslant t_0$. Select now γ such that $0 < \gamma < \delta/2$ and the γ-neighborhood of $\varphi_0(\mathcal{T}, [0, t_0])$ is contained in $\mathcal{B}_0$. We choose λ_0 in such a way that $\varphi_\lambda(x, t)$ and $\varphi_0(x, t)$ are γ-close for every $x \in \mathcal{T}$, $t \in [0, t_0]$ and $\lambda \leqslant \lambda_0$. This implies that $\varphi_\lambda(\mathcal{T}, [t_0, \infty]) \subset \mathcal{T}$ and $\varphi_\lambda(\mathcal{T}, [0, t_0]) \subset \mathcal{B}_0$ for $\lambda \leqslant \lambda_0$ and from this follows the required condition.

Consider now an arbitrary λ with $0 < \lambda \leqslant \lambda_0$. We define

$$\mathcal{T}' = \varphi_\lambda\bigl(\mathcal{T}, [0, t_0]\bigr).$$

Choose $s_0 \geqslant 0$ such that $\varphi_0(\mathcal{T}', [0, s_0])$ is a positively invariant neighborhood of S. Such an s_0 exists because $\mathcal{T}'$ is a neighborhood of S contained in the region of attraction of S. We analogously define

$$\mathcal{T}'' = \varphi_0\bigl(\mathcal{T}', [0, s_0]\bigr).$$

We consider now a Lyapunov function $f : \mathcal{B}_\lambda \to \mathbb{R}_+$ defined on the basin of attraction $\mathcal{B}_\lambda$ of A_λ for the flow φ_λ. The function f is strictly decreasing along the orbits of φ_λ and has the property that $f^{-1}(0) = A_\lambda$. Since $\mathcal{T}'$ is a positively invariant neighborhood of A_λ contained in $\mathcal{B}_\lambda$ there exists $c > 0$ such that $f^{-1}([0, c]) \subset \operatorname{int} \mathcal{T}'$. We select a decreasing sequence of positive numbers $c_k \to 0$, with $c_1 = c$ and a sequence of maps

$$\alpha_k : \mathcal{T}' \to \mathbb{R}$$

such that $\alpha_k(x)$ is the only number t with $f[\varphi_\lambda(x, t)] = c_k$ if $f(x) \geqslant c_k$ and $\alpha_k(x) = 0$ otherwise.

We define another sequence of maps

$$r_k : \mathcal{T}' \to f^{-1}\bigl([0, c_k]\bigr) \subset \mathcal{T}'$$

by

$$r_k(x) = \varphi_\lambda\bigl(x, \alpha_k(x)\bigr).$$

These maps are continuous and the sequence r_k has the following property: for every neighborhood U of A_λ there is k_0 such that the image of r_k is contained in U and the homotopy $r_k \simeq r_{k+1}$ takes place in U for every k larger than k_0. Additionally, the maps r_k are all homotopic in $\mathcal{T}'$ to the identity $i : \mathcal{T}' \to \mathcal{T}'$. Moreover all the points in A_λ are left invariant by the maps r_k and the mentioned homotopies.

In other words, A_λ is a *strong shape deformation retract of* $\mathcal{T}'$ and, consequently, the inclusion $j : A_\lambda \to \mathcal{T}'$ is a strong shape equivalence (we can use the same notation $j : A_\lambda \to \mathcal{T}'$ to indicate the induced strong shape morphism without risk of confusion). We denote the inverse strong shape morphism by

$$\mathbf{r} : \mathcal{T}' \to A_\lambda.$$

Similarly, the inclusions

$$j : S \to \mathcal{T}''$$

and

$$j : S \to \mathcal{T}$$

are strong shape equivalences (the first one for the same reasons as before and the second one because $\mathcal{T} = S \times B^{n-1}$ and hence a deformation can be defined at the homotopical level). From this it follows that the inclusion

$$j : \mathcal{T} \to \mathcal{T}''$$

induces also a strong shape isomorphism. We denote its inverse strong shape morphism by

