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Closed string tachyon potential and tt^* equation

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Abstract

Recently Dabholkar and Vafa proposed that closed string tachyon potential for non-supersymmetric orbifold $\mathbb{C}/\mathbb{Z}_3$ in terms of the solution of a tt^* equation. We extend this result to $\mathbb{C}^2/\mathbb{Z}_n$ for $n = 3, 4, 5$. Interestingly, the tachyon potentials for $n = 3$ and 4 are still given in terms of the solutions of Painlevé III type equation that appeared in the study of $\mathbb{C}^1/\mathbb{Z}_3$ with different boundary conditions. For $\mathbb{C}^2/\mathbb{Z}_5$ case, governing equations are of generalized Toda type. The potential is monotonically decreasing function of RG flow.

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1. Introduction

The study of localized tachyon condensation [1–5] has been considered with many interesting developments. The basic picture is that tachyon condensation induces cascade of decays of the orbifolds to less singular ones until the spacetime supersymmetry is restored. Therefore the localized tachyon condensation has a geometric description as the resolution of the spacetime singularities.

Following the line of Vafa's reformulation of the problem in terms of mirror Landau–Ginzburg theory, we worked out the detailed analysis on the fate of spectrum and the background geometry under the tachyon condensation as well as the question of what is the analogue of c-theorem with the GSO-projection in a series of papers [6–8]. In all these works, super-conformal invariance was used very heavily so that we were working at the string theory on-shell level.

On the other hand, recently, Dabholkar and Vafa [4] proposed that the tachyon potential is given by the maximal charge. Strictly speaking, the $U(1)$ charge in the question is defined only on the conformal point not on the off-shell. On the other hand, the decay process considered as a renormalization group flow is an off-shell process and

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one needs to extend the concept of charge in order to define the tachyon potential. The way to extend the charge is to consider it as a semi-index which is contributed by BPS objects only [9]. These are not just topological since it has antiholomorphic dependence as well as holomorphic dependence on the deformation parameters. The way to calculate this quantity is to show that it satisfies tt^* equations [11] and solve it if possible. In [4], the process where $\mathbb{C}/\mathbb{Z}_3$ decays to $\mathbb{C}^1$ was discussed.

In this Letter we extend the result of [4] to $\mathbb{C}^2/\mathbb{Z}_n$ for $n = 3, 4, 5$. Interestingly, the tachyon potential for $n = 3$ and 4 is still given in terms of the solutions of Painlevé III type that appeared in the study of $\mathbb{C}^1/\mathbb{Z}_3$ with different boundary conditions. For $\mathbb{C}^2/\mathbb{Z}_5$ the governing equation is that of generalized Toda type. The potential is monotonically decreasing function of RG flow, therefore expected so as function of real time as well. For LG model associated to $\mathbb{C}^2/\mathbb{Z}_5$ without orbifold projection, tt^* equation contains the Bullough–Dodd equation, whose solution is known [13].

In this Letter, we do not attempt to give all necessary background. For the set up of tachyon condensation in terms of mirror symmetry, see [2,6]. For the application of tt^* to the tachyon condensation, see [4].

2. $\mathbb{C}^2/\mathbb{Z}_3$

In this section we study the tachyon condensation in $\mathbb{C}^2/\mathbb{Z}_n$ with Landau–Ginzburg (LG) description by calculating and solving the corresponding tt^* equations. First we consider the simplest of all $\mathbb{C}^2/\mathbb{Z}_n$, namely $n = 3$. The mirror of this is a LG whose superpotential is given by

$$W = \frac{x^3}{3} + \frac{y^3}{3} - txy, \tag{2.1}$$

with an imposition of an orbifold constraint.

