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THE GEOMETRY OF COASTLINES: A STUDY IN FRACTALS

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Abstract—The geometry of coastlines, based on an empirical study by Lewis Richardson, is presented as a way of introducing the subject of fractals developed by Benoit Mandelbrot. It is shown how the statistically self-similar nature of coastlines can be generalized to an interesting class of point sets, curves and surfaces with the same property. Brownian and fractional Brownian motion are introduced as ways of generating statistically self-similar curves with the appearance of coastlines and mountain ranges.

1. INTRODUCTION

There can be little doubt that Euclidean geometry has had a large effect on the cultural history of the world. Not only mathematics but art, architecture and the natural sciences have created their structures either from the elements of Euclidean geometry or its generalizations to projective and non-Euclidean geometries. However, by its nature, Euclidean geometry appears to be more suitable to deal with the "ordered" aspects of phenomena or as a way to describe the artifacts of civilization than as a tool to describe the "chaotic" forms that occur in nature. For example, the concepts of point, line and plane, which serve as the primary elements of Euclidean geometry, are acceptable as models of the featureless particles of physics, the horizon line of a painting or the facade of a building. On the other hand, the usual geometries are inadequate tools with which to render geometrical expressions of a cloud formation, the turbulence of a flowing stream, the pattern of lightning, the branching of trees and alveoli of the lungs or the configuration of coastlines.

In the early 1950s, a mathematician, Benoit Mandelbrot, discovered new geometrical structures suitable for describing these irregular sets of points, curves and surfaces from the natural world. He coined the word "fractals" for these entities and invented a new branch of mathematics to deal with them, an amalgam of geometry, probability and statistics. Although there is a strong theoretical foundation to this subject, it can best be studied through the medium of the computer. In fact, fractal geometry is a subject in which the mathematical objects are generally too complex to be described analytically, but it is an area in which computer experiments can lead to theoretical formulations.

Mandelbrot created his geometry in 1974 after observing fractal patterns arise in many diverse areas of research such as the structure of noise in telephone communications, the fluctuation of prices in the options market and a statistical study of the structure of language. In 1961 Mandelbrot brought his attention to an empirical study of the geometry of coastlines carried out by a British meteorologist, Lewis Richardson. Mandelbrot was able to comprehend the theoretical structure behind Richardson's data and see how this structure could be generalized and abstracted. This article is devoted to a discussion of how Richardson's work on the geometry of coastlines led Mandelbrot to formulate his fractal geometry, and it is meant to serve as an introduction to Mandelbrot's work. Mandelbrot's recent book, *The Fractal Geometry of Nature*[1] is the primary reference for this article, and several of its figures have been reproduced.

2. HISTORICAL PERSPECTIVE

Before beginning a discussion of Mandelbrot's analysis of Richardson's data, it is useful to place the subject of fractals in historical perspective.

Calculus was invented in the latter part of the 17th century by Newton and Leibnitz in order to deal mathematically with the variation and changes observed in dynamic systems, such as the movement of the planets and mechanical devices. Through calculus, the concepts of "continuity" and "smoothness" of curves were quantified. However, since the approach to this subject was intuitive, "continuity" and "smoothness" were limited in their range of

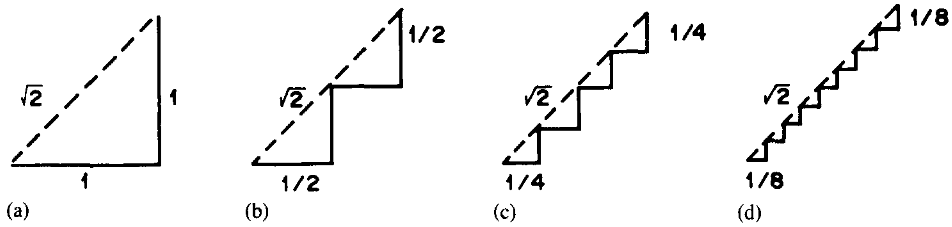


Fig. 1. Four stages in the generation of a nowhere-smooth curve.

possibilities to "ordered" or "tame" motions. It was only when the logical foundation to the subject was completed by the middle of the 19th century that mathematicians were motivated to search for extreme examples of variability with which to test the now rigorous definitions of continuity and smoothness. Much to the surprise of the mathematical community, Weirstrass, Georg Cantor and Giuseppe Peano were able to create "pathological" or "chaotic" curves and point sets that confounded intuition. Weirstrass constructed a curve spanning finite distance that was infinite in length, continuous but nowhere smooth. Peano created a curve that could fill up a square without crossovers (although segments of the curve touched). Cantor constructed a set of points as numerous as all the points within a unit interval but was so sparse as to take up negligible space or, in mathematical terms, "be of measure zero," justifying its characterization by Mandelbrot as a "dust."

Most mathematicians considered these "pathological" creations to be interesting curiosities but of little importance. However, Mandelbrot, following on the footsteps of his teacher, Paul Levy, the father of modern probability theory, saw these irregular curves and sets as models for the shapes and forms of the natural world, and he formulated his fractal geometry in order to "tame" the "monstrous" curves.

A simple example demonstrates how pathological curves can easily be constructed to confound intuition and show that basing mathematical conclusions on visual perceptions can lead to error. Compare the lengths of the sequence of zig–zag curves shown in Fig. 1. Clearly the segment lengths of each curve sum to 2 units. It is easy to see that the steps can be made small enough so that, to appearance, they are indistinguishable from the linear segment between initial and final points with length $\sqrt{2}$.

3. COASTLINES

Let us now consider the geometry of coastlines. Most people's concept of a coastline derives from two sources, observations of coastlines on a map and a vacaction visit to the seashore. However, both of these experiences yield different impressions. A coastline shown in an atlas gives the impression of the coast as a comparatively smooth and "ordered" curve of finite length easily determined from the scale of the map. However, a visit to the coast generally reveals rugged and "chaotic" terrain with rocks jutting out to the sea and an undulating shoreline dynamically varying over time and space.

Look again at maps illustrating a coastline at an increasing sequence of scales. At a scale of 1 in. to 100,000 ft, the scale of a good roadmap, the coast appears quite smooth and indicates a bay by means of a small indenture. At a scale of 1 in. to 10,000 ft we notice within the bay a small inlet not indicated on the previous map. At 1 in. to 1000 ft a small cove is demarcated within the inlet. Finally, at 1 in. to 100 ft, the scale of a highly detailed map, the cove is seen to have almost as much detail as the original map of the entire coastline. In fact, Richardson discovered that the coastline has effectively unbounded length and is nowhere smooth; this ever increasing detail is taken into account at decreasing scales. Scales below 1 in. to 50 ft are not considered since they bring into focus irrelevant details, whereas scales above an upper cutoff are also not considered since crucial details of the coastline would be deleted.

Even more remarkable than having infinite length, Mandelbrot discovered that any segment of the coastline is "statistically self-similar" to the whole coastline. By a self-similar curve, we mean a curve such that any segment is mathematically similar to the whole, i.e. a magni-

