Homoclinic orbits in generalized Liénard systems

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Abstract
This paper is devoted to the investigation on the existence of homoclinic orbits of the planar system of Liénard type \( \dot{x} = h(y) - F(x), \dot{y} = -g(x) \). Here \( h(y) \) is strictly increasing, but is not imposed \( h(\pm\infty) = \pm\infty \). Sufficient conditions are given for a positive orbit of the system starting at a point on the curve \( h(y) = F(x) \) to approach the origin without intersecting the \( x \)-axis. The obtained theorems include previous results as special cases. Our results are applied to a concrete system and their sharpness are improved.

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1. Introduction

We consider the Liénard-type system

\[
\dot{x} = h(y) - F(x), \quad \dot{y} = -g(x),
\]

(1.1)

where \( F(x) \) and \( g(x) \) are continuous on an open interval \( I \) which contains 0, and \( h(y) \) is continuous and strictly increasing on \( \mathbb{R} \). The functions \( F(x) \), \( g(x) \) and \( h(y) \) satisfy

\[ h(\pm\infty) = \pm\infty. \]
smoothness conditions for uniqueness of solutions of initial value problems. Throughout this paper, we assume that \( F(0) = 0 \),
\[
x g(x) > 0 \quad \text{if } x \neq 0
\]
and
\[
y h(y) > 0 \quad \text{if } y \neq 0,
\]
which guarantee that the origin is the unique critical point of (1.1). Let \( h^{-1}(v) \) be the inverse function of \( v = h(y) \). Note that \( h^{-1}(v) \) is also strictly increasing and satisfies \( v h^{-1}(v) > 0 \) if \( v \neq 0 \), and the composite function \( h^{-1}(F(x)) \) is defined on an open subinterval \( J \) of \( I \) which contains 0.

In system (1.1), a trajectory is said to be a homoclinic orbit if its \( \alpha \)- and \( \omega \)-limit sets are the origin. The purpose of this paper is to give some sufficient conditions on \( F(x) \), \( g(x) \) and \( h(y) \) under which system (1.1) has homoclinic orbits.

Homoclinic orbits play an important role in nonlinear dynamical systems. On this account, many efforts have been made on the existence of homoclinic orbits for various dynamical systems, such as Lagrangian and Hamiltonian systems, the Lorenz system and Schrödinger systems. For example, those results can be found in [1,3,8,11] and the references contained therein. However, there are few papers concerning homoclinic orbits of systems of Liénard type (refer to [2,14]). The study of homoclinic orbits is by no means an insignificant subject in Liénard dynamics. Our subject is closely connected with the stability of the zero solution and the center problem. If system (1.1) has a homoclinic orbit, then the zero solution is no longer stable. A homoclinic orbit and a center cannot exist together in system (1.1). Our subject also bears a near relation to global attractivity of the origin and oscillation of solutions and so on (see [6,7,12]).

Let \( J^+ = J \cap [0, \infty) \) and \( J^- = J \cap (-\infty, 0] \). We denote
\[
C^+ = \{(x, y): x \in J^+ \text{ and } h(y) = F(x)\}
\]
and
\[
C^- = \{(x, y): x \in J^- \text{ and } h(y) = F(x)\}.
\]
We also write \( \gamma^+(P) \) and \( \gamma^-(P) \) for the positive orbit and negative orbit of (1.1) starting at a point \( P \in \mathbb{R}^2 \), respectively. If an orbit of (1.1) crosses the \( y \)-axis, then its tangent is horizontal at the intersection point. Also, if an orbit of (1.1) meets the curve \( C^+ \) or \( C^- \), then its tangent is vertical at the point of intersection. System (1.1) has a homoclinic orbit if and only if there exists a point \( P \in \mathbb{R}^2 \) such that both \( \gamma^+(P) \) and \( \gamma^-(P) \) approach the origin.

Paying attention to the vector field of (1.1), we see that any homoclinic orbit is in the upper half-plane or in the lower half-plane; in other words, no homoclinic orbit crosses the \( x \)-axis. When a homoclinic orbit appears in the upper (respectively, lower) half-plane, all the other homoclinic orbits exist in the same half-plane. In either case, a homoclinic orbit of (1.1) always intersects the curves \( C^+ \) and \( C^- \). Hence, our problem resolves itself into the following questions:

(i) whether or not we can find a point \( P \in C^+ \) such that \( \gamma^+(P) \) approaches the origin without intersecting the \( x \)-axis;
(ii) whether or not we can find a point \(P \in C^-\) such that \(\gamma^-(P)\) approaches the origin without intersecting the \(x\)-axis;
(iii) whether or not we can find a point \(P \in C^-\) such that \(\gamma^+(P)\) approaches the origin without intersecting the \(x\)-axis;
(iv) whether or not we can find a point \(P \in C^+\) such that \(\gamma^-(P)\) approaches the origin without intersecting the \(x\)-axis.

