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Noise-enhanced response of nonlinear oscillators

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Abstract

Although often considered to be undesirable, noise can produce beneficial effects in a system. Here, the authors discuss two representative nonlinear systems and the influence of noise on the responses of these systems. One of these systems is a set of coupled monostable Duffing oscillators, while the second of these systems is a Rayleigh-Duffing system that has been considered in honor of Dr. Y. Ueda. For the coupled oscillators, it is shown that an appropriately chosen noise addition can be used to localize energy as well as shift energy localization locations. In the case of the Rayleigh-Duffing system, the authors illustrate how the addition of noise to a deterministic input can push the system from a periodic attractor in the case without noise to a "broken-egg attractor" in the case with noise. These representative examples serve to illustrate a range of possible noise-influenced responses, and it is expected that similar as well as a wider range of responses can be expected in other nonlinear systems.

Keywords: broken-egg attractor; energy localization; noise-enhancement; nonlinear oscillator array; Poincaré section; Fokker-Planck equation

1 Introduction

The introduction of noise into a nonlinear system can result in significant changes to the response of a system. Traditionally, these effects have been perceived as being undesirable. Following the notion of stochastic resonance [1, 2], there have been findings that noise can – and very often does – have a beneficial or enhancing effect. There are many examples of stochastic resonance in biological, physical, and chemical systems [e.g., 3]. These examples do provide a motivation to look for noise-enhanced responses in a wide variety of nonlinear systems. By the qualifier "noise-enhancement," the authors refer to one of the following, due to the introduction of noise into the system: i) amplification of the response, ii) energy transfer from one oscillator to another in an array, iii) stabilization of the response to a harmonic input, iv) sustenance of the response in an oscillatory state (even after the deterministic

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excitation input is switched off); and v) an attainment of desirable dynamics. Here, the authors will explore noise-enhancement possibilities in the context of an array of monostable Duffing oscillators and the Rayleigh-Duffing mixed type oscillator.

In the literature, intrinsic localized modes (ILMs) are known as discrete breathers or Anderson localizations; they are energy localizations that can occur in spatially extended, perfectly periodic, discrete systems [4, 5]. ILMs occurring in pure anharmonic lattices are similar to energy localizations occurring in harmonic lattices with a defect [6]. They can be considered to be a forced nonlinear vibration mode of an oscillatory system [7]. Here, an array of monostable Duffing oscillators is studied, as it has relevance to microelectromechanical systems (MEMS), which are relevant for a host of engineering applications including communications and signal processing. However, the oscillator array structure considered here is different from previous studies carried out in the authors' group [7]. At the micro-scale, stochastic effects can play a significant role in determining the system dynamics and ILMs have been studied in the context of micro-scale systems [7, 8]. ILMs can have adverse effects on the performance of a micromechanical device; for example, they could inhibit information flow or, in some cases, damage the microelectromechanical array and the associated electronic circuitry. However, if these energy localizations are better understood, they have the potential to lead to new technologies. The energy localization phenomenon in the coupled oscillator array of this study does not have the large amplitude characteristic of an ILM; however, this work is relevant to realize energy localization in coupled oscillator arrays. The second system studied here is a Rayleigh-Duffing mixed type oscillator, which is a system studied by Dr. Y. Ueda in the 1960s. This system was shown to have a chaotic attractor in the form of a broken egg [9]. Since this system has had an important influence on developing an understanding of chaos, as a tribute to Dr Y. Ueda, the authors have chosen to explore the effects of noise on the response of this oscillator.

Nomenclature

x_i	position of *i*th oscillator
$x_{i,1}$	position of *i*th oscillator in state space
$x_{i,2}$	velocity of *i*th oscillator in state space
m_i	mass of *i*th oscillator
$k_{0,i}$	linear coupling spring on left side of *i*th oscillator
$k_{1,i}$	linear spring constant of *i*th oscillator
$k_{3,i}$	nonlinear spring constant of *i*th oscillator
K_i	$= k_{0,i} + k_{0,i+1} + k_{1,i}$
c_i	damping term of *i*th oscillator
F, Ω	forcing amplitude, forcing frequency
$\dot{W}(t)$	white noise (derivative of Wiener process)
σ	noise amplitude
μ, γ	Rayleigh-Duffing mixed type equation parameters