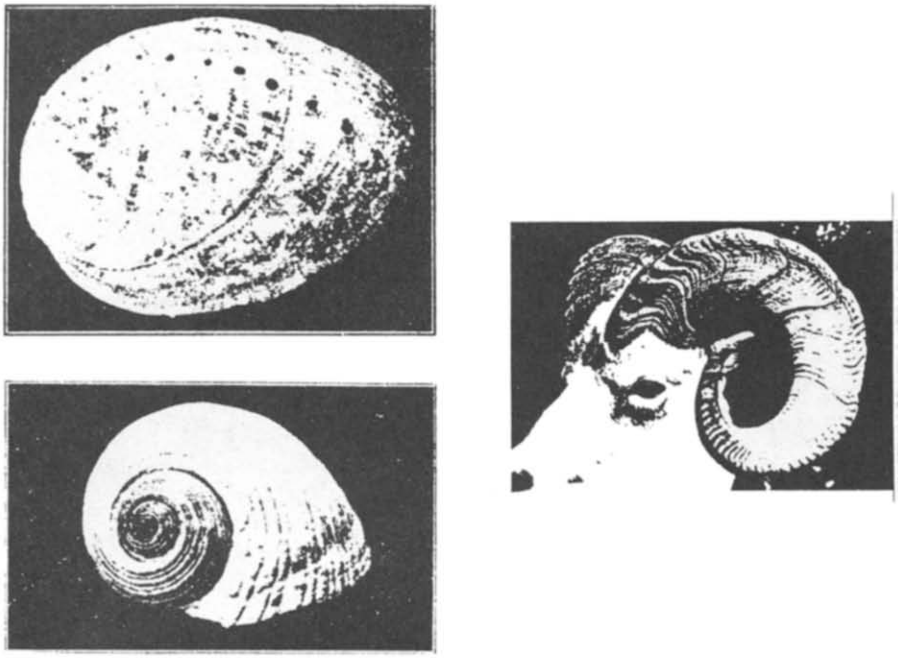


PLATE XXX
Logarithmic Spiral and Shell-Growth

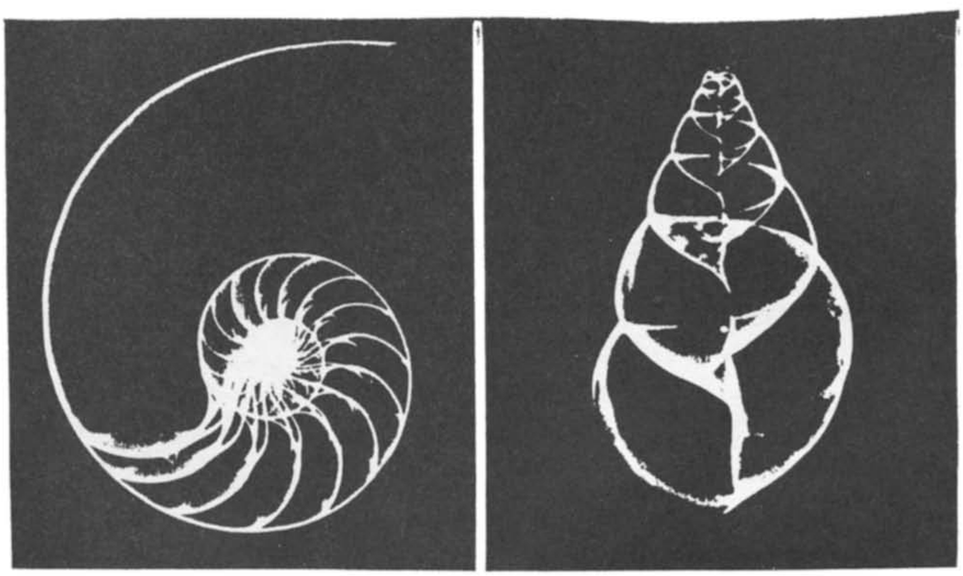


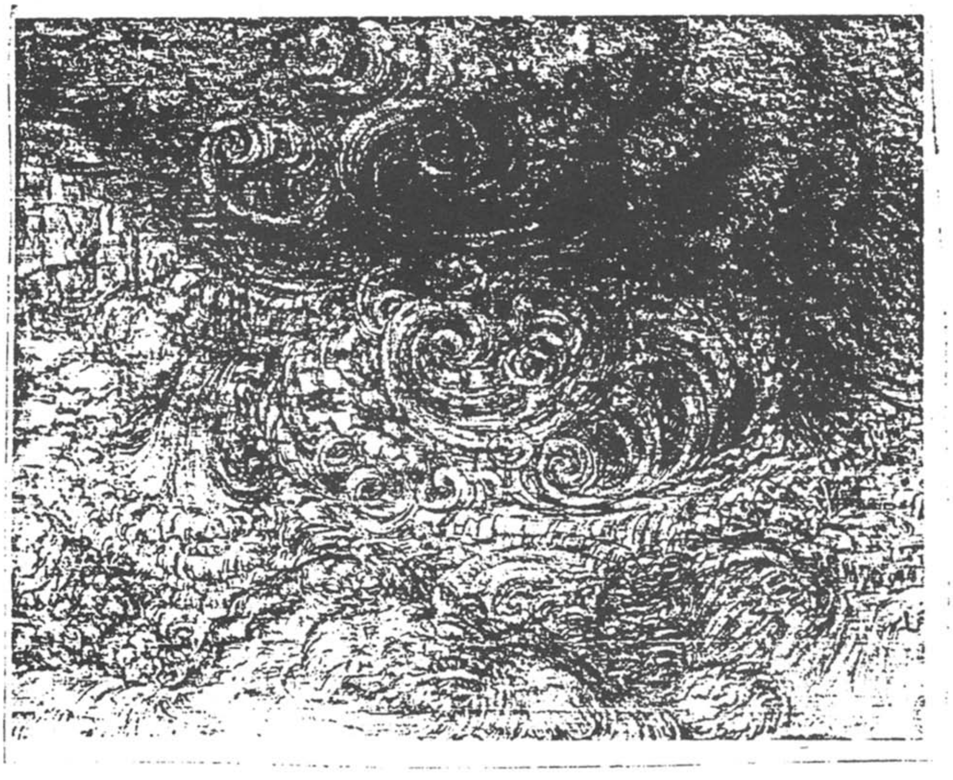
PLATE XXXV
Shell, Logarithmic Spiral and Gnomonic Growth
(Photographs: Kodak Limited)

Fig. 2. Self-similar growth in nature.

fication or contraction of the whole. It is well known that among smooth curves, only the logarithmic spiral is self-similar and this curve serves as the configuration of Nautilus shells, seashells and the horns or horned animals shown in Fig. 2.

In order to be statistically similar, two curves need only have the same statistical distribution of their features, such as ins and outs, under a magnification or contraction in their geometric scale, as will be described later in some detail. Statistical self-similarity can never be empirically verified for naturally occurring curves such as coastlines, since there are an infinite number of features associated with such curves. However, the distribution of enough aspects of these curves can be examined to conclude with a high degree of certainty whether or not two curves are statistically self-similar. Also, as we shall see, mathematical models can be formulated with exact statistical self-similarity.

The notion of self-similarity also pervades the realm of art, as shown in the hierarchal patterns of the paintings *The Great Wave* by Katsushika Hokusai and *The Deluge* by Leonardo Da Vinci, shown in Fig. 3 along with a short doggerel by Richardson reminiscent to me of the latter painting.



The Deluge, by Leonardo Da Vinci.

Big Whorls have little whorls,
which feed on their velocity;
And little whorls have lesser
whorls,
And so on to viscosity.
by L.F. Richardson

The Great Wave, by Katsushika Hokusai.

Fig. 3. Hierarchies of self-similar curves in art.

4. A GEOMETRIC MODEL OF A COASTLINE

In this section we present a mathematical model of a coastline that is geometrically self-similar at a sequence of scales and infinite in length. In the next section, we will present a more sophisticated model that also embodies statistical self-similarity. In order to describe these mathematical models, we will give operational definitions to the following basic concepts: self-similarity, scale, length of a curve, dimension, geometrical fractals and random fractals.

4.1 *Length and scale of a coastline*

Viewing a curve at a given scale and the definition of its length are two intimately connected notions. There are many different ways to represent a curve at a given scale. One method is illustrated in Fig. 3, where the curve on the left, spanning the unit interval [0, 1], is shown on the right at scales of 1, $\frac{1}{3}$ and $\frac{1}{9}$ in Figs. 3(a), (b) and (c) respectively. The scaled curves are derived from the actual curve by subdividing the curve with dividers set to intervals of length

