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A linear algebra approach to the conjecture of Collatz

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Abstract

We show that a "periodic" version of the so-called conjecture of Collatz can be reformulated in terms of a determinantal identity for certain finite-dimensional matrices M_k, for all $k \geqslant 2$. Some results on this identity are presented. In particular we prove that if this version of the Collatz's conjecture is false then there exists a number k satisfying $k \equiv 8 \pmod{18}$ for which the orbit of $\frac{k}{2}$ is periodic.

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1. Introduction

The Collatz's conjecture is a well-known open problem. This is also quoted in the literature as the $3n + 1$ problem, the Syracuse problem, Kakutani's problem, Hasse's algorithm, and Ulam's problem. In its traditional formulation the conjecture says that

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any orbit for the iteration function $f(n) = (3n + 1)/2$, for n odd, and $f(n) = n/2$, for n even, is always attracted to the value 1. A comprehensive discussion on the subject, namely equivalent formulations and related problems arising in several branches of mathematics, can be found in [4, Chapter 1] and [1,2] where relevant literature is discussed.

In this paper we only consider a "periodic" version of the Collatz conjecture which can be stated as: The unique periodic orbit of the function f is the orbit of 1. For convenience, hereafter we refer to this conjecture as being the Collatz's conjecture.

We show how to translate the conjecture of Collatz in terms of the determinant of finite-dimensional matrices denoted by M_k. Namely, in Section 2 we show that the conjecture is true if and only if $\det M_k = 1 - x^2$, for all $k \geqslant 2$.

Using elementary Linear Algebra we prove in Section 3 that $\det(M_k) = \det(M_{k-1})$ for all $k \not\equiv 8 \pmod{18}$. A tentative proof of the equality $\det M_k = 1 - x^2$, for all $k \geqslant 2$ is outlined. Unfortunately this proof is not complete since we were not able to prove that a certain tree rooted at $\frac{2k-1}{3}$, with $k \equiv 8 \pmod{18}$, has not a vertex equal to $\frac{k}{2}$. This tree is constructed in order to produce a nontrivial solution for a certain homogeneous system $\tilde{M}_{k-1}Y = 0$, when $k \equiv 8 \pmod{18}$. Let us emphasize that if such a solution does exist, then $\det(\tilde{M}_{k-1}) = 0$ and the Collatz's conjecture is true. Nevertheless, we can prove that such nontrivial solution exists for certain values of k and we give an algorithm for its construction (see Section 4). As a consequence of the discussion on the existence of this solution we show that if the Collatz's conjecture is false then there exist integers $k \equiv 8 \pmod{18}$ for which the orbit of $\frac{k}{2}$ is periodic.

2. Periodic orbits and a Jacobi formula

Consider the following Collatz's iteration function defined on the set of positive integers $\mathbb{N}$ (see [4, p. 11]):

$$f(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \text{ is odd,} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases} \tag{1}$$

For $n \in \mathbb{N}$, the orbit of n is the sequence defined as $\mathcal{O}_n = \{f^j(n) : j \geqslant 0\}$, where $f^j = f \circ f^{j-1}$ is the j-fold iterate of f. An orbit $\mathcal{O}_n$ is said to be periodic, with period $p \geqslant 1$, if p is the least positive integer verifying $f^p(n) = n$. For instance, for f given by (1) the orbit of 1 is periodic with period 2 (since $f(1) = 2$ and $f^2(1) = 1$).

A "periodic" version of the so-called Collatz's conjecture can be stated as: "$\mathcal{O}_1$ is the unique periodic orbit of f". Note that $\mathcal{O}_1 = \mathcal{O}_2 = \{1, 2\}$. Throughout this paper each time we refer to the conjecture of Collatz we mean this version of the conjecture.

For each $m \geqslant 2$, consider the following $m \times m$ matrix A_m, whose entries $A_m(i, j)$ are defined by

$$A_m(i,j) = \begin{cases} 1 & \text{if } f(i) = j \text{ and } 1 \leqslant j \leqslant m, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Remarks

(i) Note that for a fixed $m \geqslant 2$ the matrix A_m does not contain all the information on the images $f(i)$ for all $1 \leqslant i \leqslant m$. For instance, when $m = 5$, one has $f(5) = 8$ and so the fifth row of A_5 is a zero-row. Therefore the orbit of, say 3, can not be traced out from A_5, since $3 \to 5 \to 8 \to 4 \to 2 \to 1$ and $8 > 5$. However the orbit of 4 can be completely followed from the information in A_5.

(ii) Using the matrix multiplication rule it is easy to see that a power $r \geqslant 1$ of A_m is related to the values $f^r(i)$, for $i = 2, \ldots, m$ as follows:

$$A_m^r(i,j) = \begin{cases} 1 & \text{if } f^r(i) = j \text{ and} f^s(i) \leqslant m, \text{ for } 0 < s < r, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Theorem 1. *Let $P_m(x) = \det(I_m - xA_m)$, where I_m is the identity matrix of order m and A_m given in (2). For f as in (1), the orbit of the point 1 is the unique periodic orbit of f if and only if $P_m(x) = 1 - x^2$, for all $m \geqslant 2$.*

Before proving this theorem we will prove the following lemma.