$$\mathbf{r}' : \mathcal{T}'' \to \mathcal{T}.$$

We claim that

$$j : \mathcal{T} \to \mathcal{T}'$$

is also a strong shape equivalence. Indeed, a left inverse is the composition of $j : \mathcal{T}' \to \mathcal{T}''$ with $\mathbf{r}'$ and a right inverse is the composition of $\mathbf{r} : \mathcal{T}' \to A_\lambda$ with $j : A_\lambda \to \mathcal{T}$ (we use again the same notation for the inclusions j and for the strong shape morphisms induced by them). From this it follows that

$$j : A_\lambda \to \mathcal{T}$$

is a strong shape equivalence and, hence, A_λ is a strong shape deformation retract of $\mathcal{T}$.

Consider now

$$K_\lambda = A_\lambda - \mathcal{R}_\lambda.$$

Since S and K_λ define a repeller–attractor decomposition of A_λ, we can use a Lyapunov function in $A_\lambda - S$ for the attractor K_λ to show, as we did before, that there is a compact neighborhood, L, of K_λ in $A_\lambda - S$ such that the inclusion

$$j : K_\lambda \to L$$

is a strong shape equivalence and, hence, K_λ is a strong shape deformation retract of L. Moreover L can be chosen in such a way that it is a strong deformation retract of $A_\lambda - S$ (basin of attraction of K_λ for the restriction flow $\varphi_\lambda | A_\lambda$). The proof of these facts makes use of the elementary properties of the Lyapunov functions and follows a pattern similar to that established in the proof of the case considered above. For this reason we omit the details.

Since the deformations involved in the strong shape deformation retractions are defined by means of homotopies which leave all the points of K_λ invariant, we can combine them to obtain new homotopies establishing the fact that K_λ is a strong shape deformation retract of $(\mathcal{T} - A_\lambda) \cup L$. Since, in turn, $(\mathcal{T} - A_\lambda) \cup L$ is a deformation retract of $\mathcal{T} - S$ and $\mathcal{T} - S$ has the homotopy type of $S^{n-2} \times S^1$ we conclude that K_λ has the shape of $S^{n-2} \times S^1$.

The last assertion in the statement of the theorem is clear from the previous discussion. This completes the proof of the theorem. □

Remark. It is possible to give more refined versions of Theorem 2, similar to part (2) of Theorem 1, in such a way that the attractors K_λ are contained in an invariant submanifold W_0 of W. In the special case when W_0 is 3-dimensional, the attractors K_λ have the shape of a torus.

When the bifurcation originates from an attractor more general than an stationary point or a periodic orbit we still have some topological information about the attractors originated in the bifurcation.

Theorem 4 *(Bifurcations from general attractors). Let W be an orientable n-dimensional manifold. Let $\varphi_\lambda : W \times \mathbb{R} \to W$ be a parametrized family of flows with $\lambda \in I$ such that the compact connected set $A \subset W$ is an attractor of φ_0 and a repeller of φ_λ for $\lambda > 0$. Suppose that for a fixed $k \leqslant n$ the reduced homology groups $\tilde{H}_k(\mathcal{B})$ and $\tilde{H}_{k-1}(\mathcal{B})$ are trivial, where $\mathcal{B}$ is the basin of attraction of A for φ_0. Then for every compact neighborhood V of A in $\mathcal{B}$ there is $\lambda_0 > 0$ such that for every λ with $0 < \lambda \leqslant \lambda_0$ there is an attractor K_λ of φ_λ contained in $V - A$, attracting $V - A$ and such that $\check{H}^{n-k}(K_\lambda) \cong H^{n-k}(\mathcal{B})$ if $k \neq 1$ and $\check{H}^{n-1}(K_\lambda) \cong \mathbb{Z} \oplus H^{n-1}(\mathcal{B})$ if $k = 1$. In particular, if $\mathcal{B}$ is contractible then $\check{H}^{n-1}(K_\lambda) \cong \check{H}^0(K_\lambda) \cong \mathbb{Z}$ and $\check{H}^{n-k}(K_\lambda) \cong \{0\}$ for $k \neq 1, n$.*

The attractors K_λ are in arbitrarily small neighborhoods of A for values of λ close to 0.