Before we consider the orbifolded LG theory, we work out the tt^* equations for LG theory without orbifolding for later interests. The fundamental variables are not x, y but $\log x, \log y$ [4,10] so that the chiral ring consists of

$$\{xy, x^2y, xy^2, x^2y^2\} \tag{2.2}$$

from which charges of the elements can be read off to give NS-charges

$$\left\{\frac{2}{3}, 1, 1, \frac{4}{3}\right\}. \tag{2.3}$$

The R-charges in the topological strings are related to those in NS by the spectral flow

$$q_{\mathrm{R}} = q_{\mathrm{NS}} - \frac{n}{2}. \tag{2.4}$$

For $\mathbb{C}^2/\mathbb{Z}_3$, we have $n = 2$ so we get

$$\left\{-\frac{1}{3}, 0, 0, \frac{1}{3}\right\}. \tag{2.5}$$

This superpotential has symmetries

$$x \to \omega x, \quad y \to \omega^2 y; \quad \text{and} \quad x \to \omega^2 x, \quad y \to \omega y, \tag{2.6}$$

where $\omega = e^{\frac{2}{3}\pi i}$, which constrains the metric $g_{\bar{j}i} := \langle \phi_{\bar{j}} \phi_i \rangle$ ($\phi_i s$ are chiral fields) to be of the form

$$g = \begin{pmatrix} a_{11} & 0 & 0 & \bar{b} \\ 0 & a_{21} & 0 & 0 \\ 0 & 0 & a_{12} & 0 \\ b & 0 & 0 & a_{22} \end{pmatrix}. \tag{2.7}$$

The topological metric η_{ij} is given by residue paring

$$\eta_{ij} = \langle \phi_i \phi_j \rangle = \frac{1}{(2\pi i)^n} \int_\Gamma \frac{\phi_i(X)\phi_j(X) dX^1 \wedge \cdots \wedge dX^n}{\partial_1 W \partial_2 W \cdots \partial_n W}$$

with superpotential W, which in this case is calculated to be

$$\eta = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & t^2 \end{pmatrix}. \tag{2.8}$$

From the reality constraint

$$\eta^{-1} g \left(\eta^{-1} g\right)^* = I, \tag{2.9}$$

we have

$$b = \frac{t^2}{2} a_{11}, \qquad a_{12} = \frac{1}{a_{21}}, \qquad a_{22} = \frac{1}{a_{11}} + \frac{|t|^2}{4} a_{11}. \tag{2.10}$$

The chiral ring coefficient are defined by $\phi_i \phi_j = C_{ij}^k \phi_k$ and if we denote the matrix for the multiplication by xy by C_t, then

$$C_t = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & t^2 & 0 & 0 \\ 0 & 0 & t^2 & 0 \\ 0 & 0 & 0 & t^2 \end{pmatrix}. \tag{2.11}$$

Putting all these into tt^* equation

$$\partial_{\bar{t}} \left(g \partial_t g^{-1}\right) = \left[C_t, g (C_t)^\dagger g^{-1}\right], \tag{2.12}$$

we get

$$\partial_{\bar{t}} \partial_t \log a_{11} = -\frac{1}{a_{11}^2} + \frac{|t|^8}{16} a_{11}^2, \qquad \partial_{\bar{t}} \partial_t \log a_{12} = 0. \tag{2.13}$$

The exchange symmetry $x \rightleftarrows y$ due to the special form of perturbation, together with the reality condition (2.10), determines

$$a_{12} = a_{21} = 1, \tag{2.14}$$

which is consistent with the second equation of Eq. (2.13). By changes of variables

$$y = a_{11}^{-1}, \qquad \zeta = \frac{1}{3} t^3 \quad \text{and} \quad y = \frac{1}{2} |t|^2 Y, \tag{2.15}$$

the first equation of Eq. (2.13) can be transformed into

$$\partial_{\bar{\zeta}} \partial_\zeta \log Y = \frac{1}{4}\left(Y^2 - Y^{-2}\right). \tag{2.16}$$