With each of the systems considered, the authors begin with the deterministic system and then include a stochastic noise component in the input. In both cases, the system is numerically studied by using the Euler-Maruyama method [10]. By using this method, one obtains an approximate solution of the system. This method is an extension of the Euler method for ordinary differential equations, which has been adapted to perform integrations of stochastic differential equations (SDEs). In addition, the authors also use a Method of Moments analysis to study the averaged dynamics of the system. Through this analysis, they obtain an approximation to the Fokker-Planck equation (a partial differential equation) for the system, which governs the evolution of the probability density function of the states of each oscillator in the array. The moment evolution equations (an infinite set of ordinary differential equations) are derived and truncated from the Fokker-Planck equation. Numerically solving this truncated, finite set of ordinary differential equations gives an averaged (approximate) solution to the variables in the state space. In the case of the Rayleigh-Duffing mixed type oscillator, to visualize the influence of noise on the response, the authors present the Poincaré sections of this system in the form of a two-dimensional histogram. By using this means of visualization, the stochastic dynamics of the system is believed to be portrayed in a meaningful way. The authors believe that this approach might serve as a useful tool in exploring the response of other systems with noise component inclusions.

The rest of this article is organized as follows. In Section 2, the equations governing the array of coupled monostable Duffing oscillators are derived. The Euler-Maruyama simulations and the Method of Moments analysis are both presented in this section and discussed along with the results obtained. In Section 3, the Rayleigh-Duffing mixed type oscillator is presented and the response of this oscillator is examined. By using the Euler-Maruyama method and the aforementioned histograms of the Poincaré sections, the behaviour of this nonlinear system is explored and compared to that observed in the absence of noise. Concluding remarks are collected together in Section 4.

2 Monostable Duffing oscillator array

Consider the array of monostable Duffing oscillators, which is depicted in Figure 1 and described by the following equations, with the variable and parameter descriptions, following that provided in Section 1:

$$\begin{cases} \mathrm{m}_1\ddot{x}_1 + \mathrm{c}_1\dot{x}_1 + \left[\mathrm{k}_{0,1} + \mathrm{k}_{0,2} + \mathrm{k}_{1,1}\right]x_1 - \mathrm{k}_{0,2}x_2 + \mathrm{k}_{3,1}x_1^3 \\ \qquad = \mathrm{F}\sin(\Omega\, T) + \sigma\dot{W}(t) \\ \vdots \\ \mathrm{m}_i\ddot{x}_i + \mathrm{c}_i\dot{x}_i - \mathrm{k}_{0,i}x_{i-1} + \left[\mathrm{k}_{0,i} + \mathrm{k}_{0,i+1} + \mathrm{k}_{1,i}\right]x_i - \mathrm{k}_{0,i+1}x_{i+1} + \mathrm{k}_{3,i}x_i^3 \\ \qquad = \mathrm{F}\sin(\Omega\, T) + \sigma\dot{W}(t) \\ \vdots \\ \mathrm{m}_n\ddot{x}_n + \mathrm{c}_n\dot{x}_n - \mathrm{k}_{0,n}x_{n-1} + \left[\mathrm{k}_{0,n} + \mathrm{k}_{0,n+1} + \mathrm{k}_{1,n}\right]x_n + \mathrm{k}_{3,n}x_n^3 \\ \qquad = \mathrm{F}\sin(\Omega\, T) + \sigma\dot{W}(t) \end{cases} \tag{1}$$

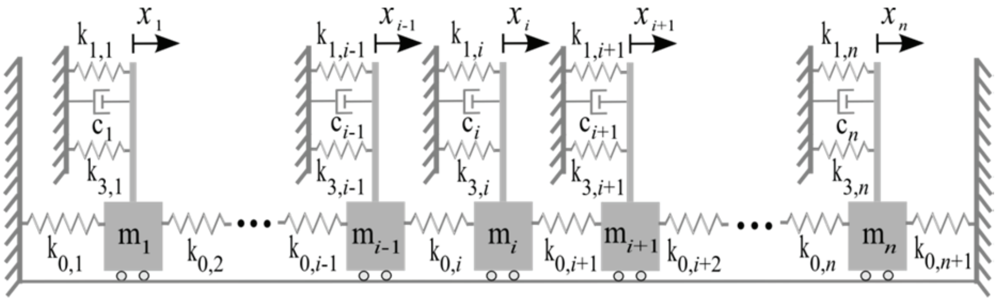

Figure 1. Array of n coupled monostable Duffing oscillators.

