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PARTITIONING THE PAIRS AND TRIPLES OF TOPOLOGICAL SPACES

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We carry out the task given by the title, introduce a combinatorial principle, and use it to prove $X \nrightarrow (\text{top } \omega+1)^2_\omega$ for all spaces X, $X \nrightarrow (Y)^3_\omega$ for all spaces X where Y is any nondiscrete countable space, and related results.

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Introduction

The partition calculus for topological spaces arises from the ordinary partition calculus for cardinals, as expounded, for example, in [2]. The notation, for X and Y topological spaces,

$$X \to (Y)^n_\lambda$$

means that whenever f is a partition of the n element subsets of X into λ pieces, $f:[X]^n \to \lambda$, there is an $H \subset X$ such that

(i) H is homogeneous for f, $|f''[H]^n| = 1$, and

(ii) H is homeomorphic to Y.

For certain Y, we sometimes use "top" to distinguish the topological properties of Y from the order-theoretic or cardinality properties; for example.

Theorem 0.1. *For any finite n and m, $\omega_1 \to (\text{top } \omega+1)^n_m$.*

Here the usual order topology is the topology on ordinals. By the way, unless otherwise stated, all topological spaces are assumed to be regular. Theorem 0.1 is well known to those who work in the partition calculus. For completeness, we shall later prove a lemma which will give Theorem 0.1 as well as the following.

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Theorem 0.2. *For any uncountable cardinal κ, there is a space X such that*

$$X \to (B_\kappa)^2_\omega.$$

Here B_κ denotes the subspace of $\kappa+1$ formed by deleting all limit ordinals less than κ. Note B_ω is $\omega+1$.

Theorem 0.3. *If κ is weakly compact and n is finite, then*

$$\kappa^+ \to (B_\kappa)^n_\omega.$$

Theorem 0.4. *It is relatively consistent with ZFC that there exists a space X such that for all $f:[X]^2 \to \omega$ there is a countable nondiscrete homogeneous set.*

Proof. All we need is a hereditarily separable space of cardinality greater than 2^ω; see [3]. Now by the Erdös-Rado Theorem $(2^\omega)^+ \to (\omega_1)^2_\omega$, so that any $f:[X^2] \to \omega$ must have a separable homogeneous set of size ω_1; hence a countable nondiscrete subset. $\square$

After verifying the above theorems, the purpose of this article will be to show that they cannot be significantly extended. In particular, we prove:

Theorem 0.5. *If $X \to (Y)^2_2$, then Y is scattered.*

Recall that $\square_\kappa$ is the following statement, originally formulated by R. Jensen.

There is a sequence $\{C_\alpha: \alpha < \kappa^+$ and α a limit ordinal$\}$ such that
(i) C_α is closed and unbounded by α;
(ii) $\mathrm{cf}(\alpha) < \kappa$ implies $|C_\alpha| < \kappa$;
(iii) if β is a limit point of C_α, then $C_\beta = \beta \cap C_\alpha$. $\square_\kappa$

Theorem 0.6. *Assume that for each cardinal κ of cofinality ω we have $\kappa^\omega = \kappa^+$ and $\square_\kappa$. For any X, $X \not\to (\mathrm{top}\ \omega+1)^2_\omega$.*

Theorem 0.7. *Assume that for each cardinal κ of cofinality ω we have $\kappa^\omega = \kappa^+$ and $\square_\kappa$. For any X, there is a partition $f:[X]^3 \to \omega$ such that any countable homogeneous subset is discrete.*

We had originally proved Theorem 0.5 for a slightly smaller class of topological spaces. We thank K. Kunen and S. Todorčević for giving us the improved version stated here. We will point out the exact topological and set-theoretic assumptions used for each of these theorems in the lemmas we prove later. In particular we introduce a new, consistent combinatorial principle, SHEL(κ).

The set-theoretic assumptions for Theorems 0.6 and 0.7 are relatively consistent; in particular following from the axiom of constructibility or the nonexistence of $0^\#$, see [1].