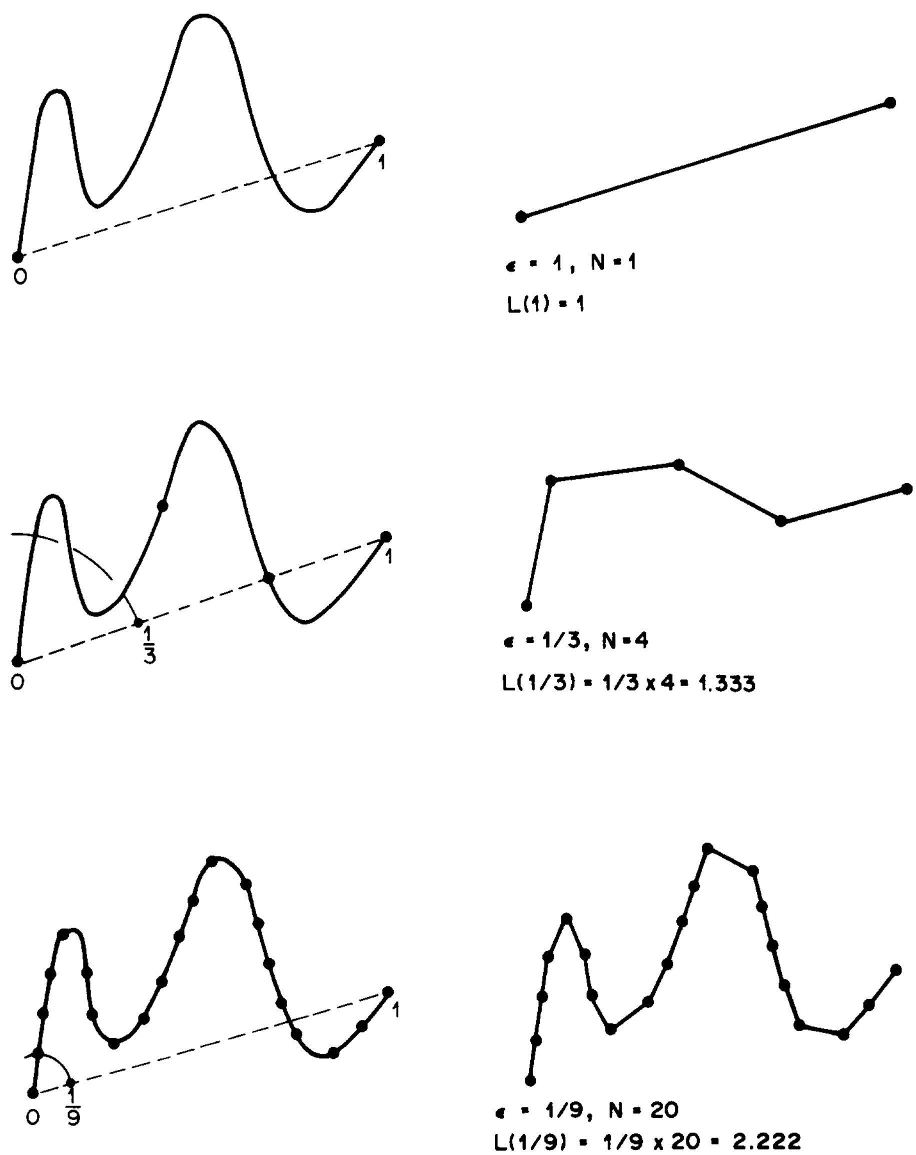


Fig. 4. Determination of the length L of a curve spanning [0, 1] by approximating the curve with N linear line segments. (a) Representation of curve at scale of $\epsilon = 1$; (b) representation of curve at scale of $\epsilon = \frac{1}{3}$; (c) representation of curve at scale of $\epsilon = \frac{1}{9}$.

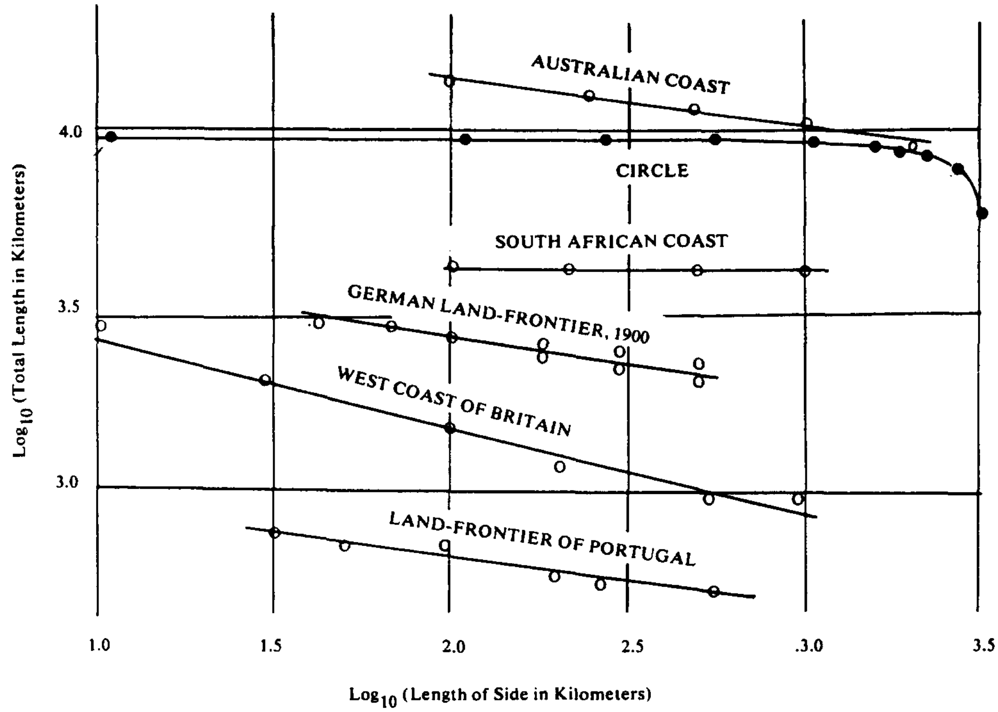


Fig. 5. Richardson's empirical data concerning the rate of increase of coastline's lengths at decreasing scales.

equal to one-third and one-ninth of the unit, starting at the beginning of the curve as illustrated by the arcs in Fig. 4. The marked points are then connected with line segments. The length of the curve, $L(\epsilon)$, at scale ϵ is then defined by,

$$L(\epsilon) = \epsilon \times N(\epsilon), \tag{1}$$

where $N(\epsilon)$ are the number of segments of length ϵ that span the curve. The total length, L, of the curve is then defined as the limiting value that $L(\epsilon)$ approaches as ϵ approaches zero or, mathematically,

$$L = \lim_{\epsilon \to 0} L(\epsilon) \tag{2}$$

Richardson applied this definition to determine the coastal length of many different countries, and he discovered that, for each of them, the number of segments at scale ϵ satisfied the empirical law

$$N(\epsilon) = K\epsilon^{-D}, \tag{3}$$

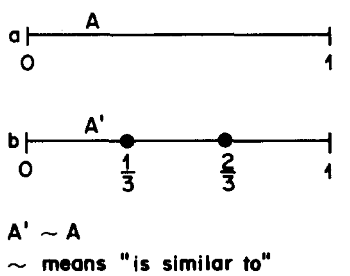


Fig. 6. The unit interval, a trivial example of a self-similar curve with dimension $D = 1$.

where K and D are constants depending on the country. Inserting eqn (3) in (1),

$$L(\epsilon) = K\epsilon^{1-D}. \tag{4}$$

The results of Richardson's analysis are shown in Fig. 5, where (4) yields a straight line when log $L(\epsilon)$ is plotted against log ϵ. Also, as a result of eqns (2) and (4), the total length of the coastline L is effectively infinite.

Richardson's data indicates that the configuration of coastlines is derived from a general law of nature, and Mandelbrot's analysis of Richardson's data led to the following expression of that law:

> Each segment of a coastline is statistically similar to the whole, i.e. the coastline is statistically self-similar.

4.2 *Geometrically self-similar curves*

In his book *The Fractal Geometry of Nature*[1], Mandelbrot presents a procedure for constructing curves that are geometrically self-similar. To understand how self-similar curves relate to Richardson's law, it is sufficient to set $K = 1$ and rewrite (4) as

$$L(\epsilon) = \epsilon\ 1/\epsilon^{D}. \tag{5}$$

First, consider a trivial example of a self-similar curve, the straight-line segment of unit length shown in Fig. 6(a). This segment is self-similar at any scale. For example, at the scale $\frac{1}{3}$, Fig. 6(b) shows that three similar editions of the segment replicates the original. Thus, from eqn (1),

$$L(\epsilon) = \tfrac{1}{3} \times 3$$

or

$$L(\epsilon) = \tfrac{1}{3} \times 1/\tfrac{1}{3}^{1},$$

and consequently $D = 1$ in eqn (5).