Proof. We can follow the same steps as in Theorems 1 and 2 to establish the following facts:

(1) There is $\lambda_0 > 0$ such that for every $\lambda \leqslant \lambda_0$ there exists an attractor $A_\lambda \subset V$ attracting V and such that $\mathrm{Sh}(A_\lambda) = \mathrm{Sh}(A)$. Moreover, since the inclusion $A \to \mathcal{B}$ is a shape equivalence, we also have that $\mathrm{Sh}(A_\lambda) = \mathrm{Sh}(\mathcal{B})$.

(2) If $\mathcal{R}_\lambda$ is the basin of repulsion of A for φ_λ then $A_\lambda \subset \mathcal{R}_\lambda$ and

$$K_\lambda = A_\lambda - \mathcal{R}_\lambda$$

is an attractor and $V - A$ is contained in its basin of attraction.

Now, by the Alexander duality theorem applied to the orientable manifold $\mathcal{B}$, we have that

$$\check{H}^{n-k}(K_\lambda) \cong H_k(\mathcal{B}, \mathcal{B} - K_\lambda).$$

If we consider the long homology sequence of the pair $(\mathcal{B}, \mathcal{B} - K_\lambda)$

$$\cdots \to \tilde{H}_k(\mathcal{B}) \to H_k(\mathcal{B}, \mathcal{B} - K_\lambda) \to \tilde{H}_{k-1}(\mathcal{B} - K_\lambda) \to \tilde{H}_{k-1}(\mathcal{B}) \to \cdots,$$

since $\tilde{H}_k(\mathcal{B}) \cong \tilde{H}_{k-1}(\mathcal{B}) \cong \{0\}$, we have that

$$H_k(\mathcal{B}, \mathcal{B} - K_\lambda) \cong \tilde{H}_{k-1}(\mathcal{B} - K_\lambda).$$

Since $\mathcal{B} - K_\lambda = \mathcal{R}_\lambda(A) \cup (\mathcal{B} - A_\lambda)$ we have that

$$\tilde{H}_{k-1}(\mathcal{B} - K_\lambda) \cong \tilde{H}_{k-1}\big(\mathcal{R}_\lambda(A)\big) \oplus \tilde{H}_{k-1}\big((\mathcal{B} - A_\lambda)\big) \quad \text{if } k \neq 1.$$

On the other hand, from the long exact sequence for the pair $(\mathcal{B}, \mathcal{B} - A_\lambda)$

$$\cdots \to \tilde{H}_k(\mathcal{B}) \to H_k(\mathcal{B}, \mathcal{B} - A_\lambda) \to \tilde{H}_{k-1}(\mathcal{B} - A_\lambda) \to \tilde{H}_{k-1}(\mathcal{B}) \to \cdots$$

we obtain that

$$\tilde{H}_{k-1}(\mathcal{B} - A_\lambda) \cong H_k(\mathcal{B}, \mathcal{B} - A_\lambda).$$

But, using again the Alexander duality theorem for the compactum A_λ in the manifold $\mathcal{B}$ and recalling that Čech homology and cohomology are shape invariants, we obtain

$$H_k(\mathcal{B}, \mathcal{B} - A_\lambda) \cong \check{H}^{n-k}(A_\lambda) \cong \check{H}^{n-k}(A) \cong H^{n-k}(\mathcal{B}).$$

Hence, from the fact that the inclusion $A \to \mathcal{R}_\lambda(A)$ is a shape equivalence we deduce that

$$\begin{aligned} \check{H}^{n-k}(K_\lambda) &\cong H_{k-1}\big(\mathcal{R}_\lambda(A)\big) \oplus H^{n-k}(\mathcal{B}) \\ &\cong \check{H}_{k-1}(A) \oplus H^{n-k}(\mathcal{B}) \cong H_{k-1}(\mathcal{B}) \oplus \check{H}^{n-k}(\mathcal{B}) \cong H^{n-k}(\mathcal{B}) \quad \text{if } k \neq 1. \end{aligned}$$