2.1 Euler-Maruyama simulations

These n oscillators all have the same deterministic and stochastic forcing ($\mathrm{F}\sin(\Omega\, T)$ and $\sigma\,\dot{W}(t)$, respectively). The term, $\dot{W}(t)$, denotes white noise, which is defined as the derivative of Brownian motion. Since Brownian motion (or in the physics literature, the Wiener process) has independent increments, its derivative does not exist with probability one [11]. Thus, $\dot{W}(t)$ is a "mnemonic" derivative. To write the equations with more formality, the authors convert the system of stochastic differential equations into Langevin form. For facilitating the analysis, the equations of motion are first cast into a state-space form. In the subsequent notation, the x's first subscript refers to the oscillator and the second subscript is used to denote the corresponding state:

$$
\begin{cases}
\dot{x}_{1,1} = x_{1,2} \\
\dot{x}_{1,2} = [-\mathrm{c}_1 x_{1,2} - [\mathrm{k}_{0,1} + \mathrm{k}_{0,2} + \mathrm{k}_{1,1}]x_{1,1} + \mathrm{k}_{0,2}x_{2,1} - \mathrm{k}_{3,1}x_{1,1}^3 \\
\quad +\mathrm{F}\sin(\Omega\, T) + \sigma\dot{W}(t)]/\mathrm{m}_1 \\
\vdots \\
\dot{x}_{i,1} = x_{i,2} \\
\dot{x}_{i,2} = [-\mathrm{c}_i x_{i,2} + \mathrm{k}_{0,i}x_{i-1,1} - [\mathrm{k}_{0,i} + \mathrm{k}_{0,i+1} + \mathrm{k}_{1,i}]x_{i,1} + \mathrm{k}_{0,i+1}x_{i+1,1} - \mathrm{k}_{3,i}x_{i,1}^3 \\
\quad +\mathrm{F}\sin(\Omega\, T) + \sigma\dot{W}(t)]/\mathrm{m}_i \\
\vdots \\
\dot{x}_{n,1} = x_{n,2} \\
\dot{x}_{n,2} = [-\mathrm{c}_n x_{n,2} + \mathrm{k}_{0,n}x_{n-1,1} - [\mathrm{k}_{0,n} + \mathrm{k}_{0,n+1} + \mathrm{k}_{1,n}]x_{n,1} - \mathrm{k}_{3,n}x_{n,1}^3 \\
\quad +\mathrm{F}\sin(\Omega\, T) + \sigma\dot{W}(t)]/\mathrm{m}_n
\end{cases}
\tag{2}
$$

Next, the system of Langevin equations from this system of stochastic differential equations is written as

$$\begin{cases} \mathrm{d}x_{1,1} = x_{1,2}\mathrm{d}t \\ \mathrm{d}x_{1,2} = [(-\mathrm{c}_1 x_{1,2} - [\mathrm{k}_{0,1} + \mathrm{k}_{0,2} + \mathrm{k}_{1,1}]x_{1,1} + \mathrm{k}_{0,2}x_{2,1} - \mathrm{k}_{3,1}x_{1,1}^3 \\ \quad +\mathrm{F}\sin(\Omega\, T))\mathrm{d}t + \sigma \mathrm{d}W]/\mathrm{m}_1 \\ \vdots \\ \dot{x}_{i,1} = x_{i,2}\mathrm{d}t \\ \mathrm{d}x_{i,2} = [(-\mathrm{c}_i x_{i,2} + \mathrm{k}_{0,i}x_{i-1,1} - [\mathrm{k}_{0,i} + \mathrm{k}_{0,i+1} + \mathrm{k}_{1,i}]x_{i,1} + \mathrm{k}_{0,i+1}x_{i+1,1} - \mathrm{k}_{3,i}x_{i,1}^3 \\ \quad +\mathrm{F}\sin(\Omega\, T))\mathrm{d}t + \sigma \mathrm{d}W]/\mathrm{m}_i \\ \vdots \\ \dot{x}_{n,1} = x_{n,2}\mathrm{d}t \\ \mathrm{d}x_{n,2} = [(-\mathrm{c}_n x_{n,2} + \mathrm{k}_{0,n}x_{n-1,1} - [\mathrm{k}_{0,n} + \mathrm{k}_{0,n+1} + \mathrm{k}_{1,n}]x_{n,1} - \mathrm{k}_{3,n}x_{n,1}^3 \\ \quad +\mathrm{F}\sin(\Omega\, T))\mathrm{d}t + \sigma \mathrm{d}W]/\mathrm{m}_n \end{cases} \tag{3}$$