Now consider a less trivial example of a curve, self-similar at a sequence of scales $(\frac{1}{3})^n$, $n = 0, 1, 2, 3, \ldots$ known as the Koch snowflake. Since the curve is infinite in length, continuous and nowhere smooth, it cannot be drawn, as we explained for the example shown in Fig. 2. However, it can be generated by an infinite process, each stage of which represents the curve as seen at one of the scales in the above sequence. Figures 7(a), (b) and (c) show views of the Koch snowflake at scales of 1, $\frac{1}{3}$ and $\frac{1}{9}$, respectively both as linear segments on the left and incorporated into triangular snowflakes on the right. The snowflake is generated iteratively by replacing each segment of one stage by four identical segments of length one-third the original in the next stage. Thus, whereas for stage 1

$$L(1) = 1,$$

for stage two

$$L(\tfrac{1}{3}) = \tfrac{1}{3} \times 4$$

or

$$L(\tfrac{1}{3}) = \tfrac{1}{3} \times 1/(\tfrac{1}{3})^{D},$$

where D is easily determined as

$$D = \frac{\log 4}{\log 3} = 1.2618.\ldots$$

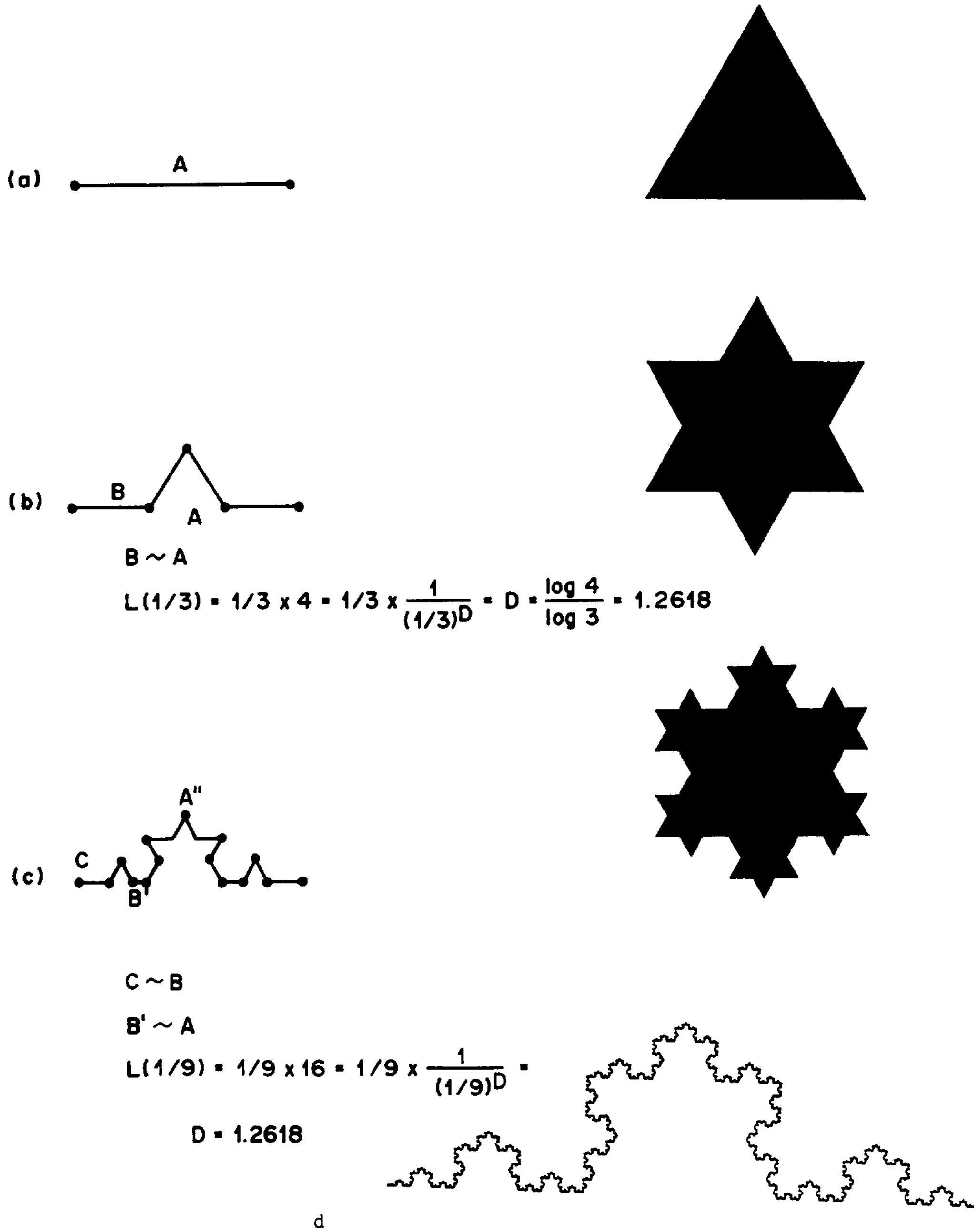


Fig. 7. The Koch snowflake, a nontrivial example of a self-similar curve with dimension $D = 1.2618$. (a) Koch snowflake at scale of $\epsilon = 1$; (b) Koch snowflake at scale of $\epsilon = \frac{1}{3}$; (c) Koch snowflake at scale of $\epsilon = \frac{1}{9}$; (d) Koch snowflake at an advanced stage in its generation.

For each successive stage,

$$L(\epsilon) = \epsilon \times 1/\epsilon^D,$$

with the same value of D. Each segment of a given stage is seen to be similar to a segment three times as large in the previous stage, as illustrated in Fig. 7. Thus, in the limit, each segment of length $(\frac{1}{3})^n$ of the Koch snowflake must be geometrically similar to the whole, satisfying both Richardson's data and Mandelbrot's interpretation of it. This property of self-similarity at a sequence of scales is more evident in Fig. 7(d) of a Koch snowflake at an advanced stage in its development.

Mandelbrot shows that, as for the Koch snowflake, any geometrically self-similar curve satisfies

$$D = \frac{\log N}{\log (1/r)}, \tag{6}$$

where N is the number of congruent segments of length r that replace the unit interval in the initial stage of the iteration. Thus, for the Koch snowflake $N = 4$ and $r = \frac{1}{3}$. Mandelbrot refers to D as the dimension of the curve, and he shows that for curves of infinite length, spanning the finite distance,

$$1 < D \leq 2,$$

The magnitude of D is a measure of the "roughness" of the curve.