For the sake of convenience, we say that system (1.1) has \(\text{property } (Z^+_1)\) (respectively, \(\text{property } (Z^-_2)\)) if there exists a point \(P \in C^+\) (respectively, \(C^-\)) such that \(\gamma^+(P)\) approaches the origin through only the first (respectively, third) quadrant. We also say that system (1.1) has \(\text{property } (Z^-_3)\) (respectively, \(\text{property } (Z^+_4)\)) if there exists a point \(P \in C^-\) (respectively, \(C^+\)) such that \(\gamma^-(P)\) approaches the origin through only the second (respectively, fourth) quadrant. In case system (1.1) has both property \(\text{(Z}_1^+)\) and property \(\text{(Z}_2^-)\), a homoclinic orbit exists in the upper half-plane; in case system (1.1) has both property \(\text{(Z}_3^+)\) and property \(\text{(Z}_4^-)\), a homoclinic orbit exists in the lower half-plane. To put it more precisely, if system (1.1) has a homoclinic orbit, then all trajectories of (1.1) in the region that is enclosed by the union of the homoclinic orbit and the origin are also homoclinic orbits.

We take a very simple example to illustrate properties \((Z^+_1)\) and \((Z^-_2)\). Consider the system
\[
\dot{x} = y - \rho |x|, \quad \dot{y} = -x, \tag{1.2}
\]
where \(\rho \in \mathbb{R}\). Since \(F(x)\) is even and \(g(x)\) is odd, \(\gamma^+(P) \cup \gamma^-(P)\) has mirror symmetry about the \(y\)-axis for any \(P \in \mathbb{R}^2\). It is clear that
(a) if \(\rho \geq 2\), then system (1.2) has property \((Z^+_1)\) and property \((Z^-_2)\);
(b) if \(|\rho| < 2\), then system (1.2) fails to have properties \((Z^+_1)\), \((Z^-_2)\), \((Z^+_3)\) and \((Z^-_4)\);
(c) if \(\rho \leq -2\), then system (1.2) has property \((Z^+_3)\) and property \((Z^-_4)\).

In [6], Hara and Yoneyama have considered the Liénard system
\[
\dot{x} = y - F(x), \quad \dot{y} = -g(x) \tag{1.3}
\]
and obtained sufficient conditions and necessary conditions under which the origin of (1.3) is a center. Using our terms, we can state one of their results as follows: if there exists a \(\delta > 0\) such that
\[
F(x) > 0 \quad \text{and} \quad \frac{1}{F(x)} \int_0^x \frac{g(\xi)}{F(\xi)} \, d\xi \leq \frac{1}{4} \tag{1.4}
\]
for \(0 < x < \delta\), then system (1.3) has property \((Z^+_1)\) (see also [4,5,9,10,16]). Let
\[
G(x) = \int_0^x g(\xi) \, d\xi.
\]
Then, by a straightforward calculation, we see that if \(F(x)\) and \(G(x)\) satisfy
\[
F(x) \geq 2\sqrt{2} G(x) \tag{1.5}
\]
for $x > 0$ sufficiently small, then condition (1.4) holds. Hence, condition (1.5) is sufficient for property $(Z_1^+)$. We can reach the above fact (a) by using condition (1.4) or (1.5). Although conditions (1.4) and (1.5) are primitive, they are very informative. We will extend conditions (1.4) and (1.5).

2. Implicit conditions

In this section, we discuss mainly the question of when system (1.1) has property $(Z_1^+)$. If $F(x)$ has an infinite number of positive zeros clustering at $x = 0$, then the curve $C^+$ meets the $x$-axis infinitely many times, and therefore, system (1.1) fails to have property $(Z_1^+)$. Of course, if there exists a $\delta > 0$ such that $F(x) < 0$ for $0 < x < \delta$, then system (1.1) does not have property $(Z_1^+)$. Hence, assuming that $F(x) > 0$ for $0 < x < \delta$,

we must proceed our argument from now on. To begin with, we present an essential criterion for judging whether system (1.1) has property $(Z_1^+)$. 

**Theorem 2.1.** Let $P = (x_0, y_0) \in C^+$. Then $\gamma^+(P)$ approaches the origin through only the region \( \{(x, y) : 0 < x < x_0 \text{ and } 0 < h(y) < F(x)\} \) if and only if there exist a constant $\delta \geq x_0$ and a continuous function $\psi(x)$ such that

$$
\psi(x) < F(x) \quad \text{and} \quad \int_0^x \frac{g(\xi)}{F(\xi) - \psi(\xi)} \, d\xi \leq h^{-1}(\psi(x)) \tag{2.1}
$$

for $0 < x < \delta$.

**Remark 2.1.** From condition (2.1) it follows that $h^{-1}(\psi(x)) > 0$ for $0 < x < \delta$, and therefore, $F(x) > \psi(x) > 0$ and $\psi(0) = 0$. Hence, the curve $y = h^{-1}(\psi(x))$ runs through the region $\{(x, y) : 0 < x < x_0 \text{ and } 0 < h(y) < F(x)\}$.