In terms of Y and $z = |\zeta|$ the last equation can be written as

$$Y'' = \frac{(Y')^2}{Y} - \frac{Y'}{z} + Y^3 - \frac{1}{Y}, \tag{2.17}$$

which is already known as Painlevé III equation. We can further rewrite Eq. (2.16) as the sinh-Gordon equation

$$\partial_{\bar{\zeta}}\partial_{\zeta} u = \sinh u, \tag{2.18}$$

by introducing u by $u = 2\log y$. Now, since we are interested in the scaling behavior of the solutions, we look at the dependence on $z = |\zeta|$ of Eq. (2.18),

$$u'' + u'/z = 4\sinh u, \tag{2.19}$$

where the prime denote the derivative in z. The solution to this is well known [12]. The real solutions are classified by their asymptotic behavior in $z \to 0$

$$u(z) \simeq r\log z + O\big(z^{2-|r|}\big) \quad \text{for } |r| < 2. \tag{2.20}$$

In our case, the regularity of a_{11} requires

$$u(z) \sim -\frac{4}{3}\log z. \tag{2.21}$$

Therefore this fixes $r = -4/3$. For $z \to \infty$

$$u(z) \sim \sqrt{\frac{3}{\pi}}\frac{e^{-2z}}{\sqrt{z}}, \quad z \to \infty. \tag{2.22}$$

The precise form of regular solution can be written in terms of a convergent expansion

$$u(z;r) = -4\sum_{n=0}^{\infty}\frac{[2\cos((2-r)\pi/4)]^{2n+1}}{2n+1}\int_{-\infty}^{\infty}\prod_{i=1}^{2n+1}\frac{d\theta_i}{4\pi}\frac{e^{-2z\cosh\theta_i}}{\cosh[(\theta_i-\theta_{i+1})/2]} \tag{2.23}$$

with $\theta_{2n+1} \equiv \theta_1$ and $r = -4/3$.

In order to define the charge matrix, we need to look at the scaling behavior of the system. The scaling $z \to \lambda z$ induces $\int d^2z\, d^2\theta \to \lambda \int d^2z\, d^2\theta$, which is equivalent to the field redefinition and coupling change such that $W \to \lambda W$. For the given superpotential

$$W = \frac{x^3}{3} + \frac{y^3}{3} - txy, \tag{2.24}$$

we can identify the necessary field redefinitions and coupling change as

$$x = \lambda^{1/3}\tilde{x}, \qquad y = \lambda^{1/3}\tilde{y} \quad \text{and} \quad t = \lambda^{1/3}t_0, \tag{2.25}$$

by which W can be written as

$$W = \lambda\left(\frac{\tilde{x}^3}{3} + \frac{\tilde{y}^3}{3} - t_0\tilde{x}\tilde{y}\right). \tag{2.26}$$

In this way changing t is equivalent to the changing the scale. The metric components in old and new basis are related by

$$a_{ij} = \langle x^{\bar{i}} y^{\bar{j}} | x^i y^j \rangle = |\lambda|^{\frac{(i+j)}{n}} \langle \tilde{x}^{\bar{i}} \tilde{y}^{\bar{j}} | \tilde{x}^i \tilde{y}^j \rangle := |\lambda|^{\frac{(i+j)}{n}} b_{ij} \tag{2.27}$$

with $n = 3$. The off-shell ‘charge’ of the system was defined [11] as

$$Q = g\partial_\tau g^{-1} - \frac{\hat{c}}{2}, \tag{2.28}$$

where $\tau = \log\lambda$ and the metric components in use are in $\tilde{x}, \tilde{y}$ basis, namely b_{ij}’s. From the regularity condition of the metric in x, y basis at $t = 0$, one can verify that the charge at the starting point $t = \lambda = 0$ is encoded correctly to give the result listed in Eq. (2.5).