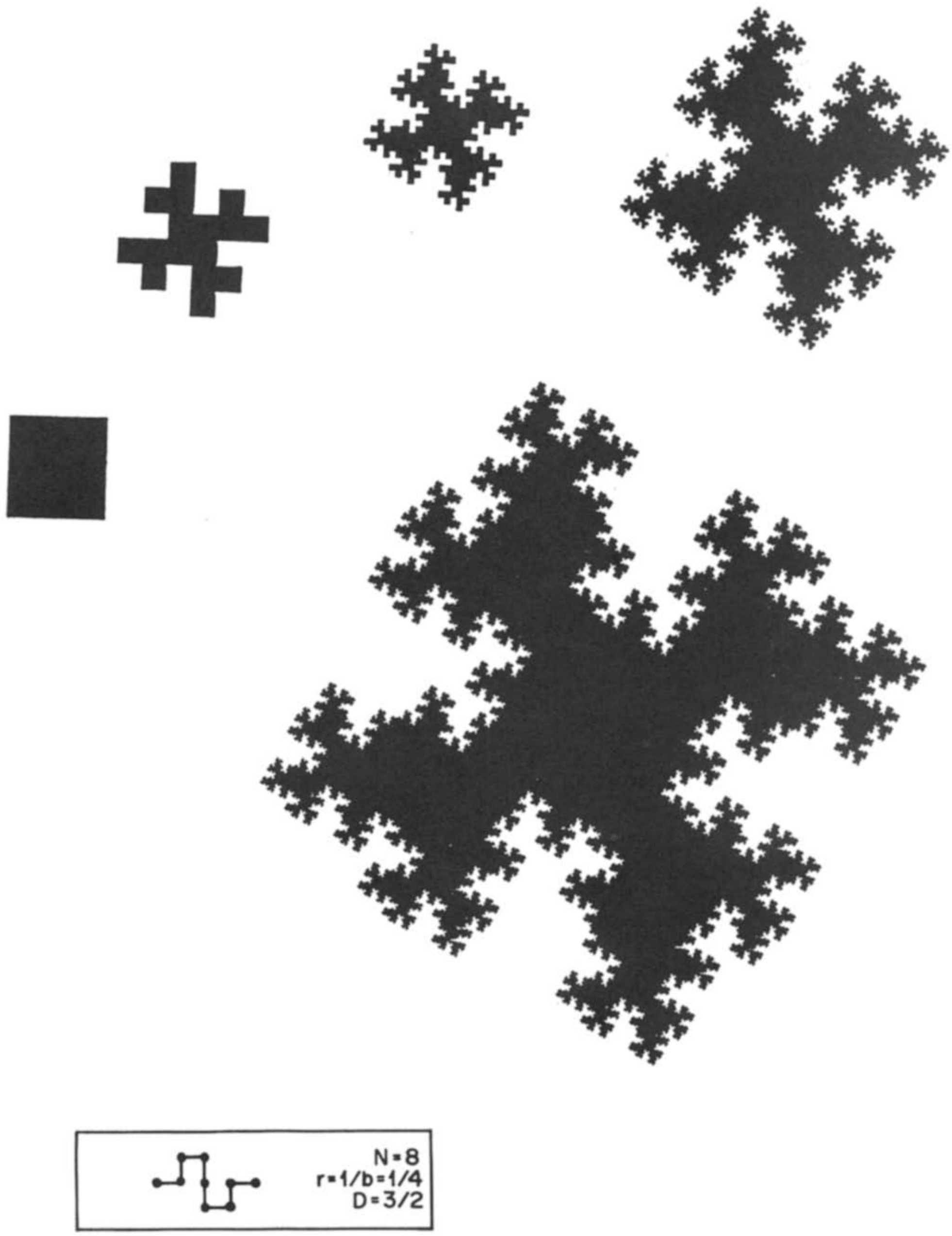


Fig. 8. Another fractal curve with dimension $D = \frac{3}{2}$.

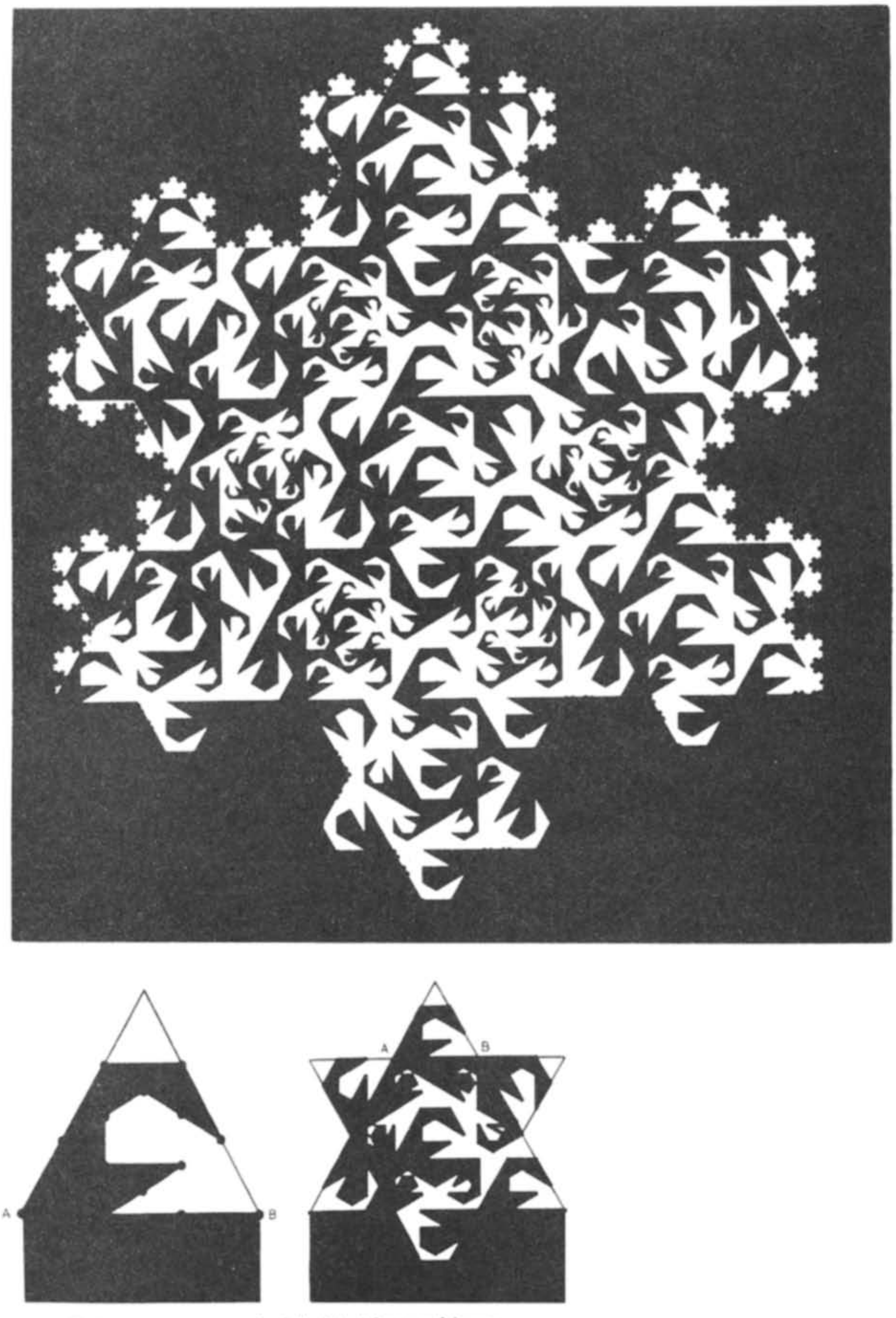


Fig. 9. The third stage in the generation of a space-filling Peano curve filling the interior of a Koch snowflake.

The relationship between N and r, expressed by eqn (6), is quite general and is illustrated for other geometrically self-similar structures in Figs. 8, 9, and 10. Figure 8 is an analogous curve to the Koch snowflake with dimension $D = \frac{3}{2}$, while Fig. 9 is the third stage of a space-filling Peano curve of dimension 2 that fills up the interior of the Koch snowflake. The point set in Fig. 11 represents six stages in the generation of what Mandelbrot calls a Cantor "dust" and has dimension $D = .6309$. Starting with the unit interval, each stage is generated from the preceding stage by removing the middle third of each remaining subinterval. What remains after an infinity of stages is an infinite set of points interspersed within empty space. The dimension is determined in the same manner as for the Koch snowflake, from eqn (6), and is

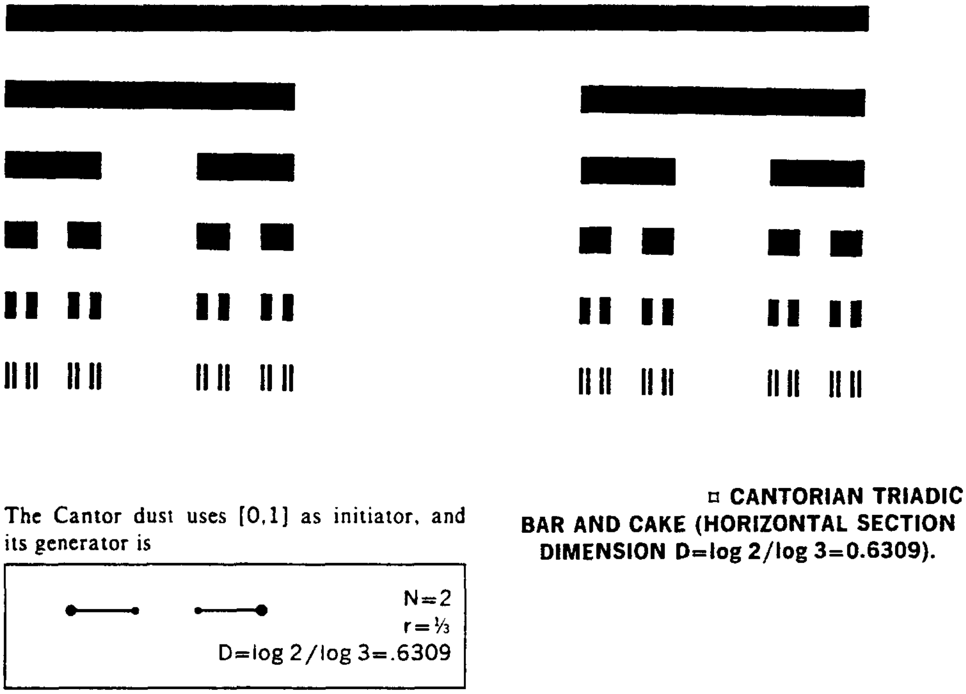


Fig. 10. Six stages in the generation of a "cantor triadic bar," an approximation to a Cantor "dust" with dimension D = .6309.

shown in Fig. 10 to be .6309. However, since it would be difficult to illustrate even a few stages in the generation of a Cantor dust, Fig. 11 shows an analogous sequence of bars with finite thickness.