Lemma 1. *Let $A = A_m$ be as in (2). Then, $\det(I_m - xA) = 1 - x^2$ if and only if*

$$\mathrm{Tr}(A^p) = \begin{cases} 0 & \textit{if } p \textit{ odd}, \\ 2 & \textit{if } p \textit{ even}. \end{cases} \tag{4}$$

Proof. Let A be a square real matrix. Recall that as a consequence of Jacobi's formula for the derivative of a determinant (see for instance [3, p. 570]), one has the well known identity between formal power series in the indeterminate x

$$\exp \sum_{n \geqslant 1} \frac{\mathrm{Tr}(A^n)}{n} x^n = \frac{1}{\det(I_m - xA)}. \tag{5}$$

If $\det(I_m - xA) = 1 - x^2$, then from (5) we get

$$\exp \sum_{n \geqslant 1} \frac{\mathrm{Tr}(A^n)}{n} x^n = \frac{1}{1 - x^2}, \tag{6}$$

or equivalently

$$\sum_{n \geqslant 1} \frac{\mathrm{Tr}(A^n)}{n} x^n = -\log(1 - x^2). \tag{7}$$

Differentiating (7) we obtain

$$\sum_{n \geqslant 1} \mathrm{Tr}(A^n) x^{n-1} = \frac{2x}{1 - x^2} = 2x + 2x^3 + 2x^5 + \cdots \tag{8}$$

So, comparing both members of Eq. (8) we get the result in (4).

Conversely, from (4) we have (8). Integrating both members of (8) we obtain (7) and equivalently (6). Finally, comparing (6) with (5), we have $\det(I_m - xA) = 1 - x^2$. $\square$

Proof (*of Theorem* 1). Suppose there exists an orbit of period p that is not the orbit of 1, say $\mathcal{O}_a$ with $a \neq 2$. That is, we are assuming that $f^p(a) = a$. Let us choose $m = \max \mathcal{O}_a$ and A_m defined as in (2).

From the expression of the powers of A_m, given in (3), we have $A^p(a, a) = 1$. Then, as $a > 2$, we have $\mathrm{Tr}(A_m^p) \geqslant 3$ if p is even, and $\mathrm{Tr}(A_m^p) \geqslant 1$ if p is odd (note that $\mathrm{Tr}(A_2^p) = 2$ if p is even and $\mathrm{Tr}(A_2^p) = 0$ if p is odd). So, by Lemma 1 we get $P_m(x) = \det(I_m - xA_m) \neq 1 - x^2$ which is a contradiction.

Conversely, suppose that the orbit of 1 is the unique periodic orbit. By definition of A_m one has

$$\mathrm{Tr}(A_m^n) = \#\big\{k \in \{1, \ldots, m\} : f^n(k) = k \text{ and } f^i(k) \leqslant m, \text{ for all } i = 1, \ldots, n\big\},$$

thus $\mathrm{Tr}(A_m^n) = 2$, if n is even and $\mathrm{Tr}(A_m^n) = 0$, if n is odd. Therefore by Lemma 1 we get $\det(I_m - xA) = 1 - x^2$ for all $m \geqslant 2$. $\square$

3. On the determinant of a Collatz matrix

The aim of this section is to discuss whether $M_k = I_k + xA_k$ satisfies the identity $\det(M_k) = 1 - x^2$, for all $k \geqslant 2$. We will call the matrix M_k a Collatz matrix. By Theorem 1, the equality $\det(M_k) = 1 - x^2$, for all $k \geqslant 2$, is equivalent to the Collatz's conjecture.

The Collatz matrix $M_k = (m_{ij})_{i,j=1,\ldots,k}$ is given by

$$m_{ij} = \begin{cases} 1 & \text{if } i = j, \\ x & \text{if } f(i) = j, \\ 0 & \text{otherwise,} \end{cases}$$

that is

$$M_k = \begin{pmatrix} 1 & x & 0 & 0 & 0 & \cdots & m_{1k} \\ x & 1 & 0 & 0 & 0 & \cdots & m_{2k} \\ 0 & 0 & 1 & 0 & x & \cdots & m_{3k} \\ 0 & x & 0 & 1 & 0 & \cdots & m_{4k} \\ 0 & 0 & 0 & 0 & 1 & \cdots & m_{5k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ m_{k1} & m_{k2} & m_{k3} & m_{k4} & m_{k5} & \cdots & 1 \end{pmatrix}. \tag{9}$$

The following lemma presents some fundamental properties of the Collatz matrix.

Lemma 2. *The Collatz matrix M_k has the following structure*:

(a) *There is no more than one value x per row.*
(b) *A column j $(2 \leqslant j \leqslant k)$ of M_k has one value x above the main diagonal, if and only if*

$$j \equiv 2 \pmod 3. \tag{10}$$

In this case x belongs to an odd row of M_k. Moreover, it cannot exist more that one x above the main diagonal in the same column.
(c) *A column j $(2 \leqslant j \leqslant k)$ of M_k has one value x below the main diagonal, if and only if*

$$j \leqslant \left\lfloor \frac{k}{2} \right\rfloor,$$

where $\lfloor\ \rfloor$ denotes the floor function. In this case, x belongs to an even row. Moreover, it cannot exist more that one x below the main diagonal in the same column.