**Remark 2.2.** Take $\psi(x) = F(x)/2$ and $h(y) = y$. Then condition (2.1) becomes condition (1.4).

**Proof of Theorem 2.1.** We first note that $\gamma^+(P)$ is considered as a solution $y(x)$ of

$$
\frac{dy}{dx} = \frac{g(x)}{F(x) - h(y)} \tag{2.2}
$$

satisfying $y(x_0) = y_0$.

**Sufficiency.** Suppose that $\gamma^+(P)$ does not tend to the origin through the region $\{(x, y) : 0 < x < x_0 \text{ and } 0 < h(y) < F(x)\}$. Then $\gamma^+(P)$ rotates in a clockwise direction about the origin. For this reason, $\gamma^+(P)$ crosses the curve $y = h^{-1}(\psi(x))$ and meets the $y$-axis at a point $(0, y_1)$ with $y_1 < 0$. Let

$$
x_1 = \inf\{x : 0 < x < \delta \text{ and } y(x) > h^{-1}(\psi(x))\},
$$

then

$$
y(x) = \begin{cases} 
\psi(x) & \text{for } 0 < x < x_1, \\
F(x) & \text{for } x_1 < x < \delta.
\end{cases}
$$

Since $\psi(x) < F(x)$ for $0 < x < x_1$ and $\psi(x_1) = F(x_1)$, the existence of $(x_1, y_1)$ follows.

References
Then, \((x_1, y(x_1))\) is the intersection point of \(\gamma^+(P)\) and the curve \(y = h^{-1}(\psi(x))\) nearest to the origin; that is, \(y(x_1) = h^{-1}(\psi(x_1))\) and \(y(x) < h^{-1}(\psi(x))\) for \(0 < x < x_1\). Hence, together with (2.1), we have

\[
h^{-1}(\psi(x_1)) < y(x_1) - y_1 = \int_0^{x_1} \frac{g(\xi)}{F(\xi) - h(y(\xi))} \, d\xi < \int_0^{x_1} \frac{g(\xi)}{F(\xi) - h(y(\xi))} \, d\xi
\]

\[
\leq h^{-1}(\psi(x_1)),
\]

which is a contradiction.

**Necessity.** Suppose that \(\gamma^+(P)\) approaches the origin through only the region \(\{(x, y): 0 < x < x_0 \text{ and } 0 < h(y) < F(x)\}\). Then its corresponding solution \(y(x)\) of (2.2) satisfies \(y(x) \searrow 0\) as \(x \to 0\).

(2.3)

Let \(\delta = x_0\) and \(\psi(x) = h(y(x))\) for \(0 < x < \delta\). Then we obtain

\[
h^{-1}(\psi(x)) = y(x) < h^{-1}(F(x)),
\]

and therefore, \(\psi(x) < F(x)\) for \(0 < x < \delta\). Also, by (2.3) we get

\[
\int_0^x \frac{g(\xi)}{F(\xi) - \psi(\xi)} \, d\xi = \int_0^x \frac{g(\xi)}{F(\xi) - h(y(\xi))} \, d\xi = y(x) - \lim_{\varepsilon \to 0} y(\varepsilon) = h^{-1}(\psi(x)).
\]

Thus, (2.1) holds. The proof is complete. \(\Box\)

**Remark 2.3.** From the proof of Theorem 2.1, we see that condition (2.1) is necessary and sufficient for system (1.1) to have property \((Z^+_1)\).

We here enumerate analogous results to Theorem 2.1, relative to properties \((Z^-_2)\), \((Z^+_3)\) and \((Z^-_4)\). We omit the proofs.

**Theorem 2.2.** Let \(P = (x_0, y_0) \in C^-\). Then \(\gamma^-(P)\) approaches the origin through only the region \(\{(x, y): x_0 < x < 0 \text{ and } 0 < h(y) < F(x)\}\) if and only if there exist a constant \(\delta \geq -x_0\) and a continuous function \(\psi(x)\) such that

\[
\psi(x) < F(x) \quad \text{and} \quad \int_0^x \frac{g(\xi)}{F(\xi) - \psi(\xi)} \, d\xi \leq h^{-1}(\psi(x))
\]

for \(-\delta < x < 0\).

**Theorem 2.3.** Let \(P = (x_0, y_0) \in C^-\). Then \(\gamma^+(P)\) approaches the origin through only the region \(\{(x, y): x_0 < x < 0 \text{ and } F(x) < h(y) < 0\}\) if and only if there exist a constant \(\delta \geq -x_0\) and a continuous function \(\psi(x)\) such that

\[
F(x) < \psi(x) \quad \text{and} \quad \int_0^x \frac{g(\xi)}{F(\xi) - \psi(\xi)} \, d\xi \geq h^{-1}(\psi(x))
\]

for \(-\delta < x < 0\).
Theorem 2.4. Let $P = (x_0, y_0) \in C^+$. Then $\gamma^-(P)$ approaches the origin through only the region $\{ (x, y) : 0 < x < x_0$ and $F(x) < h(y) < 0 \}$ if and only if there exist a constant $\delta > x_0$ and a continuous function $\psi(x)$ such that

$$F(x) < \psi(x) \quad \text{and} \quad \int_0^x \frac{g(\xi)}{F(\xi) - \psi(\xi)} \, d\xi \geq h^{-1}(\psi(x))$$

for $0 < x < \delta$.