Finally, Mandelbrot coined the term fractal curves to refer to curves with dimension $D > 1$, the term fractal surfaces to refer to surfaces with dimension $D > 2$, and the term fractal point sets for point sets with $D > 0$. Although, according to this definition, fractals need not be self-similar, the class of self-similar fractals generated by Mandelbrot's recursive procedure appear to be the ones of greatest interest and I refer to them as "geometrical fractals" in order to distinguish them from the statistically self-similar curves discussed in the next section, which I refer to as "random fractals."

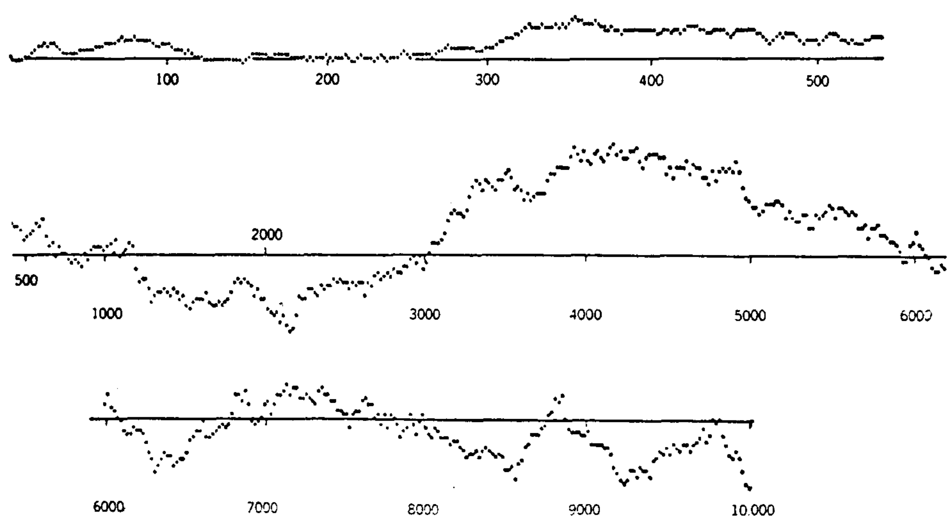


Fig. 11. A curve of one-dimensional Brownian motion. Position is plotted on the ordinate and time is plotted on the abscissa.

5. A STATISTICAL MODEL OF A COASTLINE

Although the Koch snowflake serves as a good mathematical model of the coastline, it fails to represent actual coastlines in two important respects. Its sequence of scales are bound to powers of $\frac{1}{3}$. Thus examining the curve at intervals of $\frac{1}{4}$ would yield none of its self-similar properties. Also, as irregular as the snowflake is, its structure is completely ordered, unlike coastlines. Both of these shortcomings can be removed by randomizing the fractals. Before describing how Mandelbrot introduced the element of statistics into the analysis of fractals, it is useful to revisit the linear Koch snowflake shown in Fig. 7, but this time imagine the curve to represent a spectrum of amplitudes of sound over an interval of time. Generally, a phonograph recording of sound, such as the sound of a violin, changes if the record is played fast or slow. In fact, a record of whale sounds is inaudible until the record is played at a sufficiently high speed. However, Koch snowflake music would clearly sound the same played at one-third the speed and then amplified three times. More precisely, if the amplitude at time t is represented by the function $B(t)$, the scaling property of the snowflake music is a statement about the identity of the functions $B(t)$ and $B(rt)/r$, where r is the self-similar scale. Such sounds are called scaling noises and have been studied by a colleague of Mandelbrot's, Richard Voss, at IBM Watson Research Center[2].

5.1 *Brownian motion as a generator of random fractals*

Curves that are statistically self-similar at every scale yield scaling noises that sound alike on a record played at any speed. One such curve can be simulated by playing a game in which a marker is moved forward or backwards along a line of equally spaced squares with equal

JEAN PERRIN'S CLASSIC DRAWINGS OF PHYSICAL BROWNIAN MOTION

Physical Brownian motion is described in Perrin 1909 as follows: "In a fluid mass in equilibrium, such as water in a glass, all the parts appear completely motionless. If we put into it an object of greater density, it falls. The fall, it is true, is the slower the smaller the object; but a visible object always ends at the bottom of the vessel and does not tend again to rise. However, it would be difficult to examine for long a preparation of very fine particles in a liquid without observing a perfectly irregular motion. They go, stop, start again, *mount*, descend, *mount again*, without in the least tending toward immobility."

The present plate, the only one in this book to picture a natural phenomenon, is reproduced from Perrin's *Atoms*. We see four separate tracings of the motion of a colloidal particle of radius 0.53μ, as seen under the microscope. The successive positions were marked every 30 seconds (the grid size being 3.2μ), then joined by straight intervals having no physical reality whatsoever.

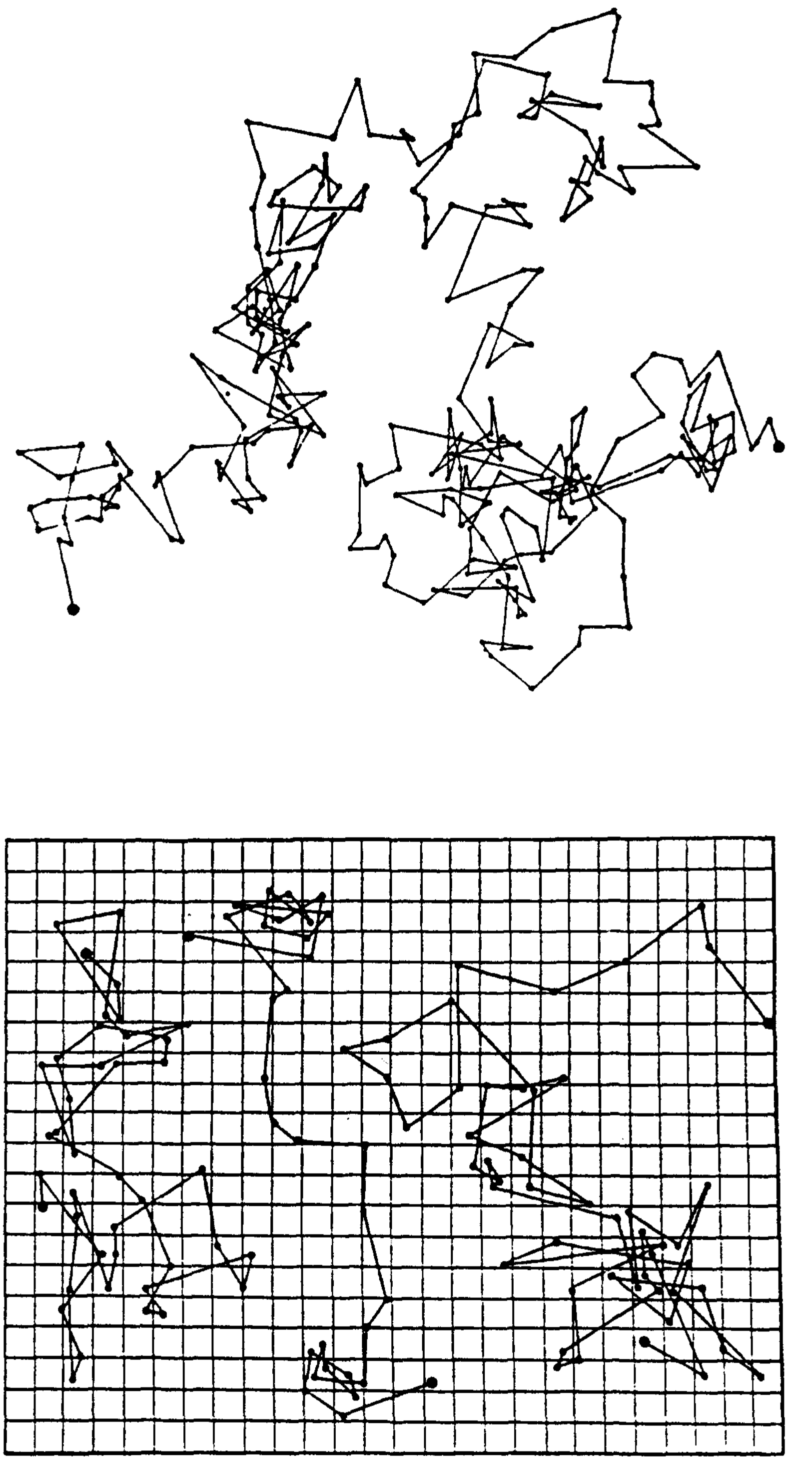
To resume our free translation from Perrin 1909, "One may be tempted to define an 'average velocity of agitation' by following a particle as accurately as possible. But such evaluations are *grossly wrong*. The apparent average velocity varies crazily in magnitude and direction. This plate gives only a weak idea of the prodigious entanglement of the real trajectory. If indeed this particle's positions were marked down 100 times more frequently, each interval would be replaced by a polygon smaller than the whole drawing but just as complicated, and so on. It is easy to see that in practice the notion of tangent is meaningless for such curves."