Orbifolded LG model: $\{W = x^3 + y^3 - txy\}//\mathbb{Z}_3$

In this case, the chiral ring generated by xy: $\{xy, x^2y^2\}$ and NS-charges are $\{\frac{2}{3}, \frac{4}{3}\}$. The metric and charges can be obtained from the previous subsection by simply discarding a_{12} and a_{21}. Following [4], the tachyon potential is proposed to be

$$V(t,\bar{t}) = 2\max\left|Q(t,\bar{t})\right| = -\frac{1}{2}z\frac{du}{dz}, \tag{2.29}$$

where we identified $\zeta = \lambda$ by Eqs. (2.15), (2.25) after setting $t_0 = 3^{1/3}$. The factor 2 in the above equation is from the asymptotic form of u is given by Eqs. (2.22), (2.21). Its exact form can be written in terms of u given in Eq. (2.23) with $r = -4/3$. The potential vanishes exponentially as $t \to \infty$. As a consequence of the tachyon condensation, the fate of the $\mathbb{C}^2/\mathbb{Z}_3$ is just $\mathbb{C}^2$ as expected.

3. $\mathbb{C}^2/\mathbb{Z}_4$

Let us now consider $n = 4$. Working in the basis with definite ordering the chiral ring consists of

$$\left\{xy, x^2y, x^3y, xy^2, x^2y^2, x^3y^2, xy^3, x^2y^3, x^3y^3\right\}. \tag{3.1}$$

The topological metric and metric are given by

$$\eta_{x^{i_1}y^{j_1},x^{i_2}y^{j_2}} = 1, \quad \text{if } i_1 + i_2 = j_1 + j_2 = 4, \qquad \eta_{x^3y^3,x^3y^3} = t \tag{3.2}$$

and

$$g_{i\bar{j}} = \begin{pmatrix} a_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \bar{b} \\ 0 & a_{21} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_{31} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{12} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{22} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & a_{32} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{13} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{23} & 0 \\ b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & a_{33} \end{pmatrix}. \tag{3.3}$$

Then, the reality condition gives

$$\begin{aligned} &a_{22} = 1, \qquad a_{32} = 1/a_{12}, \qquad a_{13} = 1/a_{31}, \qquad a_{23} = 1/a_{21}, \qquad a_{33} = 1/a_{11} + |t|^4 a_{11}/4, \\ &b = t^2 a_{11}/4. \end{aligned} \tag{3.4}$$

C_t can also be calculated easily and given by

$$C_t = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & t & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & t^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & t^2 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{3.5}$$

Then the tt^* equation becomes

$$-\partial_{\bar{t}}\partial_t \log a_{11} = 1/a_{11} - |t|^4 a_{11}/4, \tag{3.6}$$

$$-\partial_{\bar{t}}\partial_t \log a_{21} = 1/(a_{21}a_{12}) - |t|^4 a_{21}a_{12}, \tag{3.7}$$

$$-\partial_{\bar{t}}\partial_t \log a_{31} = |t|^2/a_{31}^2 - |t|^2 a_{31}^2, \tag{3.8}$$

$$-\partial_{\bar{t}}\partial_t \log a_{12} = 1/(a_{21}a_{12}) - |t|^4 a_{21}a_{12}. \tag{3.9}$$

Most of the equations are of the form

$$\partial_{\bar{t}}\partial_t \log y = y^2 - \frac{|t|^{2k}}{m^2} y^{-2}. \tag{3.10}$$

By introducing change of variables

$$\zeta = \bigl(1/(1+k/2)\bigr)\bigl(16/m^2\bigr)^{1/(2k+4)} t^{1+k/2}, \qquad y = \sqrt{1/m}|t|^{k/2} e^{u/2}, \tag{3.11}$$

above equation lead us to the sinh-Gordon equation (2.18). The value of the r in the solution of sinh-Gordon equation can be obtained from the regularity of the metric component near $t = 0$. Since $y \sim z^{k/k+2} e^{u/2}$, we have

$$u \sim -\frac{2k}{k+2} \log z + \cdots, \tag{3.12}$$

which determines the value $r = -\frac{2k}{k+2}$.