Although the simplest case of $h(y)$ is $y$, another case appears in applied science very often. For instance, we can cite the predator–prey model of Holling type

$$\frac{du}{ds} = ru \left(1 - \frac{u}{K}\right) - \frac{uv}{a + u^p}, \quad \frac{dv}{ds} = v \left(\frac{\mu u^p}{a + u^p} - D\right),$$

where $u(s)$ and $v(s)$ are the densities of the prey and predator, respectively, at given time $s \geq 0$. The numbers $r$, $k$, $a$, $\mu$, $D$ and $p$ are positive ecological parameters: $r$ is the intrinsic rate of increase for the prey; $K$ is the carrying capacity for the prey; $\mu$ is the birth rate for the predator; $D$ is the death rate for the predator; $a$ is the half-saturation constant for the predator. It is well known that the change of variables $x = u - \lambda, \quad y = \log v - \log \nu$ and $dt = \frac{u^p}{a + u^p} \, ds$ transforms the predator–prey model into system (1.1) with

$$F(x) = v - r \left(1 - \frac{x + \lambda}{k}\right) a + (x + \lambda)^p, \quad g(x) = \mu - D - \frac{aD}{(x + \lambda)^p},$$

$$h(y) = v(1 - e^{-y}),$$

where

$$\lambda = \left(\frac{aD}{\mu - D}\right)^{1/p} \quad \text{and} \quad \nu = \frac{r\lambda}{D}\left(1 - \frac{\lambda}{k}\right).$$

Note that $h(y)$ is nonlinear, but it is almost linear for $y > 0$ sufficiently small.

To fulfill the practical demands of applied science, we consider the case that there exists an $m > 0$ such that

$$h(y) \leq my$$

for $y > 0$ sufficiently small. Put $y = \psi(x)/m$. Then condition (2.4) becomes

$$h\left(\frac{\psi(x)}{m}\right) \leq \psi(x),$$

namely,

$$\frac{\psi(x)}{m} \leq h^{-1}\left(\psi(x)\right)$$

for $x > 0$ sufficiently small. Hence, from Theorem 2.1 we have the following result.
Corollary 2.1. Assume (2.4). If there exists a continuous function $\psi(x)$ such that

$$\psi(x) < F(x) \quad \text{and} \quad \int_0^x \frac{g(\xi)}{F(\xi) - \psi(\xi)} \, d\xi \leq \frac{\psi(x)}{m} \quad (2.5)$$

for $x > 0$ sufficiently small, then system (1.1) has property $(Z_1^+)$.

Under the assumptions of Corollary 2.1, we can specify a positive orbit of (1.1) which approaches the origin without intersecting the $x$-axis. By condition (2.4), there exists a number $y_1$ such that $h(y) \leq my$ for $0 < y < y_1$. Let

$$\delta_1 = \inf \{ x > 0 : \psi(x) > my_1 \}$$

(such $\delta_1$ exists because $\psi(0) = 0$ and $\delta_1$ might be $\infty$). Then $\psi(\delta_1) = my_1$ and $0 < \psi(x) < my_1$ for $0 < x < \delta_1$. Hence, we have

$$\psi(x) \leq \frac{m}{h^{-1}(\psi(x))}$$

for $0 < x < \delta_1$. Let $\delta_2$ be a positive number satisfying (2.5) for $0 < x < \delta_2$. Put

$$\delta = \min\{\delta_1, \delta_2\}$$

and let $P = (x_0, y_0) \in C^+$ with $0 < x_0 \leq \delta$. Then, from Theorem 2.1 we conclude that $\gamma(P)$ approaches the origin through only the region $\{(x, y) : 0 < x < x_0 \text{ and } 0 < h(y) < F(x)\}$.

By the same manner in Corollary 2.1, we can present an immediate consequence of Theorem 2.2 regarding property $(Z_2^+)$.

Corollary 2.2. Assume (2.4). If there exists a continuous function $\psi(x)$ such that

$$\psi(x) < F(x) \quad \text{and} \quad \int_0^x \frac{g(\xi)}{F(\xi) - \psi(\xi)} \, d\xi \leq \frac{\psi(x)}{m}$$

for $x < 0$, $|x|$ sufficiently small, then system (1.1) has property $(Z_2^+)$.

Let us assume that there exists an $m > 0$ such that

$$h(y) \geq my$$

(2.6)

for $y < 0$, $|y|$ sufficiently small, instead of condition (2.4). Then we obtain the following results.