Theorem 0.8. *Assume $0^\#$ does not exist. Then for all X, $X \to (Y)^{<\omega}_{\omega}$ implies Y is discrete.*

Proof. This uses the deep theorems of others see [1], who proved that if $0^\#$ does not exist, then for all cardinals κ of cofinality ω we have $\kappa^\omega = \kappa^+$ and $\square_\kappa$, and for all cardinals λ, $\lambda \not\to (\omega_1)^{<\omega}_{\omega}$. We can then apply Theorem 0.7. $\square$

For more on the partition calculus of topological spaces, see [4].

1. A positive partition lemma

In this section we prove a result which will give immediate proofs of Theorems 0.1–0.3. The method of proof is not new.

Lemma 1.1. *Let n and μ be cardinals greater than 1 with n finite. Let κ be any regular cardinal such that $\kappa \to (\kappa)^{n-1}_{\mu}$. If λ is any cardinal such that $\lambda^{<\kappa} = \lambda > \mu$, then*

$$\lambda^+ \to (B_\kappa)^n_\mu.$$

Proof. Let $f:[\lambda^+]^n \to \mu$ be a partition. We will first construct a particular kind of *end-homogeneous* set for f. That is, a set $S \subset \lambda^+$ and an ordinal $\alpha \in \lambda^+$ such that

(i) $S \subset \alpha$ and S is cofinal in α, $\mathrm{cf}(\alpha) = \kappa$ and

(ii) if $\{\beta_1, \beta_2, \ldots, \beta_n, \beta_{n+1}\} \subset S \cup \{\alpha\}$ and $\beta_1 < \beta_2 < \cdots < \beta_n < \beta_{n+1}$, then

$$f(\{\beta_1, \beta_2, \ldots, \beta_{n-1}, \beta_n\}) = f(\{\beta_1, \beta_2, \ldots, \beta_{n-1}, \beta_{n+1}\}).$$

Given such an α and an S we can assume that the order type of S is κ. Define a partition $g:[S]^{n-1} \to \mu$ by

$$g(\{\beta_1, \ldots, \beta_{n-1}\}) = f(\{\beta_1, \ldots, \beta_{n-1}, \alpha\}).$$

Since $\kappa \to (\kappa)^{n-1}_{\mu}$, there is a homogeneous $T \subset S$ for g of cardinality κ. Condition (ii) shows that $T \cup \{\alpha\}$ is homogeneous for f. Clearly, $T \cup \{\alpha\}$ contains a subspace homeomorphic to B_κ.

It now remains only to find S and α satisfying (i) and (ii). If no such S and α existed, then for each $\alpha < \lambda^+$ with $\mathrm{cf}(\alpha) = \kappa$, there would be a set $S_\alpha \subset \alpha$ such that (ii) holds for S_α, $|S_\alpha| < \kappa$ and if $\sup(S_\alpha) < \beta < \kappa$, then (ii) does not hold for $S_\alpha \cup \{\beta\}$. We can then construct a regressive function h by $h(\alpha) = \sup(S_\alpha)$. By Fodor's theorem there is a stationary set $X \subset \lambda^+$ such that

$$\{\alpha, \beta\} \subset X \quad \text{implies} \quad \sup(S_\alpha) = \sup(S_\beta).$$

Now, since $\{S_\alpha : \alpha \in X\}$ has cardinality at most $\lambda < \kappa$ we have a stationary $Y \subset X$ such that

$$\{\alpha, \beta\} \subset Y \quad \text{implies} \quad S_\alpha = S_\beta.$$

Similarly, it is now possible to find $\beta < \alpha$ in Y such that for all $\{\beta_1, \ldots, \beta_{n-1}\} \subset S_\alpha = S_\beta$ we have $f(\{\beta_1, \ldots, \beta_{n-1}, \beta\}) = f(\{\beta_1, \ldots, \beta_{n-1}, \alpha\})$. This means that (ii) holds for $S_\alpha \cup \{\beta\}$ and α, and completes the proof. $\square$

2. The right separation of homogeneous sets

In this section we first prove a lemma from which Theorem 0.5 follows immediately.