Now from Eq. (2.28) the components of charge matrix is given by

$$q_{ij} = b_{ij}\partial_\tau b_{ij}^{-1} - 1. \tag{3.13}$$

Using $a_{ij} = |\lambda|^{(i+j)/2} b_{ij}$, $a_{ij}^{-1} \sim y^l$ for some l and $y \sim z^{k/(k+2)} e^{u/2}$ with identification $z = |\lambda|$,

$$q_{ij}(z) = \frac{l}{4} z \frac{du(z)}{dz} + \frac{lk}{2(k+2)} + \frac{i+j}{n} - 1. \tag{3.14}$$

Notice that (ij) is not the matrix index but the vector index. Using Eq. (3.12), the value of the charge at $t = 0$ is

$$q_{ij}(0) = \frac{i+j}{n} - 1, \tag{3.15}$$

which confirms that we are in the right track. Now we apply this result to our system.

For a_{11}, by change of variables $a_{11}^{-1} = 2y^2$ Eq. (3.6) reduces to the standard form Eq. (3.10) with $k = 2, m = 4$. Therefore $r = -1$. This equation further can be reduced to sinh-Gordon equation $\partial_{\bar{\zeta}}\partial_\zeta u = \sinh u$ by setting $\zeta = t^2/2$ and $y = |t|e^{u/2}/2$. Since $l = 2$ in this case, charge is given by $q_{11}(z) = \frac{1}{2} z \frac{du(z)}{dz}$.

For a_{12} and a_{21}, first we show they are equal. From Eqs. (3.7) and (3.9) $a_{12} = |F(t)|^2 a_{21}$ for some holomorphic function $F(t)$. Since $a_{12} = a_{21}$ at $t = 0$ as well as at $t = \infty$, the only nonsingular holomorphic function with such boundary conditions is a constant function $F(t) = 1$, i.e., $a_{12} = a_{21}$. This supports the exchange symmetry $a_{ij} = a_{ji}$. With this, Eqs. (3.7), (3.9) are the case of $m = 1, k = 2$; by setting $a_{12}^{-1} = y = |t|e^{u/2}$ and $\zeta = t^2/2$, we get sinh-Gordon equation. $l = 1$ lead us to $q_{12}(z) = q_{21}(z) = \frac{1}{4} z \frac{du(z)}{dz}$.

For a_{31}, by $z = t^2$ and $y = a_3^{-1}$ we get sinh-Gordon and the solution is $y = e^{u/2}$. It is easy to see $q_{31}(z) = \frac{1}{4} z \frac{du(z)}{dz}$. Notice that $q_{31}(0) = q_{13}(0) = 0$. The monotonicity of the charge in t suggests that $q_{31}(z) = 0$. In fact the exchange symmetry $a_{31} = a_{31}$ and the reality condition $a_{31} = 1/a_{31}$ sets $a_{31} = 1$.

Mirror of $\mathbb{C}^2/\mathbb{Z}_4$

So far, we have been considering the LG model without orbifolding action. To consider the mirror of $\mathbb{C}^2/\mathbb{Z}_4$ with generator xy, we need to consider a_{ii} $i = 1, 2, 3$. Since $a_{22} = 1$ and a_{33} is given by a_{11}, we only need to consider the equation for a_{11}, which is given by Eq. (3.6).

4. $\mathbb{C}^2/\mathbb{Z}_5$

Here again, we first analyze the general LG model and at the end we comments on the orbifolded case. The superpotential is given by

$$W = \frac{x^5}{5} + \frac{y^5}{5} - txy. \tag{4.1}$$

The chiral ring is given by $x^4 - ty = 0$ and $y^4 - tx = 0$. We order the basis by dictionary order in charge pair (i, j):

$$\mathcal{R} = \{xy, x^2y, x^3y, x^4y, xy^2, x^2y^2, x^3y^2, x^4y^2, xy^3, x^2y^3, x^3y^3, x^4y^3, xy^4, x^2y^4, x^3y^4, x^4y^4\}. \tag{4.2}$$

The η_{ij} can be readily written and we avoid to writing it down. The metric $g_{i\bar{j}}$ has 16 diagonal components which we denote by $a_{ij} = \langle x^{\bar{i}} y^{\bar{j}} | x^i y^j \rangle$, $i, j = 1, 2, 3, 4$, and two nonvanishing off diagonal elements $b, \bar{b}$ as before. The coupling matrix C_t in this basis is given by

$$C_t = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & t & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & t & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & t^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}. \tag{4.3}$$