If $k = 1$, the only difference from the previous argument is that

$$\tilde{H}_0(\mathcal{B} - K_\lambda) \cong H_0\big(\mathcal{R}_\lambda(A)\big) \oplus \tilde{H}_0\big((\mathcal{B} - A_\lambda)\big)$$

and, since $\mathcal{R}_\lambda(A)$ is a connected open subset of W, we have that $H_0(\mathcal{R}_\lambda(A)) \cong \mathbb{Z}$. The rest of the argument is exactly the same and we conclude that

$$\check{H}^{n-1}(K_\lambda) \cong \mathbb{Z} \oplus H^{n-1}(\mathcal{B}) \quad \text{if } k = 1.$$

The last sentence in the statement of the theorem is proved just like in Theorem 1. This completes the proof of the theorem. □

Alexander and Yorke proved in [1] global versions of results about bifurcation of periodic orbits from an equilibrium point. In particular, they proved that, under suitable hypotheses, the family of periodic orbits satisfies one of the three following conditions: (i) the family of orbits contains elements for λ arbitrarily close to the boundary of the interval, (ii) the family contains elements of arbitrarily large period, or (iii) the family contains orbits which do not lie in any preassigned compact subset of the manifold. It would be interesting to study properties of this kind in the more general context of the present paper.

4. The Lorenz attractor near the Hopf bifurcation

We shall consider an example related to the Lorenz attractor. This attractor has been studied for a long time by many authors since E. Lorenz introduced his famous equations [22], but only recently has its existence been rigorously proved by W. Tucker [42,43]. We recommend the book by C. Sparrow [41] and the expository paper [45] by M. Viana for information about this subject. The results of C. Morales, M.J. Pacifico and E. Pujals [25] provide a unified framework for robust strange attractors in dimension 3 of which the Lorenz attractor is a particular case. See also the paper [9] by L. Díaz, E. Pujals and R. Ures for related results about discrete-time systems. The topological classification of the Lorenz attractors (for different parameter values) can be found in the paper [29] by D. Rand. More general results about the classification of Lorenz maps are due to J.H. Hubbard and C.T. Sparrow [19].

The Lorenz equations provide an example of a Hopf bifurcation which takes place at parameter values very close to those which correspond to the creation of the Lorenz attractor. The equations are

$$\begin{aligned} dx/dt &= \sigma(y-x),\\ dy/dt &= rx - y - xz,\\ dz/dt &= xy - bz, \end{aligned}$$

where σ, r and b are three real positive parameters. As we vary the parameters, we change the behaviour of the flow determined by the equations in $\mathbb{R}^3$. The values $\sigma = 10$ and $b = 8/3$ have deserved special attention in the literature. We shall fix them from now on, and we shall consider the family of flows obtained when we vary the remaining parameter, r. In the sequel we follow Sparrow [41] for the presentation of all the aspects concerning the basic properties of the Lorenz equations. Sparrow's book was written long before Tucker's work was available and some of the global statements made in it are only tentative. However, except for a few details, they have proved to agree with Tucker's results.

The origin is a stationary point for all the parameter values. If $0 < r < 1$, it is a global attractor. At $r = 1$ there is a bifurcation of a simple kind, and for $r > 1$ the origin is non-stable and there are two other stationary points,