Corollary 2.3. Assume (2.6). If there exists a continuous function $\psi(x)$ such that

$$F(x) < \psi(x) \quad \text{and} \quad \int_0^x \frac{g(\xi)}{F(\xi) - \psi(\xi)} \, d\xi \geq \frac{\psi(x)}{m}$$

for $x < 0$, $|x|$ sufficiently small, then system (1.1) has property $(Z_3^+)$.
Corollary 2.4. Assume (2.6). If there exists a continuous function \( \psi(x) \) such that
\[
F(x) < \psi(x) \quad \text{and} \quad \int_0^x \frac{g(\xi)}{F(\xi) - \psi(\xi)} \, d\xi \geq \frac{\psi(x)}{m}
\]
for \( x > 0 \) sufficiently small, then system (1.1) has property \((Z^-_4)\).

3. Explicit conditions

As we have shown, condition (2.1) is necessary and sufficient for system (1.1) to possess property \((Z^+_4)\). In a concrete case, however, it is very difficult to find a suitable function \( \psi(x) \) with a constant \( \delta \) satisfying (2.1). We intend to give explicit condition for our problem hereafter. To this end, we define
\[
H(y) = \int_0^y h(\eta) \, d\eta.
\]
From the assumption of \( h(y) \), we see that the inverse function of \( v = H(y) \) \( \text{sgn} \, y \) exists. We denote by \( H^{-1}(v) \) the inverse function. Then, we can state as follows.

Theorem 3.1. Suppose that
\[
F(x) \geq 2h(H^{-1}(G(x))) \tag{3.1}
\]
for \( x > 0 \) sufficiently small. Then system (1.1) has property \((Z^+_4)\).

Proof. Let \( \psi(x) = h(H^{-1}(G(x))) \). Then, by (3.1) we can choose a \( \delta > 0 \) such that
\[
F(x) - \psi(x) \geq h(H^{-1}(G(x))) > 0
\]
for \( 0 < x < \delta \). Since
\[
\frac{d}{dx} H^{-1}(G(x)) = \frac{g(x)}{h(H^{-1}(G(x)))},
\]
we have
\[
\int_0^x \frac{g(\xi)}{F(\xi) - \psi(\xi)} \, d\xi \leq \int_0^x \frac{g(\xi)}{h(H^{-1}(G(\xi)))} \, d\xi = \int_0^x \frac{d}{d\xi} H^{-1}(G(\xi)) \, d\xi
= H^{-1}(G(x)) - \lim_{\varepsilon \to 0} H^{-1}(G(\varepsilon))
= H^{-1}(G(x)) = h^{-1}(\psi(x))
\]
for \( 0 < x < \delta \). Hence, condition (2.1) in Theorem 2.1 is satisfied, and therefore, system (1.1) has property \((Z^+_4)\). \( \square \)
Condition (3.1) is very simple in form. Unfortunately, however, it is difficult to check condition (3.1) because we cannot construct exactly the inverse function $H^{-1}(v)$ in general. For example, we consider the case that $h(y) = \nu(1 - e^{-y})$ with $\nu > 0$. Then it is clear that $H(y) = \nu(y + e^{-y} - 1)$, but it is hard to find the inverse function. We would like to ensure that system (1.1) has property $(Z^+_1)$ without calculating $H^{-1}(v)$.

Using Corollary 2.1, we give another sufficient condition for property $(Z^+_1)$. For this purpose, we need to assume condition (2.4) again.

**Theorem 3.2.** Assume (2.4) and suppose that

$$F(x) \geq \sqrt{m} \left( 2\sqrt{2G(x)} - \phi(\sqrt{2G(x)}) \right)$$

(3.2)

for $x > 0$ sufficiently small, where $\phi(u)$ is a nonnegative continuous function satisfying

$$\frac{\phi(u)}{u} \text{ is nondecreasing}$$

(3.3)

and there exists a constant $b > 4$ such that

$$b \left( \int_0^u \frac{\phi(\xi)}{\xi^2} d\xi \right)^2 \leq \frac{\phi(u)}{u}$$

(3.4)

for $u > 0$ sufficiently small. Then system (1.1) has property $(Z^+_1)$.

Before giving the proof of Theorem 3.2, it is convenient to show examples of $\phi(u)$ satisfying conditions (3.3) and (3.4). Let $\phi(u) = u^{1+\varepsilon}$ for some $\varepsilon > 0$. Then it is clear that $\phi(u)$ is nonnegative and continuous. Since $\phi(u)/u = u^\varepsilon$, condition (3.3) holds. For arbitrary $b > 4$, we have

$$b \left( \int_0^u \frac{\phi(\xi)}{\xi^2} d\xi \right)^2 = \frac{b}{\varepsilon} u^{2\varepsilon} < u^\varepsilon = \frac{\phi(u)}{u}$$

for $u > 0$ sufficiently small. Thus, condition (3.4) is also satisfied with any $b > 4$. As another example, take

$$\phi(u) = \frac{cu}{(\log u)^2}$$

for $u > 0$ sufficiently small and $0 < c < 1/4$. This is a very sharp case because condition (3.4) is satisfied with $b = 1/c$. In fact, we obtain

$$b \left( \int_0^u \frac{\phi(\xi)}{\xi^2} d\xi \right)^2 = \frac{bc^2}{(\log u)^2} = \frac{c}{(\log u)^2} = \frac{\phi(u)}{u}.$$

Other examples abound.