Lemma 2.1. *If X is a Hausdorff (not necessarily regular) space, then there is a partition $f:[X]^2 \to 2$ which has no dense-in-itself homogeneous set.*

Proof. Let $<$ be a well ordering of X and let $<'$ be a well ordering of all disjoint ordered pairs $\langle U_0, U_1\rangle$ of open subsets of x. Define a partition $f:[X]^2 \to 2$ as follows. If $x<y$ and $\langle U_0, U_1\rangle$ is the $<'$ first pair of open sets separating x and y, then $f(\{x, y\})=0$ if and only if $x \in U_0$ and $y \in U_1$.

Now let's show that f works. Let H be a dense-in-itself homogeneous set for f such that the $<$ order type of H is minimal. Let $\langle U_0, U_1\rangle$ be the $<'$ first pair separating some points of H. Let these two points be $x \in U_i$ and $y \in U_{1-i}$ with $x<y$. Now $U_i \cap H$ is an open subset of H, hence dense-in-itself. Therefore $U_i \cap H$ must be cofinal in H, and so there is some $z \in U_i \cap H$ with $y<z$. It is now easy to complete the proof by showing that $\{x, y, z\}$ is not homogeneous for f. □

By increasing the number of pieces in the partition to ω we can get a finer result, which is necessary for the proofs of Theorems 0.6 and 0.7. Recall that a space is scattered if and only if it is right separated by some well ordering. A space X is *right separated* by the well ordering $<$ means that every $<$ initial segment of X is an open subspace.

Lemma 2.2. *Let X be any regular space and let $<$ be any well ordering of X. There is a partition $f:[X]^2 \to \omega$ such that any set homogeneous for f is right separated by $<$.*

Proof. The property of regular spaces we need here is the G_δ *partition property*: for any two points x and y there is a partition of X into two G_δ sets A and B such that $x \in A$ and $y \in B$.

Now let $<'$ be a well ordering of all pairs of the form

$$p=\left\langle \bigcap_{j<\omega} U^p(0,j), \bigcap_{j<\omega} U^p(1,j)\right\rangle.$$

which partition X into two pieces such that for each $i \in \{0,1\}$ and each $j \in \omega$, $U^p(i,j)$ is an open set. We define the partition $f:[X]^2 \to \omega$ as follows. Let $x<y$ and p be the $<'$ first G_δ partition to separate x and y. Then $f(\{x,y\})=2^i3^j$ if and only if $x \in \bigcap_{j<\omega} U^p(i,j)$ and j is the least integer such that $y \notin U^p(i,j)$ and $x \notin U^p(1-i,j)$.

Now let's show that f works. Suppose H is a set homogeneous for f which is not right separated by $<$; that is, there is some $x \in H$ which is in the closure of $I=\{y \in H: x<y\}$. Let J be an initial segment of I minimal with the property that x is in the closure of J. Let p be the $<'$ first G_δ partition which separates x and some other point of J. Call this point y and let $f(\{x,y\})=2^i3^j$. If there was some

$z \in J$ with $y < z$ and $z \in \bigcap_{j<\omega} U^p(i, j)$, then the minimality of p would imply that $\{x, y, z\}$ is not homogeneous; hence

$$K = \{z \in J: y < z\} \subset \bigcap_{j<\omega} U^p(1-i, j).$$

Now, by homogeneity, $K \cap U^p(i, j) = \emptyset$. Thus $x \notin \bar{K}$. But by the minimality of J, x is not in the closure of $J \setminus K$ either. This contradiction completes the proof. □

3. The left separation of homogeneous sets

A well ordering of a topological space is said to *left separate* the space if every initial segment of the ordering is a closed subspace. We will use this concept to prove Theorems 0.6 and 0.7. For example, for Theorem 0.6 we will construct a well ordering $<$ of the space X and a partition $f_1: [X]^2 \to \omega$ such that any homogeneous topological copy of $\omega + 1$ is left separated by $<$. We then invoke Lemma 2.2 to obtain $f_2: [X]^2 \to \omega$ such that any homogeneous subspace is right separated by $<$. For the combined partition $f: [X]^2 \to \omega$ defined by

$$f(\{x, y\}) = 2^{f_1(\{x,y\})} 3^{f_2(\{x,y\})},$$

there can be no homogeneous topological copy of $\omega + 1$ since it would have to be both left and right separated and hence discrete.