By use of the reality condition, we have 8 independent variables out of $16 + 2$ real variables and the rests can be written in terms of them:

$$\begin{aligned}
&a_{13} = 1/a_{42}, \qquad a_{23} = 1/a_{32}, \qquad a_{33} = 1/a_{22}, \qquad a_{43} = 1/a_{12}, \qquad a_{14} = 1/a_{41}, \\
&a_{24} = 1/a_{31}, \qquad a_{34} = 1/a_{21}, \qquad a_{44} = 1/a_{11} + |t|^4 a_{11}/4, \\
&b = \frac{1}{2} t^2 a_{11}.
\end{aligned} \tag{4.4}$$

Then the tt^* equation becomes

$$-\partial_{\bar{t}} \partial_t \log a_{11} = a_{11}^{-1} a_{22} - \frac{1}{4} |t|^4 a_{11} a_{22},$$

$$
\begin{aligned}
&-\partial_{\bar t}\partial_t \log a_{22} = a_{22}^{-2} - a_{11}^{-1}a_{22} - \frac{1}{4}|t|^4 a_{11}a_{22},\\
&-\partial_{\bar t}\partial_t \log a_{21} = -|t|^4 a_{21}a_{12} + a_{21}^{-1}a_{32},\\
&-\partial_{\bar t}\partial_t \log a_{12} = -|t|^4 a_{21}a_{12} + a_{12}^{-1}a_{32}^{-1},\\
&-\partial_{\bar t}\partial_t \log a_{32} = a_{12}^{-1}a_{32}^{-1} - a_{21}^{-1}a_{32},\\
&-\partial_{\bar t}\partial_t \log a_{31} = -|t|^2 a_{31}a_{41} + a_{31}^{-1}a_{42},\\
&-\partial_{\bar t}\partial_t \log a_{41} = -|t|^2 a_{31}a_{41} + |t|^2 a_{41}^{-1}a_{42}^{-1},\\
&-\partial_{\bar t}\partial_t \log a_{42} = |t|^2 a_{41}^{-1}a_{42}^{-1} - a_{31}^{-1}a_{42}.
\end{aligned}
\tag{4.5}
$$

The reality condition together with exchange symmetry gives us only 4 independent equations

$$-\partial_{\bar t}\partial_t \log a_{11} = a_{22}/a_{11} - a_{11}a_{22}|t|^4/4, \tag{4.6}$$

$$-\partial_{\bar t}\partial_t \log a_{22} = a_{22}^{-2} - a_{22}/a_{11} - |t|^4 a_{11}a_{22}/4, \tag{4.7}$$

$$-\partial_{\bar t}\partial_t \log a_{21} = -|t|^4 a_{21}^2 + a_{21}^{-1}, \tag{4.8}$$

$$-\partial_{\bar t}\partial_t \log a_{31} = -|t|^2 a_{31} + a_{31}^{-2}, \tag{4.9}$$

as well as the predetermined values of some of them

$$a_{32} = a_{23} = 1, \qquad a_{41} = a_{14} = 1. \tag{4.10}$$

Notice that all monomials x^2y, xy^2, x^3, y^3 have NS-charge 1 and these are the marginal operators. Above results show that the marginal operators do not evolve under the condensation of tachyon represented by xy.

To eliminate t from above equations, let

$$a_{ij} := |t|^{c_{ij}} b_{ij}, \qquad \zeta := at^b. \tag{4.11}$$

Then by using $\partial_{\bar t}\partial_t = (ab)^2|t|^{2b-2}\partial_{\bar\zeta}\partial_\zeta$, and by requiring that Eqs. (4.6), (4.7) are homogeneous in t,

$$2b - 2 = c_{22} - c_{11} = 4 + c_{22} + c_{11} = -2c_{22} = -c_{11} + c_{22}, \tag{4.12}$$