**Proof of Theorem 3.2.** For simplicity, let

$$\Phi(u) = \int_0^u \frac{\phi(\xi)}{\xi^2} d\xi.$$
It follows from (3.3) that there exists a $\sigma \geq 0$ such that
\[ \frac{\phi(u)}{u} \searrow \sigma \quad \text{as } u \to 0. \]
Suppose that $\sigma$ is positive. Then we have
\[ \Phi(u_0) \geq \sigma \int_0^{u_0} \frac{1}{\xi} d\xi = \infty \]
for some $u_0 > 0$. This contradicts (3.4), and therefore, $\sigma = 0$. Hence, by (3.4) again, we see that
\[ \Phi(u) \to 0 \quad \text{as } u \to 0. \]

It is enough to show that condition (2.5) in Corollary 2.1 is satisfied. Let
\[ u(x) = \sqrt{2G(x)}. \]
Then $u(x)$ tends to zero as $x \to 0$. Since $b > 4$, we can choose a constant $d$ satisfying
\[ 0 < d < 1 - \frac{4}{b}. \quad (3.5) \]
From the above properties of $\phi(u)/u$ and $\Phi(u)$, we conclude that
\[ \frac{\phi(u(x))}{u(x)} < \frac{d}{4} \quad \text{and} \quad \Phi(u(x)) < \frac{d}{12} \quad (3.6) \]
for $x > 0$ sufficiently small. Let $\delta$ be chosen so that (3.2) and (3.6) are satisfied for $0 < x < \delta$. Define
\[ \psi(x) = \sqrt{m} \{ u(x) + 2u(x)\Phi(u(x)) \}. \quad (3.7) \]
Then, by (3.2) and (3.7) we obtain
\[ F(x) - \psi(x) \geq \sqrt{m} u(x) \left\{ 1 - \frac{\phi(u(x))}{u(x)} - 2\Phi(u(x)) \right\} \]
\[ > \sqrt{m} u(x) \left( 1 - \frac{d}{4} - \frac{d}{6} \right) > 0 \quad (3.8) \]
for $0 < x < \delta$. Let $k(x)$ be a function defined by
\[ k(x) = \frac{\psi(x)}{m} - \int_0^x \frac{g(\xi)}{F(\xi) - \psi(\xi)} d\xi \]
for $x > 0$ and $k(0) = 0$. Using (3.2) and (3.7), we get
\[ \frac{d}{dx} k(x) = \frac{g(x)}{\sqrt{m} u(x)} \left\{ 1 + \frac{2\phi(u(x))}{u(x)} + 2\Phi(u(x)) \right\} - \frac{g(x)}{F(x) - \psi(x)} \]
\[ \geq \frac{g(x)}{\sqrt{m} u(x)} \left\{ 1 + \frac{2\phi(u(x))}{u(x)} + 2\Phi(u(x)) - \frac{u(x)}{l(x)} \right\} \]
\[ = \frac{g(x)\phi(u(x))}{\sqrt{m} u(x)l(x)} \left\{ 1 - \frac{2\phi(u(x))}{u(x)} - 6\Phi(u(x)) - \frac{4u(x)\Phi(u(x))}{\phi(u(x))} \right\}, \]
where
\[ l(x) = u(x) - \phi(u(x)) - 2u(x)\Phi(u(x)) \]

As shown in (3.8), the function \( l(x) \) is positive for \( 0 < x < \delta \). By (3.4)–(3.6) we can estimate that
\[
1 - \frac{2\phi(u(x))}{u(x)} - 6\Phi(u(x)) - \frac{4u(x)\Phi(u(x))^2}{\phi(u(x))} > 1 - \frac{d}{2} - \frac{d}{2} - \frac{4}{b} > 0.
\]

Hence, we see that \( k(x) \) is increasing for \( 0 < x < \delta_2 \). We therefore conclude that
\[
\int_0^x \frac{g(\xi)}{F(\xi) - \psi(\xi)} d\xi \leq \frac{\psi(x)}{m}
\]
for \( 0 < x < \delta \). Thus, condition (2.5) holds. This completes the proof. \( \square \)

When \( h(y) = y \), we can choose 1 as a constant \( m \) satisfying condition (2.4). Hence, condition (3.2) becomes
\[
F(x) \geq 2\sqrt{2G(x)} - \phi(2\sqrt{2G(x)}) \quad (3.2^*)
\]
for \( x > 0 \) sufficiently small, and therefore, we have the following result.

**Corollary 3.1.** Assume (3.2*), (3.3) and (3.4). Then system (1.3) has property \((Z_+^+)\).

**Remark 3.1.** Corollary 3.1 was already proved in [14, Theorem 2.1].

We can formulate some results for property \((Z_-^-), (Z_+^+)\) or \((Z_4^-)\) similar to Theorems 3.1, 3.2 and Corollary 3.1. We leave the details to the reader.