This Essay shares Perrin's concern, but attacks irregularity from a different angle. We stress the fact that when a Brownian trajectory is examined increasingly closely, Chapter 25, its length increases without bound.

Furthermore, the trail left behind by Brownian motion ends up by nearly filling the whole plane. Is it not tempting to conclude that in some sense still to be defined, this peculiar curve has the same dimension as the plane? Indeed, it does. A principal aim of this Essay will be to show that the loose notion of dimension splits into several distinct components. The Brownian motion's trail is *topologically* a curve, of dimension 1. However, being practically plane filling, it is *fractally* of dimension 2. The discrepancy between these two values will, in the terminology introduced in this Essay, qualify Brownian motion as being a fractal. ■

(a)

Fig. 12. Jean Perrin's drawings of physical Brownian motion. (a) Text from *The Fractal Geometry of Nature* by Mandelbrot. (b) Perrin's data.



(b)

Fig. 12. *(Continued)*

probability (by flipping a coin). If each move is considered to take place during one unit of time, the movement of the marker in time, also known as a random walk, is shown in Fig. 11 for 10,000 time units or moves. In fact, this motion follows the scaling law that $B(t)$ and $B(rt)/\sqrt{r}$ have the same probability distribution for all scales r. Following a plausibility argument of Mandelbrot's, we show in the Appendix that this curve has dimension $D = \frac{3}{2}$. Also, Mandelbrot has shown that the points where the curve $B(t)$ intersects the time axis forms a set analogous to the Cantor dust shown in Fig. 10. The set has dimension $D = \frac{1}{2}$ and any interval of the time axis has the same statistical distribution of gaps between successive zeros of $B(t)$ when the gaps are adjusted according to the scale.

Actually, the previous motion approximates a one dimensional Brownian motion of a particle. One of the first scientists to observe Brownian motion of a particle suspended in a fluid was Jean Perrin. Figure 12 and the accompanying text from the *The Fractal Geometry of Nature* is a good introduction to Brownian motion along with its property of statistical self-similarity at every scale. Thus, Brownian music is an example of a scaling noise at any scale. A corollary of the self-similarity is the fact that there is no way to statistically distinguish segments of a Brownian curve over equal intervals of time from each other. Thus, in the language of statistics, Brownian motion is a "stationary" process. Another feature of Brownian motion exhibited by these curves is the complete independence of past and future positions of the particle with respect to the present position. In other words, there is no "correlation" between where the particle has come from during the interval $[-t, 0]$ and where it is headed in the interval $[0, t]$.

In terms of statistics, the change of position of a particle undergoing Brownian motion over time, $\Delta B(t) = B(t + \Delta t) - B(t)$, is distributed with a Gaussian distribution (the famous normal distribution) with mean $\langle \Delta B(t) \rangle = 0$ and variance (the square of the standard deviation) $\langle \Delta B(t)^2 \rangle = |\Delta t|$. This information permits Brownian motion to be generated continuously in time.

Brownian motion in the plane can also be generated approximately by a random walk through a lattice of squares (graph paper) in which over each interval of time Δt there is an equal chance for the marker to move up–down and right–left. The path in the plane generated by this motion has dimension $D = 2$, which indicates that eventually every point in the plane will be crossed by the curve and the motion is plane filling. Such a path is shown in Fig. 13. It will have many self-intersections and is far too erratic to serve as a model for coastlines. As a matter of fact, Brownian motion along the line illustrated in Fig. 11 with $D = \frac{3}{2}$ is also too irregular to serve as the model of a coastline which, according to Richardson and Fig. 5, tends to have a dimension $D \simeq 1.2$ with no self-intersections.

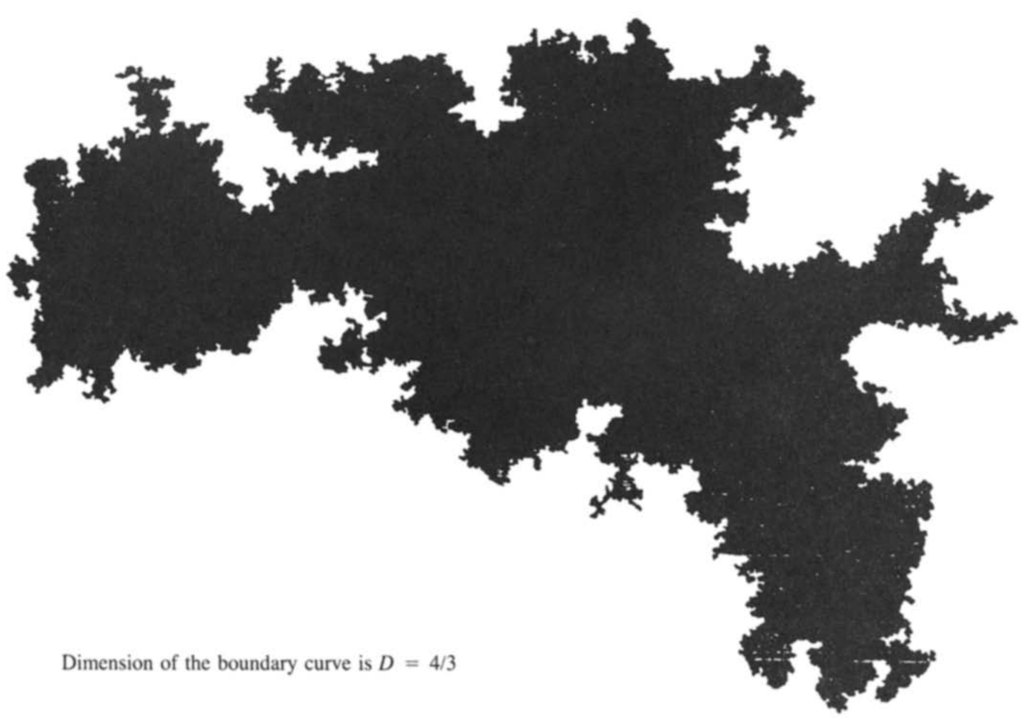


Fig. 13. A curve drawn by two-dimensional Brownian motion. The outer boundary of the curve constitutes a "random fractal" curve of dimension $D = \frac{4}{3}$. The random walk continues inside the outer boundary but is not visible in the diagram.

5.2 *Fractional Brownian motion: an improved model of coastlines*

In search of better statistically self-similar models for coastlines, Mandelbrot rediscovered a generalization of Brownian motion which he calls fractional Brownian motion, where

$$\langle \Delta B(t) \rangle = 0, \qquad \langle \Delta B(t)^2 \rangle = \Delta t^{2H}$$

and ΔB again follows a Gaussian distribution, while the resulting traces in the plane have dimension $1/H$ for $0 < H < 1$ and are statistically self-similar with $B(t) = B(rt)/r^H$. It follows that $H = \frac{1}{2}$ reproduces ordinary Brownian motion in which a particle at any given time has an equal chance to move up–down or left–right. On the other hand, the movement of a particle under fractional Brownian motion tends to persist in the direction the particle is moving when $\frac{1}{2} < H < 1$. As a result, past and present are "correlated" with respect to the present position of the particle. This motion results in few self-intersections and fractal curves of greater regularity reminiscent of coastlines. Values of H such that $0 < H < \frac{1}{2}$ result in antipersistent motions in which the curve tends to reverse its direction from moment to moment.