Proof. (a) If there are two different values of x in the same row of M_k, say i, then $f(i) = j_1$ and $f(i) = j_2$ for $j_1 \neq j_2$, which is a contradiction.

(b) If a column j of M_k has x above the main diagonal, then there exists $i < j$ such that $m_{i,j} = x$. That is, $f(i) = j$ with $1 \leqslant i, j \leqslant k$. The index i cannot be even since in this case $f(i) = \frac{i}{2} = j$ giving the contradiction $i > j$. So, if there exists x in the column j above the main diagonal it must belong to an odd row and consequently $f(i) = (3i+1)/2 = j \Longleftrightarrow 3i - 2j = -1$. The Diophantine equation $3i - 2j = -1$ has general solution $j \equiv 2 \pmod 3$, as stated. Furthermore, there exists only one value x above the main diagonal of M_k, otherwise $f(i_1) = f(i_2) = j$ for $i_1 \neq i_2$. But, as showed before, both i_1 and i_2 must be odd numbers, and $f(i_1) = f(i_2) = j$ gives $i_1 = i_2$.

(c) If the column j of M_k has x below the main diagonal, this means that there exists $i > j$ such that $f(i) = j$. For i even one has $i = 2j$ and, as $1 \leqslant i, j \leqslant k$, we get $j \leqslant \lfloor \frac{k}{2} \rfloor$.

On other hand there is no odd i with $i > j$ such that $f(i) = j$, since for i odd one has $f(i) = j \Longleftrightarrow \frac{3i+1}{2} = j \Longleftrightarrow i = \frac{2j-1}{3}$, and $i = \frac{2j-1}{3} < j$ which is a contradiction.

Similarly as in (b) there cannot exist more than one x below the main diagonal in the same column of M_k, since these values must belong to even rows. □

The matrix M_k given in (9) is highly sparse and so one is attempted to use directly the definition of determinant to compute $\det(M_k)$. By definition

$$\det(M_k) = \sum_{j \in \Pi} \operatorname{sign}(j) m_{1,j_1} m_{2,j_2} \cdots m_{k,j_k}, \tag{11}$$

where Π denotes the set of all $k!$ permutations of $\{1, \ldots, k\}$. It seems that the only permutations leading to nonzero terms in the sum (11) are $(1, 2, 3, \ldots, k)$ and $(2, 1, 3, 4, \ldots, k)$ giving $\det M_k = 1 - x^2$. However we were not able to show that these are the only possible permutations for all $k \geqslant 2$ and so we will try hereafter an indirect approach.

The next proposition gives some properties of the determinant of M_k.

Proposition 1. *Let M_k be defined as in (9) and $k \geqslant 3$.*

(a) *If $k \not\equiv 8 \pmod{18}$, then $\det(M_k) = \det(M_{k-1})$.*
(b) *If $k \equiv 8 \pmod{18}$ then, $\det(M_k) = \det(M_{k-1}) + (-1)^{\frac{5k+2}{3}} x \det(\tilde{M}_{k-1})$, where $\tilde{M}_{k-1}$ is the matrix that has all its rows equal to those of M_{k-1} except the $(2k-1)/3$ row which has x at the position $k/2$ and zeros elsewhere.*

Proof. If k is odd then by Lemma 2(c) the last row of M_k has no x. That is, this row has the form $r_k = (0, 0, \ldots, 0, 1)$. So, $\det(M_k) = \det(M_{k-1})$.

If k is even then by Lemma 2(b) the last column of M_k has a value x if and only if $k \equiv 2 \pmod 3$. As $k > 2$ is an even number and $k \equiv 2 \pmod 3$ then $k \equiv 8 \pmod 6$. Thus, if $k \not\equiv 8 \pmod 6$ then the last column of M_k has the form $c_k = (0, 0, \ldots, 0, 1)^{\mathrm{T}}$ and so $\det(M_k) = \det(M_{k-1})$.

In the case $k \equiv 8 \pmod 6$ the last column of M_k has one x. By Lemma 2(b) the value x belongs to an odd row of M_k, say i. So,

$$m_{i,k} = x \iff f(i) = \frac{3i+1}{2} = k \iff i = \frac{2k-1}{3}.$$

Therefore the last column of M_k has x in the row $(2k-1)/3$. Using a cofactor expansion along the last column of M_k one has

$$\det(M_k) = \det(M_{k-1}) + x C_{\frac{2k-1}{3},k}, \tag{12}$$

where $C_{\frac{2k-1}{3},k}$ is the $(\frac{2k-1}{3}, k)$-cofactor of M_k.

Let $\tilde{M}_{k-1} = (\tilde{m}_{ij})_{i,j=1,\ldots,k-1}$ be the matrix obtained from M_k by suppressing its last column and interchanging the $\frac{2k-1}{3}$-row of the resulting matrix with its last row. So $C_{\frac{2k-1}{3},k} = (-1)^{\frac{5k+2}{3}} \det(\tilde{M}_{k-1})$, and by (12),

$$\det(M_k) = \det(M_{k-1}) + (-1)^{\frac{5k+2}{3}} x \det(\tilde{M}_{k-1}). \tag{13}$$

Note that by construction $\tilde{M}_{k-1}$ has all rows equal to the corresponding rows of the Collatz matrix M_{k-1} with the exception of the $\frac{2k-1}{3}$-row. The row $(2k-1)/3$ of $\tilde{M}_{k-1}$ has all its entries equal to zero except the $k/2$ entry which is equal to x. This also means that all diagonal entries of $\tilde{M}_{k-1}$ are equal to 1 except the entry $(2k-1)/3$ which is equal to zero.