$$C_1 = \bigl(-\sqrt{b(r-1)}, -\sqrt{b(r-1)}, (r-1)\bigr)$$

and

$$C_2 = \bigl(+\sqrt{b(r-1)}, +\sqrt{b(r-1)}, (r-1)\bigr)$$

both of them attractors in the parameter range $1 < r < \frac{470}{19} \approx 24.74$. When r is slightly larger than one, the unstable manifold of the origin is a one-dimensional manifold composed of the origin and two trajectories α_1 and α_2 spiraling towards C_1 and C_2, respectively. For a larger value of r, approximately equal to $13.926\ldots$, the behaviour of the flow changes in an important way: the trajectories started on the unstable manifold of the origin will also lie in the stable manifold of the origin producing two homoclinic orbits. For values of r larger than the critical value $r_0 = 13.926\ldots$ the trajectories are again attracted by the stationary points but α_1 is now spiraling towards C_2 and α_2 is spiraling towards C_1. We say that a *homoclinic explosion* has taken place at this critical value of the parameter. As a consequence, a "strange invariant set" has been created. This set consists of a countable infinity of periodic orbits, an uncountable infinity of aperiodic orbits, and an uncountable infinity of trajectories which terminate in the origin. For values of r close to r_0 the strange invariant set is non-stable: trajectories of many points close to it escape, spiraling towards C_1 or C_2. However, at the critical r-value $r_A \approx 24.06$ this set becomes attracting. The resulting attractor, called the Lorenz attractor, coexists with the two attracting points C_1 and C_2 until the r-value $r_H \approx 24.74$, when a Hopf bifurcation takes place and C_1 and C_2 lose their stability. This bifurcation is subcritical, i.e. C_1 and C_2 lose their stability by absorbing a non-stable periodic orbit.

The numerically computed solutions to the Lorenz equations projected onto the xz plane give a visual image of the attractor with its characteristic butterfly aspect. In fact, the stable manifold of the origin divides the phase space into points that first go to one wing of the butterfly and those that first go to the other wing when approaching the attractor. See [26] for very suggestive computer images.

4.1. The global attractor E_∞

Lorenz [22] proved that for every value of r there is an ellipsoid E in $\mathbb{R}^3$ which all trajectories eventually enter. Sparrow [41] describes the situation in this way:

"At times $1, 2, 3, \ldots$ the surface of the ellipsoid E is taken by the flow into surfaces $S_1, S_2, S_3, \ldots$ which enclose regions $E_1, E_2, E_3, \ldots$ such that the volumes of E_i decrease exponentially to zero as i increases. Because all trajectories cross the boundary inwards we know that

$$E \supset E_1 \supset E_2 \supset \cdots \supset E_i \supset \cdots$$

and hence every trajectory is ultimately trapped in a region, E_∞, of zero volume given by

$$E_\infty = \bigcap_{i \in Z^+} E_i.\text{"}$$

E_∞ is therefore a global attractor.

By using elementary notions of shape theory we infer from this that the flow $\varphi : \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}^3$ induces, in a natural way, maps

$$\begin{aligned} r_k : E &\to E_k, \\ x &\to \varphi(x, k) \end{aligned}$$

that define an *approximative sequence*

$$\mathbf{r} = \{r_k,\ E \to E_\infty\}$$

in the sense of K. Borsuk [4]. This means that for every neighborhood V of E_∞ in $\mathbb{R}^3$ there is k_0 such that $r_k \simeq r_{k+1}$ in V for $k \geqslant k_0$. The approximative sequence $\mathbf{r}$ induces a shape isomorphism whose inverse is induced by the inclusion $i : E_\infty \to E$. *This proves that E_∞ has trivial shape.* This is also a consequence of the fact that the shape of a global attractor agrees with that of the phase space [3,13,20,36]. *We remark that, at least for some values of the parameter r, E_∞*

is not homotopically trivial since there are trajectories spiraling into C_1 (as well as trajectories spiraling into C_2) which lie in a path-component of E_∞ not containing C_1 (respectively C_2).

For r-values $r < r_H$ close to the Hopf bifurcation, the non-wandering set of the flow, Ω, is the union of the Lorenz attractor L, the stationary points C_1 and C_2 and two periodic orbits γ_1 and γ_2 which are responsible for the Hopf bifurcation at the critical value r_H. The non-wandering set defines in a natural way a Morse decomposition of the global attractor E_∞, and we are interested in studying this decomposition. Before that we study the global topological structure of the Lorenz attractor.