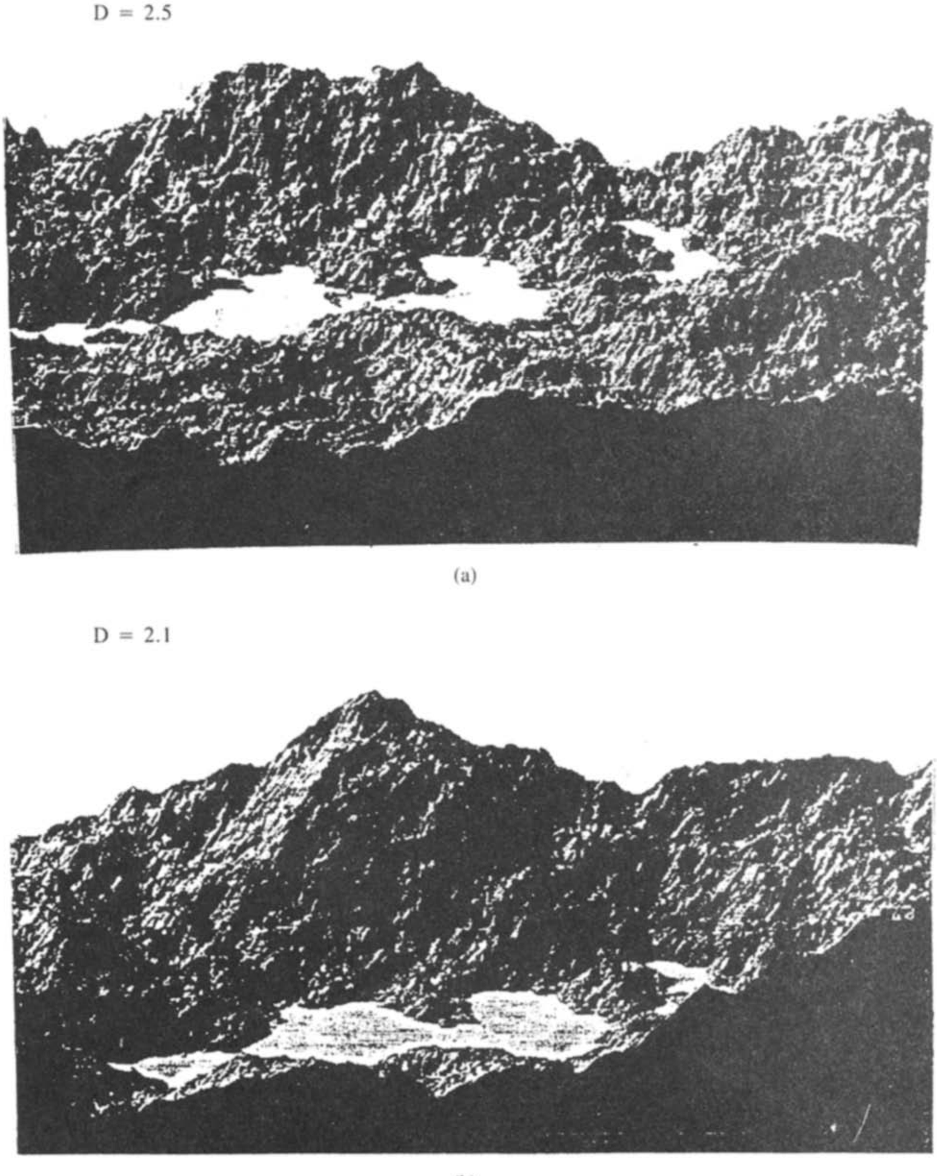


Fig. 14. Two Brownian landscapes. (a) Rough landscape is generated by Brownian motion ($D = 2.5$). (b) Smooth landscape is generated by fractional Brownian motion ($D = 2.1$).

Mandelbrot calls the tendency of a motion to "persist" in time, as persistent fractional Brownian motion does, the "Joseph effect." After all it was Joseph, during biblical times in ancient Egypt, who enhanced his position and fortunes by realizing the propensity for feast and famine to persist over seven-year cycles. Much later, in 1906, a British civil servant Harold Hurst discovered empirically, contrary to the conventional wisdom of the time, that the fluctuations in the discharges of the river Nile were not random from year to year. Hurst applied his data to the operations of the newly built Aswan dam. Mandelbrot made the important discovery that Hurst's data followed a curve of fractional Brownian motion. Since these motions were self-similar at every scale, Mandelbrot refers to noises generated with amplitude spectra that follows a fractional Brownian curve as Hurst noises.

Finally, Fig. 14 depicts mathematical models of two Brownian landscapes. The upper landscape is an ordinary Brownian model while the lower one is fractional Brownian. Both landscapes represent wrinkled surfaces with fractional dimensions between 2 and 3. As for Brownian motion on a line and in the plane, the ordinary Brownian landscape is quite rough, more like a typical mountain scene. Analogous to the way the line and plane Brownian motion is generated, the ordinary Brownian scene is generated by starting with a plane. On one side of a randomly placed line in the plane, the plane is raised or lowered one unit with respect to the other with equal probability. Another line is randomly placed on the plane, and again one side is raised or lowered with respect to the other. This morphology is repeated iteratively with each successive elevation or depression diminished in scale by a factor of $\sqrt{k}$, where k is the number of the iteration. Generation of the fractional Brownian landscape is quite technical and is not described.

Brownian curves and surfaces are fractals, self-similar at every scale, or what Mandelbrot refers to as "creaseless." However, Mandelbrot's book is filled with many practical ways to construct statistically random fractals self-similar at a sequence of scales as for the Koch snowflake.

6. CONCLUSION

In certain ways, fractal geometry has already found useful application in describing certain aspects of the natural world. It has served as a tool for geographers to study the branching of rivers and the geometry of river basins. It is being considered by land surveyers as a way to characterize patterns of vegetation and land quality. It has given some insight to astronomers as to the distribution of stars and galaxies. It is being used as a tool for the study of the turbulence of fluids. There is potential for medical researchers to begin characterizing mammalian tumors through their fractal dimensions. Realistic scenery is now being generated by computers as backdrops to movies. Most recently, a conference on fractals was convened to discuss applications of fractals to statistical physics[3]. As the ideas become more familiar, important applications to every field are expected to emerge.

Acknowledgments—I wish to thank Dr. Mandelbrot for reading this manuscript, making several corrections and helping to place the subject in the correct historical perspective. I would also like to express my appreciation to him for the extraordinary work that he has done to create a new branch of mathematics that sheds light on so much in our world.

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REFERENCES

1. B. B. Mandelbrot, *The Fractal Geometry of Nature*. W. H. Freeman, New York (1983).
2. R. F. Voss, $1/f$ noise in music; music from $1/f$ noise. *J. Accoust. Soc. Am.* **63**, 258–263 (1978).
3. M. Shlesinger (Ed.), Proceedings of the Gaithesburg Conference on Fractals. *J. Statist. Phys.* **36** (1984).

APPENDIX

A plausibility proof that the dimension of Brownian motion on a line $B(t)$ has dimension $D = \frac{3}{2}$.
Proof: The unit interval, [0, 1], is covered by $1/\Delta t$ squares of length Δt, where $r = \Delta t$ is the scale of the curve. Since the standard deviation of a Brownian motion over a unit of time Δt was given by

$$\langle \Delta B^2 \rangle^{1/2} = \sqrt{\Delta t},$$

the expected variation of the curve $B(t)$ during Δt is

$$B_{\max} - B_{\min} \simeq \sqrt{\Delta t}.$$

But the number of scale lengths equivalent to $\sqrt{\Delta t}$ is

$$\frac{\sqrt{\Delta t}}{\Delta t} = \frac{1}{\sqrt{\Delta t}}.$$

Since there are $1/\Delta t$ scale lengths,

$$N = \frac{1}{\Delta t} \quad \frac{1}{\sqrt{\Delta t}} = \Delta t^{-3/2}.$$

Thus,

$$D = \frac{\log N}{\log (1/r)} = \frac{\log \Delta t^{-3/2}}{\log \Delta t^{-1}} = 3/2.$$