Let us prove that $\tilde{M}_{k-1}$ has a zero column when $k \not\equiv 8 \pmod{18}$ which implies that $\det(\tilde{M}_{k-1}) = 0$, and by (13) the results in (a) and (b) follow.

As $\frac{k}{2} < \frac{2k-1}{3}$, then by Lemma 2(b) we know that if the column $(2k-1)/3$ of $\tilde{M}_{k-1}$ has x above the main diagonal one must have $\frac{2k-1}{3} \equiv 2 \pmod 3$. The conditions $k \equiv 8 \pmod 6$ and $\frac{2k-1}{3} \equiv 2 \pmod 3$ imply $k \equiv 8 \pmod{18}$.

Therefore if $k \not\equiv 8 \pmod{18}$ then $\tilde{M}_{k-1}$ has its $(2k-1)/3$ column equal to zero. □

Proposition 2. *If there exists a* $k \equiv 8 \pmod{18}$ *such that* $\det(\tilde{M}_{k-1}) \neq 0$, *then the Collatz's conjecture is false*.

Proof. Let $\bar{k} \equiv 8 \pmod{18}$ be the first matrix order for which $\det(\tilde{M}_{\bar{k}-1}) \neq 0$. By Proposition 1 one has $\det(M_j) = 1 - x^2$ for all $2 \leqslant j \leqslant \bar{k} - 1$, and $\det(M_{\bar{k}}) = \det(M_{\bar{k}-1}) + x \det(\tilde{M}_{\bar{k}-1}) \neq (1 - x^2)$. So the result follows by Theorem 1. □

4. On the solutions of $\tilde{M}_{k-1}Y = 0$

If one can prove that $\det(\tilde{M}_{k-1}) = 0$ for all $k \equiv 8 \pmod{18}$, then using mathematical induction on k and Proposition 1 we get $\det(M_k) = 1 - x^2$ for all $k \geqslant 2$, which by Theorem 1 is equivalent to say that Collatz's conjecture holds.

A strategy to show that $\det(\tilde{M}_{k-1}) = 0$ is to find a nontrivial solution for the system $\tilde{M}_{k-1}Y = 0$. As an outcome of our research we conjecture that this homogeneous system has a (one-dimensional) general solution and we describe an algorithm for the computation of such a solution. This algorithm produces a binary tree, which we shall call a Collatz tree, rooted at $(2k-1)/3$, from which the solution Y can be obtained. It is shown that if no vertex of this tree is equal to $k/2$ then the referred homogeneous system has a nontrivial solution.

We also have been able to prove that most of the vertices of the Collatz tree are different from $k/2$ and, at the same time, we identify which vertices can eventually be equal to $k/2$.

Hereafter we assume $k \equiv 8 \pmod{18}$, $Y = (y_1, \ldots, y_{k-1})^{\mathrm{T}}$ and $\tilde{M}_{k-1}$ defined as in Proposition 1(b). The system $\tilde{M}_{k-1}Y = 0$ is given by the following equations:

$$\begin{cases} y_i + xy_{f(i)} = 0, & 1 \leqslant i \leqslant (2k-4)/3, \\ xy_{k/2} = 0, & \\ y_i = 0, & i \text{ odd and } (2k-1)/3 < i \leqslant k-1, \\ y_i + xy_{f(i)} = 0, & i \text{ even and } (2k-1)/3 < i \leqslant k-2. \end{cases} \tag{14}$$

So, Y is a solution only if

$$\begin{cases} y_{k/2} = 0, & \\ y_i = 0, & \text{for } i \text{ odd and } (2k-1)/3 < i \leqslant k-1. \end{cases} \tag{15}$$

Conjecture 1. *For $k \equiv 8 \pmod{18}$, the general solution of the system $\tilde{M}_{k-1}Y = 0$ is one dimensional and parametrized by the component $y_{\frac{2k-1}{3}}$.*

Note that, from (14) and (15), it is necessary for the existence of a solution of the type referred in Conjecture 1 that $y_{\frac{2k-1}{3}}$ is not related neither to $y_{k/2}$ nor to any y_i with i odd and $(2k-1)/3 < i \leqslant k-1$.