4.2. The shape of the Lorenz attractor

In this section we calculate the Čech homotopy type (or shape) of the Lorenz attractor. For this we need to know the evolution of the flow inside the ellipsoid E. At $r = r_H$ the periodic orbits γ_1 and γ_2 are absorbed by the stationary points C_1 and C_2 and for $r \geqslant r_H$ the points C_1 and C_2 lose their stability. The non-wandering set becomes simpler. In fact, Ω is

$$\Omega = L \cup \{C_1\} \cup \{C_2\}.$$

For r-values $r \geqslant r_H$ near the Hopf bifurcation the flow defines a semi-dynamical system in the ellipsoid E whose trajectories are all attracted by L except those which compose the stable manifolds of C_1 and C_2. These are one-dimensional manifolds whose intersection with E consists of closed arcs, l_1 and l_2, respectively, with their ends in the boundary of E and such that $l_1 \cap l_2 = \emptyset$. In other words, the Lorenz attractor L is an attractor of a semi-dynamical system in E whose basin of attraction is $E - (l_1 \cup l_2)$. We shall now make use of the following result of Kapitanski and Rodnianski [20].

Theorem 5. *Let $\mathcal{U}$ be a global attractor of a semi-dynamical system defined in the metric space X. Then the inclusion map $i : \mathcal{U} \to X$ is a shape equivalence.*

The notion of attractor used by Kapitanski and Rodnianski requires attraction of all bounded sets. However, there are versions of this result that only require attraction of compact sets when, for instance, X is a metric ANR [11]. If we apply this result to the semi-dynamical system induced by the Lorenz flow in $X = E - (l_1 \cup l_2)$ we deduce that the inclusion $i : L \to E - (l_1 \cup l_2)$ is a shape equivalence and, therefore, the shape of the Lorenz attractor is that of a disc with two holes or, equivalently, that of a wedge of two circles. We have only considered r-values $r \geqslant r_H$, hence our conclusion is limited, for the moment, to those r-values. We now apply the following result that we have proved in [37].

Theorem 6. *Let $\varphi_\lambda : X \times \mathbb{R} \to X$, $\lambda \in I$, be a parametrized family of flows defined on a locally compact ANR, X. If K is an attractor of φ_0 then for every neighborhood V of K contained in the basin of attraction of K there exists λ_0, with $0 < \lambda_0 \leqslant 1$, such that for every $\lambda \leqslant \lambda_0$ there exists an attractor $K_\lambda \subset V$ of the flow φ_λ with $\mathrm{Sh}(K_\lambda) = \mathrm{Sh}(K)$. Moreover V is contained in the basin of attraction of K_λ.*

It follows from Theorem 5 that the shape of attractors is preserved by continuation and, hence, the shape of the Lorenz attractor for r-values $r < r_H$ is the same as the one at r_H. Moreover, the cohomology Conley index of an attractor is also determined by its shape. In conclusion we have the following result.

Theorem 7. *The Lorenz attractor, L, has the shape of $S^1 \vee S^1$ (a wedge of two circles) for r-values close to r_H (the critical value of the Hopf bifurcation). As a consequence, the cohomology Conley indexes of L are $CH^0(L) \cong \mathbb{Z}$, $CH^1(L) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $CH^q(L) \cong 0$ for $q > 1$.*

4.3. The Morse decomposition of E_∞

If $(M_1, M_2, \ldots, M_n)$ is a Morse decomposition of an isolated invariant set K of a flow $\varphi : X \times \mathbb{R} \to X$, we can define the associated sequence of attractors (for the flow restricted to K)

$$A_1 = M_1 \subset A_2 \subset \cdots \subset A_n = K$$

where

$$A_j = \left\{x \in K \;\middle|\; \exists i \leqslant j \text{ with } \omega^*(x) \subset M_i\right\}.$$