### 4. Sharp conditions for a concrete system

As a typical case of (1.1), we consider the system
\[
\dot{x} = m[y]^{p-1}y - F(x), \quad \dot{y} = -g(x) \tag{4.1}
\]
in which \( m > 0 \) and \( p \geq 1 \). System (4.1) contains pseudolinear systems as a special case. For example, let \( p = 3 \) and \( F(x) = g(x) = x^3 \). Then system (4.1) is pseudolinear. The pseudolinear systems had been proposed by R. Conti and were studied by several authors (see [2,7,12,13,15]). In their works, sufficient conditions are given for the origin to be a global attractor, a global weak attractor or a center-focus. A necessary and sufficient condition is also presented for the zero solution to be globally asymptotically stable. Our results are linked closely with those subjects.

Since \( h(y) = m[y]^{p-1}y \) in system (4.1), we have
\[
H(y) = \frac{m}{1 + p} |y|^{1+p} \quad \text{and} \quad H^{-1}(v) = \left( \frac{(1 + p)|v|}{m} \right)^{1/(1+p)} \text{sgn} \, v.
\]

Hence, by means of Theorem 3.1, we conclude that if
\[
F(x) \geq 2m^{1/(1+p)} \left( (1 + p)G(x) \right)^{p/(1+p)} \tag{4.2}
\]
for \( x > 0 \) sufficiently small, then system (4.1) has property \((Z_+^+)\).
Recall that we arrived at Theorem 3.1 by the use of Theorem 2.1. Applying Theorem 2.1 directly to system (4.1), we can improve the fact above as follows.

**Theorem 4.1.** Suppose that

\[ F(x) \geq m^{1/(1+p)}(1 + p) \left( \frac{1 + p}{p} G(x) \right)^{p/(1+p)} \]  
\[ \text{for } x > 0 \text{ sufficiently small. Then system (4.1) has property } (Z^+_1). \]

**Remark 4.1.** It is easy to check that

\[ \frac{1 + p}{mp/(1+p)} \leq 2 \]

for \( p \geq 1 \). Hence, condition (4.3) is better than condition (4.2).

**Proof of Theorem 4.1.** Note that

\[ h^{-1}(v) = \left( \frac{|v|}{m} \right)^{1/p} \text{sgn } v \]

in system (4.1). Let \( \delta \) be chosen so that (4.3) is satisfied for \( 0 < x < \delta \). Take

\[ \psi(x) = m^{1/(1+p)} \left( \frac{1 + p}{p} G(x) \right)^{p/(1+p)}. \]

Then condition (2.1) in Theorem 2.1 holds. In fact, by (4.3) we have

\[ F(x) - \psi(x) \geq m^{1/(1+p)} \left( \frac{1 + p}{p} G(x) \right)^{p/(1+p)} > 0 \]

for \( 0 < x < \delta \), and therefore,

\[ \int_0^x \frac{g(\xi)}{F(\xi) - \psi(\xi)} d\xi \leq \frac{1}{(mp)^{1/(1+p)}(1 + p)^{p/(1+p)}} \int_0^x \frac{g(\xi)}{G(\xi)^{p/(1+p)}} d\xi \]

\[ = \left( \frac{1 + p}{mp} \right)^{1/(1+p)} \int_0^x \frac{g(\xi)}{(1 + p)G(\xi)^{p/(1+p)}} d\xi \]

\[ = \left( \frac{1 + p}{mp} \right)^{1/(1+p)} \left\{ G(x)^{1/(1+p)} - \lim_{\varepsilon \to 0} G(\varepsilon)^{1/(1+p)} \right\} \]

\[ = \frac{1}{m^{1/(1+p)}} \left( \frac{1 + p}{p} G(x) \right)^{1/(1+p)} \]

\[ = \left( \frac{\psi(x)}{m} \right)^{1/p} = h^{-1}(\psi(x)) \]

for \( 0 < x < \delta \). Thus, system (4.1) has property \((Z^+_1)\). \( \Box \)
In case $p > 1$, we can give the following result which is sharper than Theorem 4.1.

**Theorem 4.2.** Let $p > 1$. Suppose that

$$F(x) \geq m^{1/(1+p)} \left\{ (1 + p)u(x)^p - \phi(u(x)) \right\}$$

for $x > 0$ sufficiently small, where

$$u(x) = \left( \frac{1 + p}{p} G(x) \right)^{1/(1+p)}$$

and $\phi(u)$ is a nonnegative continuous function satisfying

$$\frac{\phi(u)}{u^p} \text{ is nondecreasing}$$

and there exists a constant $b > 2p(1 + p)$ such that

$$b \left( \int_0^u \frac{\phi(\xi)}{\xi^{1+p}} d\xi \right)^2 \leq \frac{\phi(u)}{u^p}$$

for $u > 0$ sufficiently small. Then system (4.1) has property $(Z^+_1)$.

**Remark 4.2.** Theorem 4.2 is true even the case that $p = 1$. If $p = 1$, then condition (2.4) naturally holds and condition (4.4) coincides with condition (3.2). Hence, from Theorem 3.2 we see that system (4.1) has property $(Z^+_1)$.