which give $c_{22} = -2/3$, $c_{11} = -2$ and $b = 5/3$ with $ab = 1$. Based on this, we introduce q_{11}, q_{22} by

$$a_{11} = 2(3/5)^2|\zeta|^{-6/5}e^{-q_{11}}, \qquad a_{22} = 2^{1/3}(3/5)^{2/3}|\zeta|^{-2/5}e^{-q_{22}}, \tag{4.13}$$

and rescale by $\zeta \to 2^{1/3}\zeta$ to get

$$
\begin{aligned}
&\partial_{\bar\zeta}\partial_\zeta q_{11} = e^{q_{11}-q_{22}} - e^{-(q_{11}+q_{22})},\\
&\partial_{\bar\zeta}\partial_\zeta q_{22} = e^{2q_{22}} - e^{q_{11}-q_{22}} - e^{-(q_{11}+q_{22})}.
\end{aligned}
\tag{4.14}
$$

For Eqs. (4.8) and (4.9),

$$4 + 2c_{21} = -c_{21} = 2b - 2 = 2 + c_{31} = -2c_{31}. \tag{4.15}$$

Then we have $c_{31} = -2/3$, $c_{21} = -4/3$ and $b = 5/3$. We introduce τ and $Y(\tau)$, $Z(\tau)$ by

$$\tau = |\zeta|^2, \qquad a_{21} = (5/3)^{-4/5}\tau^{-2/5}e^{-Y} \quad \text{and} \quad a_{31} = (5/3)^{-2/5}\tau^{-1/5}e^{Z}. \tag{4.16}$$

Then, both Eqs. (4.8), (4.9) are reduced to

$$\partial_\tau(\tau\partial_\tau Y) = e^Y - e^{-2Y}, \tag{4.17}$$

$$\partial_\tau(\tau\partial_\tau Z) = e^Z - e^{-2Z}, \tag{4.18}$$

which are known as Bullough–Dodd equation which is a degenerate Painlevé III, which also appears in the case $\mathbb{C}^1/\mathbb{Z}_4 \to \mathbb{C}^1$ transition with $W = x^4 - tx$.

The Bullough–Dodd equation

$$\partial_\tau(\tau\partial_\tau u) = e^u - e^{-2u} \tag{4.19}$$

has been studied extensively and the properties of the asymptotically regular solutions were given in [13]. The solutions are parametrized by four complex numbers g_1, g_2, g_3, and s satisfying

$$g_1 + g_2(1-s) + g_3 = 1, \qquad g_2^2 - g_1 g_3 = g_2. \tag{4.20}$$

The asymptotic forms are given by

$$\begin{aligned}
&\tau \to \infty: \quad e^u \sim 1 + \sqrt{\frac{3}{\pi}}\frac{s}{2}(3\tau)^{-\frac{1}{4}} e^{-2\sqrt{3\tau}}, \qquad g_1 = g_2 = 0, \qquad g_3 = 1, \\
&\tau \to 0: \quad e^u = -\frac{\mu^2}{2\tau\sin^2\{\frac{i}{2}[\mu\ln\tau + \ln(r_1\frac{C_2}{C_0})]\}} \sim 2\mu^2 r_1 \frac{C_2}{C_0}\tau^{\mu-1}, \\
&s = 1 + \cos\left[\frac{2\pi\mu}{3}\right], \qquad r_1 = g_3 - g_1 + (1+\omega)(g_1 - g_2), \qquad \omega^{\mp} = e^{\mp\frac{2\pi i\mu}{3}}, \\
&\frac{C_2}{C_0} = 3^{-2\mu}\frac{\Gamma(1-\frac{\mu}{3})\Gamma(1-\frac{2\mu}{3})}{\Gamma(1+\frac{\mu}{3})\Gamma(1+\frac{2\mu}{3})}.
\end{aligned} \tag{4.21}$$

Regularity of the metric as $\tau \to 0$ can fix s.