The left separating well orders will be determined by an inductive process. Critical to the proofs will be the following *sheltering principle*. For an uncountable cardinal κ, we define SHEL(κ) to be the following statement:

> For each limit ordinal $\alpha < \kappa^+$ there exists S_α such that:
> (i) $S_\alpha \subset [\alpha]^\omega$ and $|S_\alpha| = \kappa$,
> (ii) there is a covering $\alpha = \bigcup\{P_\alpha^m: m < \omega\}$ such that for each m, $[P_\alpha^m]^\omega \subset \bigcup\{S_\beta: \beta < \alpha\}$.

The name comes from the fact that we can shelter "most" of the countable subsets of κ^+ with the sequence of collections S_α. This principle is particularly useful whenever κ has countable cofinality. Warning! SHEL(ω) is false.

Lemma 3.1. *If* $\kappa^\omega = \kappa$, *then* SHEL(κ) *holds.*

Proof. Just let $S_\alpha = [\alpha]^\omega$ and each $P_\alpha^m = \alpha$ if $\mathrm{cf}(\alpha) > \omega$ and each P_α^m be bounded in α if $\mathrm{cf}(\alpha) = \omega$. □

Lemma 3.2. *Suppose* $\mathrm{cf}(\kappa) = \omega$, *and* $\lambda < \kappa$ *implies* $\lambda^\omega < \kappa$, *and* $\square_\kappa$ *holds. Then* SHEL(κ) *holds.*

Proof. Let $\{\kappa_m: m < \omega\}$ be a sequence of cardinals cofinal in κ. We shall first construct a matrix $\{A_\alpha^m: m < \omega, \alpha < \kappa^+\}$ of subsets of κ^+ such that for all $m < \omega$ and all $\alpha < \kappa^+$:

(1) $A_\alpha^m \subset A_\alpha^{m+1}$ and $\alpha = \bigcup\{A_\alpha^m: m < \omega\}$ and $|A_\alpha^m| \leq \kappa_m$,

(2) if $B \in [A_\alpha^m]^\omega$ and $\mathrm{cf}(\alpha) > \omega$, then there is some $\beta < a$ such that $B \in [A_\beta^m]^\omega$.

For the construction we will use the $\square_\kappa$ sequence $\{C_\alpha: \alpha < \kappa^+, \alpha \text{ a limit ordinal}\}$, and build recursively on $\alpha < \kappa^+$. In particular, for stage $\alpha + 1$, we let each $A^m_{\alpha+1} = A^m_\alpha \cup \{\alpha\}$.

Now suppose α is a limit ordinal and $\{A^m_\beta: m < \omega, \beta < \alpha\}$ has been constructed. Consider C_α; by property (ii) from $\square_\kappa$ we can choose the least n such that $|C_\alpha| \leq \kappa_n$. If $m < n$, let $A^m_\alpha = \emptyset$; but if $m \geq n$, let

$$A^m_\alpha = \bigcup\{A^m_\beta: \beta \in C_\alpha\}.$$

It is easy to see that condition (1) is satisfied.

Considering condition (2), suppose $\mathrm{cf}(\alpha) > \omega$ and $B \in [A^m_\beta]^\omega$; there exists a limit ordinal $\beta < \alpha$ such that $B \subset \bigcup\{A^m_\eta: \eta < \beta\}$, and by property (iii) of $\square_\kappa$ this set is A^m_β.