Notice that in this differential form, one no longer has the derivative of Brownian motion (which does not exist) but a differential white noise which does exist. Now, the Euler-Maruyama method can be used to obtain numerical solutions of the following system:

$$\begin{cases} x_{1,1}(j+1) = x_{1,1}(j) + x_{1,2}(\mathrm{j})\mathrm{d}t \\ x_{1,2}(j+1) = x_{1,2}(j) + [(-\mathrm{c}_1 x_{1,2}(j) - [\mathrm{k}_{0,1} + \mathrm{k}_{0,2} + \mathrm{k}_{1,1}]x_{1,1}(j) + \mathrm{k}_{0,2}x_{2,1}(j) \\ \quad -\mathrm{k}_{3,1}x_{1,1}^3(j) + \mathrm{F}\sin(\Omega\, t_j))\mathrm{d}t + \sigma \Delta W_j]/\mathrm{m}_1 \\ \vdots \\ \dot{x}_{i,1} = x_{i,2} \\ x_{i,2}(\mathrm{j}+1) = x_{i,2}(j) + [(-\mathrm{c}_i x_{i,2}(j) + \mathrm{k}_{0,i}x_{i-1,1}(\mathrm{j}) - [\mathrm{k}_{0,i} + \mathrm{k}_{0,i+1} + \mathrm{k}_{1,i}]x_{i,1}(j) \\ \quad +\mathrm{k}_{0,i+1}x_{i+1,1}(j) - \mathrm{k}_{3,i}x_{i,1}^3(j) + \mathrm{F}\sin(\Omega\, t_j))\mathrm{d}t + \sigma \Delta W_j]/\mathrm{m}_i \\ \vdots \\ \dot{x}_{n,1} = x_{n,2} \\ x_{n,2}(\mathrm{j}+1) = x_{n,2}(j) + [(-\mathrm{c}_n x_{n,2}(j) + \mathrm{k}_{0,n}x_{n-1,1}(j) - [\mathrm{k}_{0,n} + \mathrm{k}_{0,n+1} + \mathrm{k}_{1,n}]x_{n,1}(j) \\ \quad -\mathrm{k}_{3,n}x_{n,1}^3(j) + \mathrm{F}\sin(\Omega\, t_j))\mathrm{d}t + \sigma \Delta W_j]/\mathrm{m}_n \end{cases} \tag{4}$$

In this form, j is associated with the time step in the solver. The quantity, ΔW_j, is the incremental noise; it has mean equal to zero and standard deviation equal to $\sqrt{\mathrm{d}t}$. In these simulations, the forcing frequency was ramped up, starting at the highest linear natural frequency and progressing to a value 0.5% higher than the highest linear natural frequency for the system of oscillators, in a manner similar to that carried

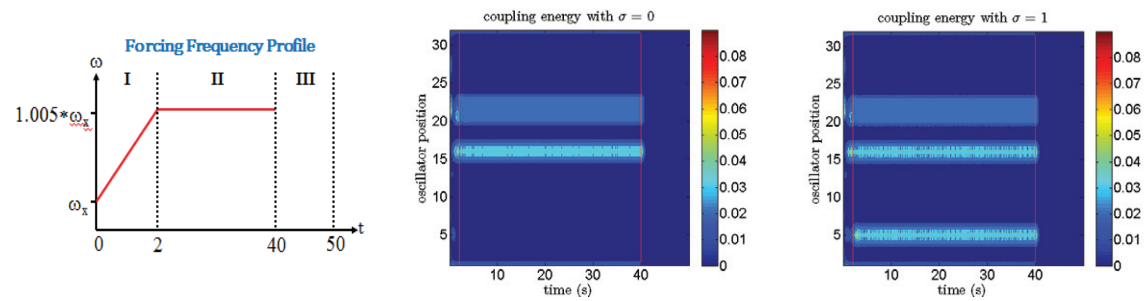

Figure 2. a) The forcing frequency profile. b) Without noise, two energy localizations form. c) With noise, three energy localizations form.