Conley and Zehnder [8] proved that there is a filtration $N_0 \subset N_1 \subset \cdots \subset N_n$ of compact sets in X such that (N_j, N_{j-1}) is an index pair for M_j and (N_j, N_0) is an index pair for A_j. They introduced the notation

$$p(t, N_j, N_{j-1}) = \sum_{k \geqslant 0} r^k(N_j, N_{j-1}) t^k$$

and

$$q(t, N_j, N_{j-1}, N_0) = \sum_{k \geqslant 0} d^k(N_j, N_{j-1}, N_0) t^k$$

where

$$r^k(N_j, N_{j-1}) = \text{rank of } H^k(N_j, N_{j-1}) \quad \text{(Čech cohomology)}$$

and

$$\begin{aligned} d^k(N_j, N_{j-1}, N_0) = {} & \text{rank of the image of the coboundary operator} \\ & \delta^k : H^k(N_{j-1}, N_0) \to H^{k+1}(N_j, N_{j-1}) \text{ in the long} \\ & \text{cohomology sequence of the triple } (N_j, N_{j-1}, N_0). \end{aligned}$$

All the cohomology groups are assumed to be of finite rank in the former expressions. These groups are the cohomology Conley indexes of the Morse sets M_j and the attractors A_j.

In these conditions, Conley and Zehnder proved that the following equation holds

$$\sum_{j=1}^{n} p(t, N_j, N_{j-1}) = p(t, N_n, N_0) + (1 + t) Q(t)$$

where

$$Q(t)=\sum_{j=2}^{n} q(t, N_j, N_{j-1}, N_0).$$

This is the Morse equation of the decomposition $(M_1, M_2, \ldots, M_n)$.

The non-wandering set Ω of the flow defined in $\mathbb{R}^3$ by the Lorenz equations induces the following Morse decomposition of the global attractor E_∞ for values of the parameter $r < r_H$ close to the Hopf bifurcation

$$M_1 = L \quad \text{(the Lorenz attractor)}, \qquad M_2 = \{C_1\} \cup \{C_2\}, \qquad M_3 = \gamma_1 \cup \gamma_2. \tag{1}$$

We are interested in studying the shape indexes of the Morse sets [30,38] and the Morse equation of this decomposition. The finer decomposition

$$M'_1 = L, \qquad M'_2 = \{C_1\}, \qquad M'_3 = \{C_2\}, \qquad M'_4 = \gamma_1, \qquad M_{5'} = \gamma_2 \tag{2}$$

is only slightly different and its Morse equation is the same.

Concerning the Morse decomposition (1), we can choose N_3 to be a positively invariant compact neighborhood of E_∞ in $\mathbb{R}^3$ and N_2 a compact subset in the interior of N_3 consisting of three connected components N_2^1, N_2^2 and N_2^3 which are positively invariant neighborhoods of the attractors C_1, C_2 and L, respectively, and contained in their corresponding basins of attraction. We choose $N_1 = N_2^3$ and $N_0 = \emptyset$. We can assume that N_3, N_2^1 and N_2^2 are topological 3-cells but we do not know all the details about the topological structure of N_1. However, this is not a problem since we can use other methods to calculate the indexes and the Morse equation.

From Theorem 6 we have that the ranks of the cohomology indices of the Lorenz attractor are

$$\begin{gathered}
r^0(N_1, N_0) = \operatorname{rank} H^0(N_1, N_0) = \operatorname{rank} H^0\big(S^1 \vee S^1 \cup \{*\}, \{*\}\big) = 1, \\
r^1(N_1, N_0) = \operatorname{rank} H^1(N_1, N_0) = \operatorname{rank} H^1\big(S^1 \vee S^1 \cup \{*\}, \{*\}\big) = 2, \\
r^q(N_1, N_0) = 0 \quad \text{for } q \geqslant 2,
\end{gathered}$$

since $H^q(N_1, N_0)$ is the cohomology index of L, which agrees with the cohomology of the shape index.