Let us illustrate what we mean by the assertion "$y_{\frac{2k-1}{3}}$ is not related to y_i", by taking $k = 8$. In this case we have $\frac{2k-1}{3} = 5$ and the system (14) is

$$\begin{cases} y_1 + xy_2 = 0, \\ y_2 + xy_1 = 0, \\ y_3 + xy_5 = 0, \\ y_4 + xy_2 = 0, \\ xy_4 = 0, \\ y_6 + xy_3 = 0, \\ y_7 = 0. \end{cases} \tag{16}$$

This means that $y_{\frac{2k-1}{3}} = y_5$ is related to y_3 and to y_6 and not related to $y_{k/2} = y_4$ nor to y_7. Indeed this fact depends on the structure of $\tilde{M}_7$ in particular because the column number 5 of this matrix has one x above the main diagonal in the (odd) row number 3 (giving the third equation in (16)) and the third column has one x in the sixth row (giving the sixth equation in (16)). Therefore the sequences of indices is $5 \to 3 \to 6$, and the solution Y of the homogeneous system $\tilde{M}_7 Y = 0$ is given by

$$y_3 = -xy_5, \quad y_6 = -xy_3 = x^2 y_5 \quad \text{and} \quad y_1 = y_2 = y_4 = y_7 = 0.$$

In the general case, for $k \equiv 8 \pmod{18}$, the unknowns related to $y_{(2k-1)/3}$ are deduced from the structure of $\tilde{M}_{k-1}$ namely due to the following properties (see Lemma 2):

(1) If $j > k/2$ then the column j has no x below the main diagonal and has x above the main diagonal (in the position $\frac{2j-1}{3}$) if and only if $j \equiv 2 \pmod 3$.
(2) If $j < k/2$ then the column j has always one x below the main diagonal (in the position $2j$) and it has also one x above the main diagonal (in the position $\frac{2j-1}{3}$) if and only if $j \equiv 2 \pmod 3$.

So, in the case Conjecture 1 holds, a nontrivial solution Y of the system $\tilde{M}_{k-1}Y = 0$ can be constructed from what we call a Collatz tree:

Definition 1 (*Collatz tree*). For $k \equiv 8 \pmod{18}$, a Collatz tree is rooted at $R = \frac{2k-1}{3}$ and constructed in the following way: given a vertex A, then

(i) If $A < \frac{k}{2}$ then $D(A) = 2A$ is a child of A.
(ii) If $A \equiv 2 \pmod 3$ and $A < \frac{k}{2}$ then $U(A) = \frac{2A-1}{3}$ is a child of A.
(iii) If $A \equiv 2 \pmod 3$ and $A > \frac{k}{2}$ then $U(A) = \frac{2A-1}{3}$ is the unique child of A.

Note that the definition given here for a Collatz tree does not coincides with the common definition used in the literature (see [4, Chapter 2]). For a fixed k, our Collatz tree is finite and has all its vertices less than k, since the children of each vertex are obtained as images of the functions D or U. The tree ends when there is no child congruent to 2 module 3. The vertices of our Collatz tree are all positive integers less than k whose trajectory contains the value $\frac{2k-1}{3}$. That is, starting from a certain vertex in the Collatz tree one can follows the corresponding trajectory along the tree, walking upwards, up to the vertex $\frac{2k-1}{3}$.

A solution $Y = (y_1, y_2, \ldots, y_{k-1})^{\mathrm{T}}$ of the system $\tilde{M}_{k-1}Y = 0$ can be read out from the Collatz tree as follows: For $i = 1, \ldots, k-1$,

$$y_i = \begin{cases} y_R & \text{for } i = R = \frac{2k-1}{3}, \\ -xy_A & \text{if } i \text{ is a child of the vertex } A, \\ 0 & \text{otherwise.} \end{cases}$$

For $y_{\frac{2k-1}{3}} \neq 0$, the system $\tilde{M}_{k-1}Y = 0$ has a nontrivial solution whenever the Collatz tree has no vertex equal to $\frac{k}{2}$ nor equal to some odd i satisfying $\frac{2k-1}{3} < i < k-1$.

All the vertices of a Collatz tree are constructed using the functions

$$U(x) = \frac{2x-1}{3} \quad \text{and} \quad D(x) = 2x. \tag{17}$$

Let us state some straightforward properties of the function U.

Lemma 3 (Basic properties of the function U). *Let U be defined as in* (17). *The following properties hold*:

(P1) *If $x < k$ then $U(x) = \frac{2x-1}{3} < \frac{2k-1}{3}$.*
(P2) *If $x > \frac{k}{n}$ $(2 \leqslant n < \frac{k}{2})$ then $U(x) = \frac{2x-1}{3} > \frac{2k-n}{3n} > \frac{k}{2n}$.*
(P3) *If $x < \frac{k}{2}$ then $U(x) = \frac{2x-1}{3} < \frac{k-1}{3} < \frac{k}{3} < \frac{k}{2}$.*
(P4) *If $x < \frac{2k}{3}$ then $U(x) = \frac{2x-1}{3} < \frac{4k}{9} < \frac{k}{2}$.*
(P5) *If $x \in \{1, 2, \ldots, k-1\}$ then whenever $U(x)$ is an integer it is an odd number less than $\frac{2k-1}{3}$.*

Proof. The properties (P1)–(P4) are obvious. The property (P5) follows also easily when one solves $U(x) = \frac{2x-1}{3} = j$ for x and j positive integers. □

In the next proposition we show that for all $k \equiv 8 \pmod{54}$ Conjecture 1 holds.