First let us consider $a_{21} \sim \tau^{-2/5}e^{-y}$. From the regularity of a_{21}, we have

$$e^u = e^y \sim \tau^{-2/5}, \tag{4.22}$$

which gives

$$\mu = 3/5, \qquad s = 1 + 2\cos(2\pi/5) \simeq 1.618, \qquad r_1 = 1, \qquad \frac{C_2}{C_0} = 3^{-6/5}(25/2)\frac{\Gamma(4/5)\Gamma(3/5)}{\Gamma(1/5)\Gamma(2/5)}. \tag{4.23}$$

Similarly, for $a_{31} \sim \tau^{-1/5}e^{Z}$, we have

$$e^u = e^{-Z} \sim \tau^{-1/5}, \tag{4.24}$$

from which we get

$$\begin{aligned}
&\mu = 4/5, \qquad s = 1 + \cos(8\pi/15) \simeq 0.791, \qquad r_1 = 1, \\
&\frac{C_2}{C_0} = 3^{-8/5}\big((15)^2/32\big)\frac{\Gamma(11/15)\Gamma(7/15)}{\Gamma(4/5)\Gamma(8/15)}.
\end{aligned} \tag{4.25}$$

These completely fixes behaviors of solutions at both ends.

Mirror of $\mathbb{C}^2/\mathbb{Z}_5$

In this case, we only need to consider a_{11}, a_{22} since a_{33} and a_{44} are determined in terms of a_{22} and a_{11} respectively by the reality condition. The tt^* equations for a_{11}, a_{22} are given by Eqs. (4.5) or Eqs. (4.14). They can be identified as the $\tilde{B}_2 = D^T(SO(5))$ Toda system. We will investigate the relation between the tt^* equations in the orbifold geometry and various Toda systems elsewhere. The charge matrix $Q = g\partial_\tau g^{-1} - 1$ with $\tau = \log\lambda$ can be calculated to be given by

$$Q = \begin{pmatrix} -\frac{3}{5} + a_{11}\partial_\tau a_{11}^{-1} & 0 & 0 & 0 \\ 0 & -\frac{1}{5} + a_{22}\partial_\tau a_{22}^{-1} & 0 & 0 \\ 0 & 0 & \frac{1}{5} - a_{22}\partial_\tau a_{22}^{-1} & 0 \\ t^2 a_{11}\partial_\tau a_{11}^{-1} & 0 & 0 & \frac{3}{5} - a_{11}\partial_\tau a_{11}^{-1} \end{pmatrix}. \tag{4.26}$$

Notice that in terms of q_{ij} and λ $(= \zeta)$, and if we look at the $|\lambda|$ dependence only, the tachyon potential can be identified as

$$V = 2Q_{\max} = -\zeta \partial_\zeta q_{11}(\zeta). \tag{4.27}$$

We expect that this is monotonically decreasing from the value $5/3$ at $t = 0$ to 0 at $t \to \infty$. So far, the mathematical literature on the solution to the equation is not available and the qualitative behavior we suggested above is from physical intuition that in the final stage of tachyon condensation there is no nontrivial chiral primaries with charge other than 0.

5. Discussion

In this Letter, we calculated tt^* equations for $\mathbb{C}^2/\mathbb{Z}_n$ with $n = 3, 4, 5$. In $n = 3, 4$ cases, they reduce to a Painlevé III equation with different boundary conditions. In $n = 5$ case, they reduce to a simple Toda system whose explicit solutions are not known yet. Nonorbifolded LG model associated to $n = 5$ case involves a Bullough–Dodd equation.

As a limitation of this Letter, we mention that we considered the string theories without GSO projection only. Since GSO projection does not provide a supersymmetry immediately in the orbifold background, there is not much point on restricting ourselves to GSO projected theory. According to the rule given in [8], xy term considered in this Letter is projected out for type II, and we need to consider the deformation by higher operator, which result in highly nontrivial equations due to the algebraic complexity of reality condition. Another very interesting case is the one where the daughter theory is also an orbifold. This also results in a highly nontrivial equations even for $\mathbb{C}^1/\mathbb{Z}_n$ background. We wish to report on these issues in later publications.

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