out in the group's previous studies [7, 12]. Thirty-two oscillators were simulated, while varying the noise intensity as a system parameter.

In Figure 2a), the forcing frequency profile is provided. The frequency is ramped from the highest natural frequency to 0.5% above the highest natural frequency over two seconds. After this time, the frequency is held constant until 40 seconds is reached, where the forcing is turned off completely. In Figure 2b), two energy localizations are shown. In Figure 2(c), with the addition of noise, additional energy localization forms. In Figure 2, one of the localizations is more spatially distributed than the other: the localization at oscillators 15-16 has denser energy than the localization at oscillators 20-23. Further research is necessary in order to determine whether the noise has a deterministic effect on where an ILM forms. This is an important question if noise can be used to control the formation of ILMs at different locations in a micro-oscillator array. The present results can be compared to those previously obtained by Ramakrishnan and Balachandran [12], for micro-cantilever arrays, in which each oscillator pair had two different types of oscillators. Although the oscillator systems of references [7, 12] are different from the current one, the authors believe that noise can produce the following two differing effects in array systems; that is, attenuation of energy localizations and creation of energy localizations. If fully understood, noise might be harnessed to create and/or destroy energy localizations in an array.

2.2 Fokker-Planck equation and Method of Moments analysis

In the preceding subsection, the authors demonstrated the use of noise to realize energy localization through a direct numerical simulation. In this subsection, the authors aim to obtain an approximate solution on the basis of a formalism based on the Fokker-Planck equation [13]. The solution of this partial differential equation is the time evolution of the probability density function, which is a function of the variables in state space and of time. In general, the Fokker-Planck equation can be written as

$$\partial_t p = -\sum_j \partial_j [A_j(\boldsymbol{x},t)p] + \frac{1}{2}\sum_{j,k} \partial_j \partial_k \{[B(\boldsymbol{x},t)B^T(\boldsymbol{x},t)]_{jk} p\} \tag{5}$$

where, p is the probability density function and $\boldsymbol{x}$ is the vector of variables in state space. The Fokker-Planck equation for the i^{th} oscillator can be constructed as

$$\partial_t p = -[\partial_{x_{i,1}} x_{i,2} p + \frac{1}{m_i} \partial_{x_{i,2}} p(-\mathrm{c}_i x_{i,2} + \mathrm{k}_{0,i} x_{i-1,1} - [\mathrm{k}_{0,i} + \mathrm{k}_{0,i+1} + \mathrm{k}_{1,i}] x_{i,1}$$
$$+\mathrm{k}_{0,i+1} x_{i+1,1} - \mathrm{k}_{3,i} x_{i,1}^3 + \mathrm{F}\sin(\Omega\, t))] + \mathrm{D}\frac{\partial^2}{\partial {x_{i,2}}^2} p \tag{6}$$

To find an approximate solution for equation (6), the Method of Moments is employed [14]. To briefly explain this method, first take the general moment equation:

$$\langle s \rangle = \iint s\, p\, dx_1 dx_2 \tag{7}$$

Then, obtaining the moment as it evolves through time, it is found that

$$\frac{d\langle s \rangle}{dt} = \iint s \frac{dp}{dt} dx_1 dx_2 \tag{8}$$

Now, considering the r$^{\text{th}}$ moment of position and s$^{\text{th}}$ moment of velocity of the i^{th} oscillator, the result is

$$\frac{d\langle x^r v^s\rangle}{dt} = \iint x^r v^s \frac{dp}{dt} dxdv \tag{9}$$

After substitution and rearrangement, the authors obtain the moment evolution equation for the i^{th} oscillator:

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle x^r v^s\rangle = r\langle x^{r-1}\rangle\langle v^{s+1}\rangle - \frac{s\mathrm{c}}{\mathrm{m}}\langle x^r\rangle\langle v^s\rangle + \frac{s\mathrm{k}_{0,i}}{\mathrm{m}}\langle x^r\rangle\langle v^{s-1}\rangle\langle x_{i-1}\rangle - \frac{s\mathrm{K}_i}{\mathrm{m}}\langle x^{r+1}\rangle\langle v^{s-1}\rangle + \frac{s\mathrm{k}_{0,i}}{\mathrm{m}}\langle x^r\rangle\langle v^{s-1}\rangle\langle x_{i+1}\rangle - \frac{s\mathrm{k}_{3,i}}{\mathrm{m}}\langle x^{r+3}\rangle\langle v^{s-1}\rangle + \frac{s\mathrm{F}\sin(\Omega\, t)}{\mathrm{m}}\langle x^r\rangle\langle v^{s-1}\rangle + s(s-1)D\langle x^r\rangle\langle v^s\rangle \tag{10}$$

The x's and v's without subscripts refer to the position and velocity of the i^{th} oscillator, and $\mathrm{K}(i) = \mathrm{k}_0(i) + \mathrm{k}_0(\mathrm{i}+1) + \mathrm{k}_1(\mathrm{i})$. This moment evolution equation gives an infinite set of ODEs, as different values of r and s are substituted. The following three approximations are made, in order to solve this infinite set of ODEs: 1) the Fokker-Planck equation is written for the i^{th} oscillator, considering only the neighboring two oscillators, 2) the states are assumed to be independent (i.e. $\langle x^r v^s\rangle = \langle x^r\rangle\langle v^s\rangle$), and 3) moments of order 4 and higher are neglected. These approximations yield a set of 6n ODEs, from the previous SDE. The numerical results obtained with this reduced-order system are presented in Figure 3.

Note that in using the Fokker-Planck equation, it is customary to use $\sigma = \sqrt{2D}$; thus, this is indeed the same noise level as that used to generate the results shown in Figure 1. Also, the same initial conditions were used to generate the results shown in Figure 2 as well as that shown in Figure 3. It seems that *on average*, the third energy localization does not occur. However, through the previous Euler-Maruyama simulations, the localization is shown to occur sometimes. By taking higher order moments, the third energy localization observed in Figure 2c) might be better predicted by the Method of Moments analysis. The authors are currently exploring this further.

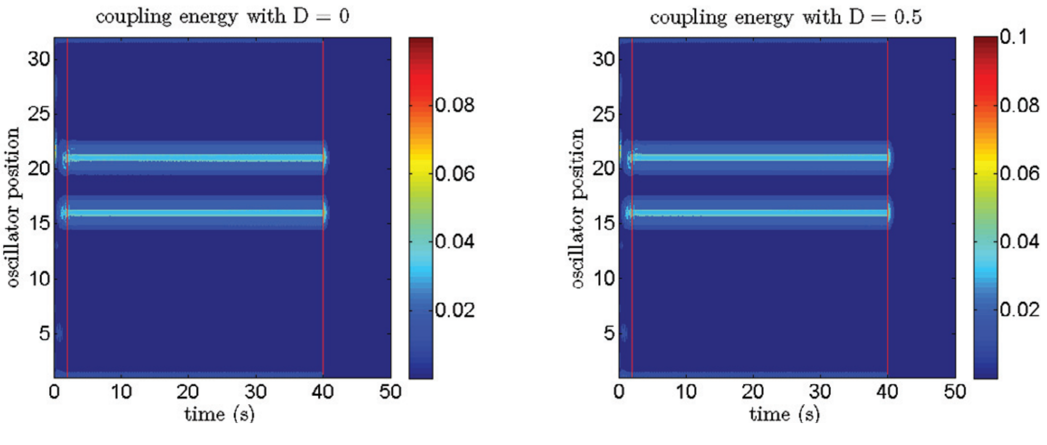

Figure 3. The numerical solution to the Fokker-Planck equation, using the same parameters as in Figure 1. a) For no noise, two energy localizations form. b) Using the same noise level as in Figure 1(b), there are still only two localizations.