Proposition 3. *When $k \equiv 8 \pmod{54}$ the Collatz tree has only three vertices none of them equal to $\frac{k}{2}$ or equal to an odd number greater than $\frac{2k-1}{3}$. So Conjecture 1 is true for $k \equiv 8 \pmod{54}$.*

Proof. As the root $R = \frac{2k-1}{3} > \frac{k}{2}$ and since $k \equiv 8 \pmod{18}$ implies $\frac{2k-1}{3} \equiv 2 \pmod 3$, then (by definition 1(iii)) R has only one child $U(\frac{2k-1}{3}) = \frac{4k-5}{9}$.

Furthermore as $\frac{k}{2} < \frac{2k-1}{3} < \frac{2k}{3}$ then by properties (P4) and (P2) (with $n = 2$) in Lemma 3, we have that the vertex $\frac{4k-5}{9}$ satisfies $\frac{k}{4} < \frac{4k-5}{9} < \frac{k}{2}$.

By definition 1(i) the vertex $\frac{4k-5}{9}$ has always at least one child $D(\frac{4k-5}{9}) = \frac{8k-10}{9}$ which verifies $\frac{k}{2} < \frac{8k-10}{9} < k-1$. However, $\frac{4k-5}{9}$ has another child given by $U(\frac{4k-5}{9}) = \frac{8k-19}{27}$ whenever $\frac{4k-5}{9} \equiv 2 \pmod 3$. As the conditions $\frac{4k-5}{9} \equiv 2 \pmod 3$ and $k \equiv 8 \pmod{18}$ imply $k \equiv 26 \pmod{54}$, then $\frac{8k-19}{27}$ is a child of $\frac{4k-5}{9}$ if and only if $k \equiv 26 \pmod{54}$. Furthermore as $\frac{k}{4} < \frac{4k-5}{9} < \frac{k}{2}$ then by properties (P3) and (P2) (with $n = 4$) in Lemma 3 we have $\frac{k}{8} < \frac{8k-19}{27} < \frac{k}{2}$.

The vertex $\frac{8k-10}{9}$ is always greater than $\frac{k}{2}$. So, this vertex has at most one child $U(\frac{8k-10}{9}) = \frac{16k-29}{27}$. This happens if and only if both congruences $\frac{8k-10}{9} \equiv 2 \pmod 3$ and $k \equiv 8 \pmod{18}$ hold simultaneously which imply $k \equiv 44 \pmod{54}$. Moreover as $\frac{k}{2} < \frac{8k-10}{9} < k$ then by properties (P1) and (P2) (with $n = 2$) we have $\frac{k}{4} < \frac{16k-29}{27} < \frac{2k-1}{3}$.

Noting that the arithmetic progression $k \equiv 8 \pmod{18}$ is the union of the three disjoint arithmetic progressions $k \equiv 8 \pmod{54}$, $k \equiv 26 \pmod{54}$ and $k \equiv 44 \pmod{54}$, thus the Collatz tree for $k \equiv 8 \pmod{54}$ has only three vertices $\frac{2k-1}{3} \to \frac{4k-5}{9} \to \frac{8k-10}{9}$, which are all different from $\frac{k}{2}$. The first two are odd numbers by (P5) and less than $\frac{2k-1}{3}$. The third vertex is an even number since it is the image of the function $D(x) = 2x$. $\square$

The results of Proposition 3 are illustrated in Fig. 1.

Remark. By property (P5) of Lemma 3 we can conclude that there is no odd vertex of the Collatz tree greater than $\frac{2k-1}{3}$ since all the odd vertices are images of the function U. Therefore in order to show that the system $\tilde{M}_{k-1}Y = 0$ has a nontrivial solution it is necessary to prove that all the vertices are different from $\frac{k}{2}$.

Denote by $F = U \circ D \circ U$, $G = D \circ U \circ U$, with D and U defined as in (17), and the root of a Collatz tree by $R = \frac{2k-1}{3}$. The properties (P1)–(P4) of function U given in Lemma 3 do not enable us to predict whether the vertices $F(R)$ and $G(R)$ (where R denotes the root) are different from $\frac{k}{2}$. As we will see in Proposition 4 these vertices are indeed greater than $\frac{k}{2}$ since they do exist only for those k verifying a certain congruence. The tree from these vertices onwards will repeat the pattern presented in Fig. 1. On the other hand, it is interesting to note that the vertices obtained from $F(R)$ and $G(R)$ by composition with F and G exhibit a regular behaviour with respect to the congruences they do verify.

At a first glance it seems that the vertices obtained from $F(R)$ and $G(R)$ by composing F and G any number of times will be greater than $k/2$. Unfortunately this is not true since for instance $G^2(R) > \frac{k}{2}$ and $G^3(R) < \frac{k}{2}$, as shown in Proposition 4. This is a drawback that prevents us to use mathematical induction in order to show