We now define the P^m_α and S_α for SHEL(κ). If $\mathrm{cf}(\alpha) \neq \omega$ we let $P^m_\alpha = A^m_\alpha$ and $S_\alpha = \bigcup\{[P^m_\alpha]^\omega: m < \omega\}$. If $\mathrm{cf}(\alpha) = \omega$, we choose $\{\alpha_n: n < \omega\}$ cofinal in C_α and let

$$\{P^m_\alpha: m < \omega\} = \{A^m_{\alpha_n}: m < \omega, n < \omega\},$$

and let S_α be defined as before. The cardinal arithmetic hypothesis shows that (i) for SHEL(κ) is satisfied, while condition (2) takes care of (ii). $\square$

Definition. A subset of a topological space X is said to be *σ-sequentially closed* whenever it can be partitioned into countably many sets, each of which is sequentially closed (i.e., contains all limit points of all of its sequences). A subset of a topological space is said to be *σ-countably closed* whenever it can be partitioned into countably many pieces, each of which contains all limit points of all its countable subsets.

Lemma 3.3. *Assume* SHEL(κ) *and τ is a Hausdorff topology on the set κ^+. Then $\{\alpha \in \kappa^+$: α is σ-sequentially closed$\}$ contains a c.u.b.*

Proof. Notice that κ must be at least 2^ω. For any set $X \subset \kappa^+$ denote by $\mathrm{sq}(X)$ the sequential closure of X with respect to the topology τ. If $|X| \leq \omega$, then $|\mathrm{sq}(X)| \leq 2^\omega$. For each S_α in the SHEL(κ) sequence, let $\mathrm{SQ}(S_\alpha)$ denote $\bigcup\{\mathrm{sq}(X): X \in S_\alpha\}$; each $|\mathrm{SQ}(S_\alpha)| \leq \kappa$.

It is now straightforward to show that

$$\{\alpha < \kappa^+: \text{for all } \beta < \alpha,\ \mathrm{SQ}(S_\beta) \subset \alpha\}$$

is a c.u.b. By condition (ii) of SHEL(κ), any α in this c.u.b. is σ-sequentially closed. $\square$

A similar proof gives the following.

Lemma 3.4. *Assume* SHEL(κ) *and $\kappa > 2^{(2^\omega)}$, and that τ is a Hausdorff topology on κ^+. Then the set $\{\alpha \in \kappa^+$: α is σ-countably closed$\}$ contains a c.u.b.*

The next result completes the proof of Theorem 0.6.

Lemma 3.5. *Assume that for each cardinal κ of cofinality ω and greater than 2^ω we have $\kappa^\omega = \kappa^+$ and $\square_\kappa$. For any Hausdorff space X, we can find a well ordering $<$ of*

X and a partition $f:[X]^2 \to \omega$ such that any topological copy of $\omega+1$ homogeneous for f is left separated by $<$.

Proof. We prove this by induction on the cardinality of the space X. For $|X| \leq 2^\omega$ any well ordering $<$ will suffice since $2^\omega \not\to (3)^2_\omega$.

Note that the hypotheses imply that SHEL(κ) holds for all $\kappa \leq 2^\omega$. We use this to do the successor case of the induction; we use Lemma 3.3 to find σ-sequentially closed subspaces $X_\alpha \subset X$ for each $\alpha <$ [illegible] such that each $|X_\alpha| \leq \kappa$ and

(1) if $\alpha < \alpha'$, then $X_\alpha \subset X_{\alpha'}$;

(2) if β is a limit ordinal, then $X_\beta = \bigcup\{X_\alpha : \alpha < \beta\}$, and such that $X = \bigcup\{X_\alpha : \alpha < \kappa^+\}$.