3 Rayleigh-Duffing mixed type oscillator

In the previous section, the authors studied the effects of noise on the response of an array of nonlinear oscillators. Here, in honor of Dr. Y. Ueda, the authors explore the effects of noise on the Rayleigh-Duffing oscillator, which is described by

$$\frac{d^2x}{dt^2} - \mu\left(1 - \gamma\left(\frac{dx}{dt}\right)^2\right)\frac{dx}{dt} + x^3 = F\sin(\Omega\, t) + \sigma\, \dot{W}(t) \quad (11)$$

As before, the system has a superposition of deterministic and stochastic forcing ($F\sin(\Omega\, t)$ and $\sigma\, \dot{W}(t)$, respectively). For the parameter values $\mu = 0.2$, $\gamma = 4.0$, $F = 0.3$, and $\Omega = 1.1$, the deterministic counterpart of this system (i.e., the system without noise) exhibits the broken-egg chaotic attractor [10]. As discussed in Section 2, this equation can be written in Langevin form, and using this form, an Euler-Maruyama simulation can be implemented. By keeping the other parameters at the aforementioned

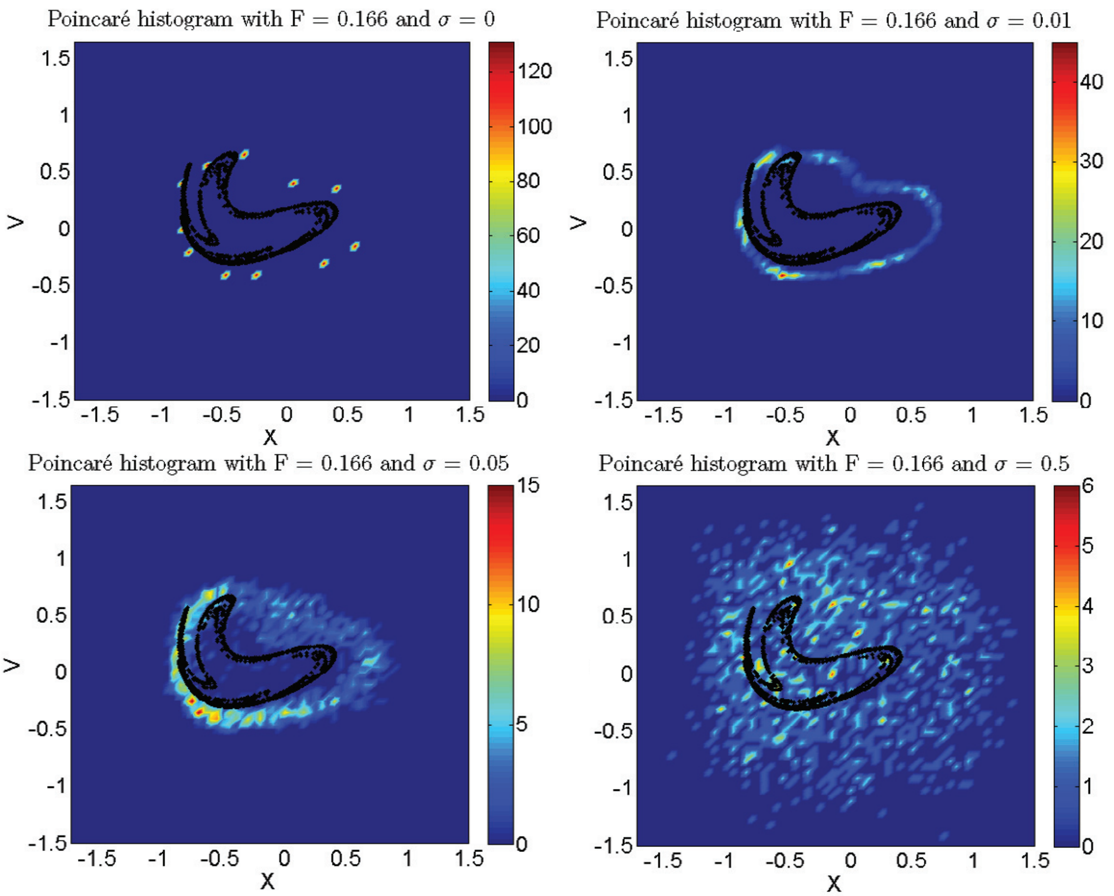

Figure 4. Illustration of the broken-egg attractor along with the different responses on the Poincaré section constructed by using the forcing frequency as the clock frequency: a) With no noise, the motion of the oscillator is periodic, as illustrated by the discrete set of points. b) With a low level of noise, a curve in the state space is traced. c) With the noise level at σ = 0.05, the Poincaré sections are, on average, on an egg-like shape which encircles the broken-egg attractor. d) With higher levels of noise, the oscillator's response fills a large, sparsely populated area.