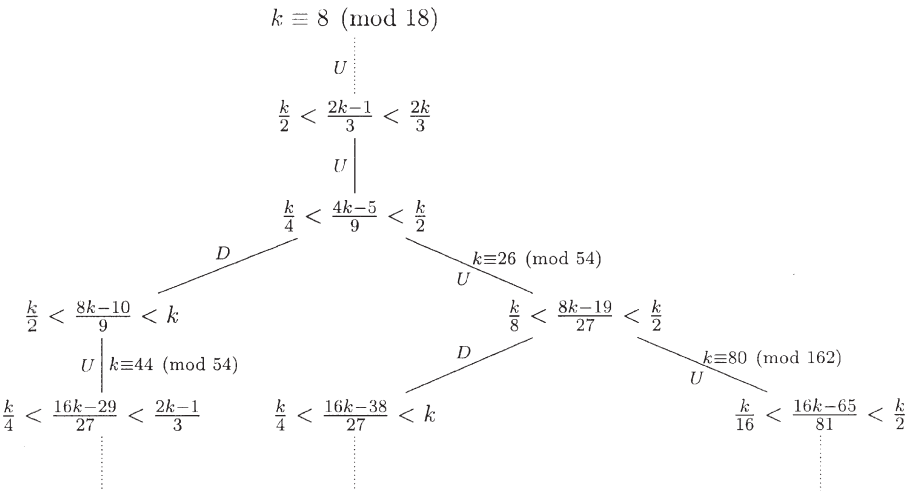

Fig. 1. Some properties for the vertices of the Collatz tree.

that for all $k \equiv 8 \pmod{18}$ the system $\tilde{M}_{k-1}Y = 0$ has a nontrivial solution. However the results presented in Proposition 4 give some hints pointing out a direction for the possible existence of a counterexample for the Collatz conjecture (see Proposition 5). The results presented in Proposition 4 can be easily improved using any symbolic computational tool. This direction of research is however out of the scope of this work.

Proposition 4. *Consider $F = U \circ D \circ U$ and $G = D \circ U \circ U$ with D and U defined as in (17). Let $k = 8 + 18p$, $p \geqslant 0$ and $R = \frac{2k-1}{3} = 5 + 12p$.*

(1) *If $F^2(R)$ and $G(F(R))$ exist then $F(R) \equiv 5 \pmod{12}$ and $F(R) > \frac{k}{2}$.*
(2) *If $F(G(R))$ and $G^2(R)$ exist then $G(R) \equiv 2 \pmod{12}$ and $G(R) > \frac{k}{2}$.*
(3) *If $F^3(R), G(F^2(R)), F(G(F(R)))$ and $G^2(F(R))$ exist, then $F^2(R) \equiv 5 \pmod{12}$ and $G(F(R)) \equiv 2 \pmod{12}$. Furthermore $G(F(R)) > \frac{k}{2}$ and $F^2(R) > \frac{k}{2}$.*
(4) *If $F^2(G((R)), G(F(G((R))), F(G^2(R))$ and $G^3(R)$ exist, then $F(G(R)) \equiv 5 \pmod{12}$ and $G^2(R) \equiv 2 \pmod{12}$. Furthermore $F(G(R)) > \frac{k}{2}$ and $G^2(R) > \frac{k}{2}$.*
(5) *If $G^3(R)$ and $F(G^2(R))$ exist, then $G^3(R) < \frac{k}{2}$ and $F(G^2(R)) < \frac{k}{2}$.*

Some of the results in Proposition 4 are illustrated in Fig. 2.

Proof. (1) Let $R = 5 + 12p$. Note that $U(R)$ is defined only if $R \equiv 2 \pmod 3$ which is always true for any p. On the other hand $U(R) = \frac{2R-1}{3} \equiv 3 + 8p$ and

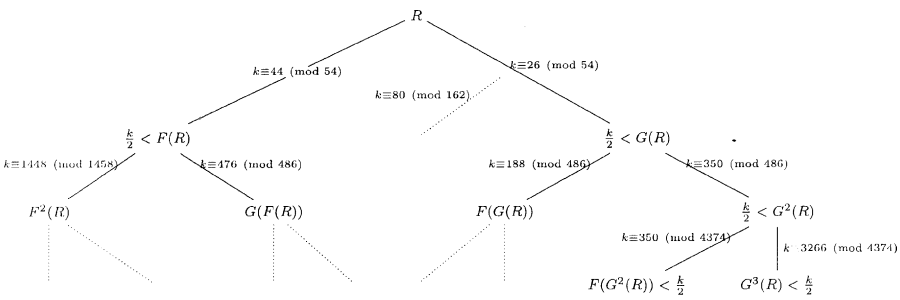

Fig. 2. Some properties of the Collatz tree given in Proposition 4.

$D(U(R)) = 2U(w) \equiv 6 + 16p$. In order that $F(R)$ to be defined we must have $(D \circ U)(R) \equiv 2 \pmod 3$. So, the Diophantine equation $2 + 3r = 6 + 16p$ implies $p = 2 + 3p'$. Thus $F(R) = 25 + 32p'$ and $\frac{k}{2} = 22 + 27p'$. So $F(R) > \frac{k}{2}$.

From the definition of the Collatz tree $F^2(R)$ and $(G \circ F)(R)$ exist only if $F(R) \equiv 2 \pmod 3$ which implies $p' = 2 + 3p''$ and so $F(R) = 89 + 96p''$ satisfying $F(R) \equiv 5 \pmod{12}$. Note that in this case $k = 152 + 162p''$.