The successor step will be completed if we can show the following statement:

> *If* for each $\alpha < \kappa^+$ there is a well ordering $<_\alpha$ on X_α and a partition $f_\alpha : [X_\alpha]^2 \to \omega$ such that each topological copy of $\omega+1$ homogeneous for f_α is left separated by $<_\alpha$, *then* there is a well ordering $<$ on X and a partition $f:[X]^2 \to \omega$ such that each topological copy of $\omega+1$ homogeneous for f is left separated by $<$. (*)

We prove (*) by building f and $<$ by recursively constructing f'_α's and $<'_\alpha$'s for $a < \kappa^+$. Let $f'_0 = f_0$ and $<'_0 = <_0$; for limit α we let $f'_\alpha = \bigcup\{f'_\beta : \beta < \alpha\}$ and $<'_\alpha = \bigcup\{<'_\beta : \beta < \alpha\}$. For the successor step we build $f'_{\alpha+1}$ from f'_α and $f_{\alpha+1}$ as follows. Let $X_\alpha = \bigcup\{X^n_\alpha : n < \omega\}$, where each X^n_α is σ-sequentially closed. Then

$$f'_{\alpha+1}(\{x,y\}) = \begin{cases} f'_\alpha(\{x,y\}), & \text{if } \{x,y\} \subset X_\alpha, \\ f_{\alpha+1}(\{x,y\}), & \text{if } \{x,y\} \subset X_{\alpha+1} \setminus X_\alpha, \\ n, & \text{if } x \in X_{\alpha+1} \setminus X_\alpha \text{ and } n \text{ is minimal} \\ & \text{such that } y \in X^n_\alpha. \end{cases}$$

We define $<'_{\alpha+1}$ from $<'_\alpha$ and $<_{\alpha+1}$ as follows:

$$x <'_{\alpha+1} y \quad \text{iff} \quad \begin{cases} x <'_\alpha y & \text{and} \quad \{x,y\} \subset X_\alpha, \\ x <'_{\alpha+1} y & \text{and} \quad \{x,y\} \subset X_{\alpha+1} \setminus X_\alpha, \\ x \in X_\alpha & \text{and} \quad y \in X_{\alpha+1} \setminus X_\alpha. \end{cases}$$

If we define $f = f'_{\kappa^+}$ and $< \; = \; <'_{\kappa^+}$, it is straightforward to show that they satisfy (*).

For the limit cardinal case, notice that the cardinal arithmetic allows us to construct σ-sequentially closed subspaces $X_\alpha \subset X$ for $\alpha < \mathrm{cf}(\kappa)$ such that properties (1) and (2) above hold, each $|X_\alpha| < \kappa$, and $\bigcup\{X_\alpha : \alpha < \mathrm{cf}(\kappa)\} = X$. The statement (*) now completes the proof of the limit case as well. $\square$

Lemma 3.6. *Assume that for each cardinal κ of cofinality ω and greater than $2^{(2^\omega)}$ we have $\kappa^\omega = \kappa^+$ and $\square_\kappa$. For any Hausdorff space X we can find a well ordering $<$ of X and a partition $f:[X]^3 \to \omega$ such that any countable homogeneous set is left separated by $<$.*

Proof. The proof is almost identical to that of the previous lemma except that we invoke

$$2^{(2^\omega)} \nrightarrow (4)^3_\omega$$

and the notion of σ-countably closed as in Lemma 3.4. □

4. Further remarks

It may not have escaped the reader's attention that we are mixing a partition of pairs and a partition of triples to get a partition of triples in the proof of Theorem 0.7. For this the following lemma is necessary.

Lemma 4.1. *If $f:[X]^2 \to \omega$, then there is $g:[X]^3 \to \omega$ such that if $|H| \geq 4$ and H is homogeneous for g, then H is homogeneous for f.*

Proof. We leave it to the reader to check that the "obvious" proof, of dropping one element to make a triple into a pair, does not work. However, let $<$ be any well ordering of X and for $x<y<z$ define

$$g(\{x, y, z\}) = 2^{f(\{x,y\})} 3^{f(\{y,z\})} 5^{f(\{x,z\})}.$$

This works. □

The proofs of Theorem 0.6 and 0.7 did not require that X be a regular space, only that it was Hausdorff and satisfied the G_δ partition property of Lemma 2.2. We do not, at this time, know if a more extensive use of topological regularity could help eliminate some of the set-theoretic hypotheses in these theorems.

Also it is now unknown whether it is relatively consistent that there is some finite n such that for any regular space X, $X \to (Y)^n_\omega$ implies Y is discrete.

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