**Remark 4.3.** It is clear that conditions (4.5) and (4.6) are satisfied for the case $\phi(x) \equiv 0$. In this case, condition (4.4) becomes condition (4.3). Hence, Theorem 4.2 is a generalization of Theorem 4.1.

**Proof of Theorem 4.2.** We will show that condition (2.1) in Theorem 2.1 is satisfied. Its proof goes through in a similar process to that of Theorem 3.2. Let

$$\psi(x) = m^{1/(1+p)} \left\{ u(x) + 2u(x)\Phi(u(x)) \right\}^p,$$

where

$$\Phi(u) = \int_0^u \frac{\phi(\xi)}{\xi^{1+p}} d\xi.$$

Then, as in the proof of Theorem 3.2, it follows from (4.5) and (4.6) that both $\phi(u)/u^p$ and $\Phi(u)$ converge to 0 as $u \to 0$. From the assumption that $b > 2p(1 + p)$, we can choose an $\varepsilon > 0$ such that

$$b - \varepsilon > 2p(1 + p).$$

Hence, together with (4.6), we have

$$\frac{(2p(1 + p) + \varepsilon)\phi(u)^2}{u^p} \leq \frac{\phi(u)}{u^p}$$

for $u > 0$ sufficiently small.
Since $u(x)$ tends to 0 as $x \to 0$, there exists a $\delta > 0$ such that
\[ \frac{\phi(u(x))}{u(x)^p} < \frac{p - 1}{4} \quad \text{and} \quad \Phi(u(x)) < \frac{p - 1}{4(1 + p)} \] (4.8)
and
\[ \psi(x) = m^{1/(1+p)}u(x)^p \{ 1 + 2\Phi(u(x)) \}^p \]
\[ < m^{1/(1+p)}u(x)^p \{ 1 + 2p\Phi(u(x)) + (2p(p - 1) + \varepsilon)\Phi(u(x))^2 \} \]
for $0 < x < \delta$. Hence, by (4.4) and (4.7) we obtain
\[ F(x) - \psi(x) > m^{1/(1+p)}u(x)^p \left\{ p - \frac{\phi(u(x))}{u(x)^p} - 2p\Phi(u(x)) - (2p(p - 1) + \varepsilon)\Phi(u(x))^2 \right\} \]
\[ > m^{1/(1+p)}u(x)^p \left\{ p - \frac{\phi(u(x))}{u(x)^p} - 2p\Phi(u(x)) - b\Phi(u(x))^2 \right\} \]
\[ > m^{1/(1+p)}u(x)^p \left\{ p - \frac{1}{2} \right\} \frac{p(p - 1)}{2(1 + p)} > 0 \]
for $0 < x < \delta$. We also have
\[ \frac{1}{m^{1/(1+p)}} \left\{ u(x) + 2u(x)\Phi(u(x)) \right\} - \int_0^x g(\xi) \frac{f(\xi)}{\psi(\xi)} d\xi > k(x), \]
for $0 < x < \delta$, where
\[ k(x) = \frac{1}{m^{1/(1+p)}} \left\{ u(x) + 2u(x)\Phi(u(x)) \right\} - \int_0^x g(\xi) \frac{f(\xi)}{\psi(\xi)} d\xi \]
\[ - \int_0^x \frac{f(\xi)}{u(\xi)^p} \left\{ p - \frac{\phi(u(\xi))}{u(\xi)^p} - 2p\Phi(u(\xi)) - (2p(p - 1) + \varepsilon)\Phi(u(\xi))^2 \right\} d\xi \].

Using
\[ \frac{d}{dx} u(x) = \frac{g(x)}{pu(x)^p}, \]
we get
\[ \frac{d}{dx} k(x) = \frac{g(x)}{m^{1/(1+p)}pu(x)^p} \left\{ 1 + 2\frac{\phi(u(x))}{u(x)^p} + 2\Phi(u(x)) - \frac{pu(x)^p}{l(x)} \right\}, \]
where
\[ l(x) = pu(x)^p - \phi(u(x)) - 2pu(x)^p\Phi(u(x)) - (2p(p - 1) + \varepsilon)u(x)^p\Phi(u(x))^2 \]
which is positive for $0 \leq x < \delta$. From (4.7) and (4.8), we see that

\[
1 + \frac{2\phi(u(x))}{u(x)p} + 2\Phi(u(x)) - \frac{pu(x)p}{l(x)} = \frac{\phi(u(x))}{l(x)} \left(2p - 1 - \frac{2\phi(u(x))}{u(x)p} - 2(1 + 2p)\Phi(u(x)) - 2(2p(p - 1) + \epsilon)\Phi(u(x))^2\right)
\]

for $0 \leq x < \delta$. We therefore conclude that $k(x)$ is increasing and

\[
h^{-1}(\psi(x)) - \int_0^x \frac{g(\xi)}{F(\xi) - \psi(\xi)} d\xi > k(x) \geq k(0) = 0
\]

for $0 \leq x < \delta$. Hence, condition (2.1) holds. Thus, system (4.1) has property $(Z_{+}^{+})$ by Theorem 2.1. The proof is now complete. \[\square\]

References