The discussion for the pair (N_2, N_1) is trivial since it is an index pair for the attractor $M_2 = \{C_1\} \cup \{C_2\}$. We obviously have

$$r^0(N_2, N_1) = 2$$

and

$$r^q(N_2, N_1) = 0 \quad \text{for } q \geqslant 1.$$

The pair (N_3, N_2) corresponds to the repeller $M_3 = \gamma_1 \cup \gamma_2$. By a result in [38] the shape index can be calculated considering the unstable manifold $W^u(M_3)$ and "truncating" it. This idea was originally used by Kapitanski and Rodnianski [20] to calculate the cohomology index.

The truncated unstable manifold of the union of the periodic orbits γ_1 and γ_2 is, topologically, the disjoint union of two planar annuli

$$W^* = A_1 \cup A_2$$

and the shape index is

$$\mathrm{Sh}\big(W^*/\partial W^*, *\big),$$

where ∂W^* is the union of the four circles in the boundary of W^* and $*$ is obtained after collapsing ∂W^* to a point. We conclude that the shape index of M_3 is

$$\mathrm{Sh}\big(S^2 \vee S^2 \vee S^1 \vee S^1, *\big),$$

where $*$ is the base point of the wedge $S^2 \vee S^2 \vee S^1 \vee S^1$.

Once we know the shape index we immediately calculate the cohomology index $H^k(N_3, N_2)$ of M_3, obtaining the polynomial

$$p(t, N_3, N_2) = \sum_{k \geqslant 0} r^k(N_3, N_2) t^k.$$

It is possible, however, to calculate the coefficients $r^k(N_3, N_2)$ without making use of the mentioned result. We can apply standard methods of algebraic topology to calculate the right-hand side of the Morse equation and from that and the rest of results previously obtained we can deduce the expression of the polynomial $p(t, N_3, N_2)$. We indicate how this can be done.

We must first consider the coboundary operator

$$\delta^k : H^k(N_2, N_0) \to H^{k+1}(N_3, N_2).$$

Since (N_2, N_0) is an index pair for the attractor $A_2 = L \cup \{C_1\} \cup \{C_2\}$ we have that $H^k(N_2, N_0) \cong \{0\}$ for $k \geqslant 2$, $H^1(N_2, N_0) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $H^0(N_2, N_0) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Considering now the relevant part of the long cohomology sequence

$$\cdots \to H^1(N_3, N_0) \to H^1(N_2, N_0) \to H^2(N_3, N_2) \to \cdots,$$

since $H^1(N_3, N_0)$ (cohomology index of the global attractor) is zero we have that the rank of the image of δ^1 is 2. Moreover, from the exactness of

$$\cdots \to H^0(N_3, N_0) \cong \mathbb{Z} \to H^0(N_2, N_0) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \to H^1(N_3, N_2) \to \cdots,$$

we easily deduce that the rank of the image of δ^0 is also 2. On the other hand, we readily see that the contribution of the triple (N_2, N_1, N_0) is trivial and that $p(N_3, N_0)$ (corresponding to the global attractor) is 1.

Summing up, we conclude that the Morse equation of the decomposition (M_1, M_2, M_3) is

$$1 + 2t + 2 + \sum_{k \geqslant 0} r^k(N_3, N_2) t^k = 1 + (1 + t)(2 + 2t)$$

and, hence, $r^0(N_3, N_2) = 0$, $r^1(N_3, N_2) = 2$, $r^2(N_3, N_2) = 2$, $r^k(N_3, N_2) = 0$ for $k \geqslant 0$.

The Morse equation undergoes a change after the Hopf bifurcation takes place. For r-values $r \geqslant r_H$ the non-wandering set Ω consists of the Lorenz attractor together with the (now non-stable) stationary points. If we consider the natural decomposition of E_∞ induced by Ω

$$M_1 = L, \qquad M_2 = \{C_1\}, \qquad M_3 = \{C_2\}$$

a similar (but easier) calculation shows that the corresponding Morse equation takes the form

$$1 + 2t + 2t^2 = 1 + (1 + t)2t.$$

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