values, but reducing the forcing constant to F = 0.166, it is found that the motion of the oscillator is periodic.

It is noted that in Figure 4, a two-dimensional histogram is taken of the Poincaré sections of the Euler-Maruyama simulation. The clock frequency used to construct the Poincaré section is taken to be the forcing frequency. In the histograms presented, the color coding shows how many points from the Poincaré sections lie on each small box of the state space in a given amount of time, where x refers to position and v refers to velocity. On top of this histogram, the broken-egg attractor with the parameter values given above is graphed also for direct comparison. As noise is added to the Rayleigh-Duffing mixed type oscillator, the oscillator progresses from periodic motion to something that appears to be "quasiperiodic" or "chaotic," in an average sense. This method of visualization might be useful for other systems that have stochastic components. Since there are many attracting sets between the parameter values F = 0.166 and F =0.3, the response of the oscillator is best thought of in an average sense: the noise pushes the response between many different attracting sets, which are near each other. In an average sense, in going from Figure 4b) to Figure 4c), the noise pushes the system response to what also looks like a broken-egg form; the noise breaks the egg shape form observed in Figure 4b). It is recalled that in the absence of noise, the response is periodic as noted in Figure 4a). For relatively large levels of noise, the state of the system begins to fill a portion of the state space, centering on the origin.

4 Concluding remarks

The present work is intended to be an exploration into two different nonlinear systems, to illustrate the influence of noise on the response of nonlinear oscillators. In one of the considered cases, the authors have studied how noise affects the response of a homogeneous array of monostable Duffing oscillators and a Rayleigh-Duffing mixed type oscillator. The Euler-Maruyama method was employed to simulate these stochastic systems. In the case of the coupled oscillator array, the Fokker-Planck equation was derived for a representative oscillator. Assumptions regarding independence of moments and a truncation approximation were made in order to find a numerical approximation for the solution of the Fokker-Planck equation. The results suggest that a white noise addition *can* create an energy localization in the array, but in an average sense, this usually does not happen. In other studies in the authors' group, with a different set of coupled oscillators, it was shown that noise can be used to attenuate an energy localization [12]. Between the present and previous studies, it is shown that noise can be used to facilitate as well as to suppress energy localizations in coupled oscillator array systems.

Through studies of the Rayleigh-Duffing mixed type oscillator, the authors have shown that the addition of noise can promote an early appearance of response in an average sense, which previously did not exist in the corresponding noise-free case. Poincaré sections were constructed for the numerically obtained responses from the Euler-Maruyama simulation. A two-dimensional histogram was used to visualize this data and compare it to the broken-egg chaotic attractor which is known to occur in this system. It is noted that the noise inclusion caused the system response to progress from a periodic motion to a motion that appears to be "quasiperiodic" or "chaotic". The histograms can be taken to show the average dynamics of the system. For the deterministic case, the histogram is simply the same as that obtained with the usual constructions of Poincaré sections. As noise is added, the system dynamics moves between different attracting sets (there are many attracting sets between F = 0.166 and F = 0.3). Eventually, the noise overpowers the attracting sets, and the system response fills a portion of the state space. These results suggest that an optimal level of noise may exist, which can be used to control/confine the system response to different attracting sets.

Whereas noise is typically considered to be undesirable, in the two examples discussed in this chapter, the noise pushes the system into areas of desirable dynamics. In physical systems, noise is always present. Understanding the stochastic aspects of a system could allow designs which utilize noise in new and advantageous ways. For systems that are capable of chaotic behavior, understanding the stochastic aspects of the system in a statistical sense can be appropriate and useful.

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