(2) From (1), $U(R) = 3 + 8p$. Since $(U \circ U)(R)$ exists only if $U(R) \equiv 2 \pmod 3$, then $p = 1 + 3p'$. Thus $G(R) = 14 + 32p'$ and $\frac{k}{2} = 13 + 27p''$, and so $G(R) > \frac{k}{2}$.

As before, $(F \circ G)(R)$ and $G^2(R)$ exist only if $G(R) \equiv 2 \pmod 3$ which implies $p' = 3p''$ and so $G(R) = 14 + 96p''$ (and $k = 26 + 162p''$) yielding $G(R) \equiv 2 \pmod{12}$.

(3) From (1), we have $F(R) = 89 + 96p$ and $k = 152 + 162p$. So $D(U(F(R))) = 118 + 128p$.

In order $F^2(R)$ to exist we must have $D(U(F(R))) \equiv 2 \pmod 3$, which implies $p = 2 + 3p'$. Thus, $\frac{k}{2} = 238 + 243p'$ and $F^2(R) = 249 + 256p' > \frac{k}{2}$.

In order that $F^3(R)$ and $G(F^2(R))$ to exist it is necessary that $F^2(R) \equiv 2 \pmod 3$, which implies $p' = 2 + 3p''$ (and $k = 1448 + 1458p''$), and so $F^2(R) = 761 + 768p'' \equiv 5 \pmod{12}$.

As $U(F(R)) = 59 + 64p$ must be congruent to 2 module 3, in order $G(F(R))$ to exist, then $p = 3p'$ which gives $\frac{k}{2} = 76 + 243p'$ and $G(F(R)) = 78 + 256p' > \frac{k}{2}$.

The existence of $F(G(F(R)))$ and $G^2(F(R))$ implies $G(F(R)) \equiv 2 \pmod 3$ and therefore $p' = 2 + 3p''$. Thus, $G(F(R)) = 590 + 768p'' \equiv 2 \pmod{12}$ and $k = 476 + 486p''$.

(4) From (2) we have $G(R) = 14 + 96p$ and $k = 26 + 162p$. Thus $D(U(G(R))) = 18 + 128p$. In order $F(G(R))$ to exist we must have $D(U(G(R))) \equiv 2 \pmod 3$, which implies $p = 1 + 3p'$. Thus, $\frac{k}{2} = 94 + 243p'$ and $F(G(R)) = 97 + 256p' > \frac{k}{2}$.

Analogously, in order $U(U(G(R)))$ to exist we must have $U(G(R)) \equiv 2 \pmod 3$, which implies $p = 2 + 3p'$. So $\frac{k}{2} = 175 + 243p'$ and $G^2(R) = 182 + 256p' > \frac{k}{2}$. In order $F^2(G(R))$ and $G(F(G(R)))$ to exist it is necessary that $F(G(R)) \equiv 2 \pmod 3$, which implies $p' = 1 + 3p''$ and $F(G(R)) = 353 + 768p'' \equiv 5 \pmod{12}$.

Similar computations show that $F(G^2(R))$ and $G^3(R)$ exist only if $p' = 3p''$, that is $G^2(R) = 182 + 768p'' \equiv 2 \pmod{12}$. Note that in this case $k = 350 + 1458p''$.

(5) From (4), $G^2(R) = 182 + 768p$, $k = 350 + 1458p$ and so $D(U(G^2(R))) = 242 + 1024p$. Therefore $F(G^2(R))$ exist only if $D(U(G^2(R))) \equiv 2 \pmod 3$ which implies $p = 3p'$. Thus $\frac{k}{2} = 175 + 2187p'$ and $F(G^2(R)) = 161 + 2048p' < \frac{k}{2}$.

Similar computations for $G^3(R)$ yield to $p = 2 + 3p'$, $\frac{k}{2} = 1633 + 2187p'$ and $G^3(R) = 1526 + 2048p' < \frac{k}{2}$. $\square$

Proposition 5. *If Conjecture 1 fails for some* $k \equiv 8 \pmod{18}$, *then the orbit of* $k/2$ *is periodic*.

Proof. The failure of Conjecture 1 for some $k \equiv 8 \pmod{18}$ means that the corresponding Collatz tree has a vertex equal to $\frac{k}{2}$. This is equivalent to say that the orbit of $\frac{k}{2}$ contains $\frac{2k-1}{3}$. On the other hand, applying the Collatz function f to the (odd) number $\frac{2k-1}{3}$ we get $f(\frac{2k-1}{3}) = k$ and $f(k) = \frac{k}{2}$ since k is even. $\square$

Similar analysis to the one presented in Propositions 3 and 4 can be carried out for the next vertices of the Collatz tree, leading us to the conclusion that if there is a subtree of the Collatz tree ending in the vertex $\frac{k}{2}$ it would have at least 12 vertices. Therefore if a periodic orbit of the Collatz function, different from the orbit of 1, exists it can not have a period less than 13. For instance the subtree ending in $G^3(R)$ has 10 vertices all different from $\frac{k}{2}$. As $G^3(R) < \frac{k}{2}$ any subsequent vertex, say V, is different from $\frac{k}{2}$ by Lemma 3. Thus if a child of V is equal to $\frac{k}{2}$ the subtree ending in this vertex has 12 vertices and so the orbit of $\frac{k}{2}$ has period 13.

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