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Cyclic q-MZSV sum ☆

Yasuo Ohno [a], Jun-ichi Okuda [b], Wadim Zudilin [c,*]

[a] *Department of Mathematics, Kinki University, Higashi-Osaka, Osaka 577-8502, Japan*
[b] *CrossTrust, Inc., 1-5-1, Otemach, Chiyoda-ku, Tokyo 100-0004, Japan*
[c] *School of Mathematical and Physical Sciences, The University of Newcastle, Callaghan, NSW 2308, Australia*

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On Don's 60th birthday: many more multiple (zeta) years, Don!

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ABSTRACT

We present a family of identities 'cyclic sum formula' and 'sum formula' for a version of multiple q-zeta star values. We also discuss a problem of q-generalization of shuffle products.

0. Introduction and notation

The classical idea of introducing an additional parameter to an expression or formula we wish to deal with, is quite fruitful in many situations. This may simplify a proof of the corresponding identity or lead to a more general identity which has several other useful specializations of the introduced

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* Corresponding author.
E-mail addresses: ohno@math.kindai.ac.jp (Y. Ohno), wadim.zudilin@newcastle.edu.au (W. Zudilin).

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parameter. The story of introducing the parameter q (or, the 'quantum' parameter) often has a different flavor. Our motivation to study q-analogues of multiple zeta values (MZVs)

$$\zeta(\boldsymbol{k}) = \zeta(k_1, \ldots, k_r) = \sum_{n_1 > \cdots > n_r \geqslant 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}, \quad k_1, \ldots, k_r \in \{1, 2, \ldots\},\ k_1 \geqslant 2, \tag{1}$$

and multiple zeta star values (MZSVs)

$$\zeta^\star(\boldsymbol{k}) = \zeta^\star(k_1, \ldots, k_r) = \sum_{n_1 \geqslant \cdots \geqslant n_r \geqslant 1} \frac{1}{n_1^{k_1} \cdots n_r^{k_r}}, \quad k_1, \ldots, k_r \in \{1, 2, \ldots\},\ k_1 \geqslant 2, \tag{2}$$

is to a better understanding the structure of linear and algebraic relations between the numbers (1) (or (2)). An important advantage of the q-model is that proving the absence of such relations is a much easier task (cf. [18]): the functional case is normally not as hard as the numerical one. On the other hand, showing that some relations hold is normally easier for numbers than for functions. The main problem here is finding an appropriate q-analogue which is often dictated by already existing proofs of the corresponding original identities. In this paper we hope to convince the reader that there is no uniform q-generalization of the multiple zeta (star) values, but having several q-analogues in mind and a simple way to pass from one q-model to another gives one a very natural parallel between the numbers and their q-analogues.

Throughout the article we assume that $q \in \mathbb{C}$ satisfies $|q| < 1$. Let us first recall the definition of the q-MZVs and q-MZSVs which is already accepted to be dominating [1,2,13,17]:

$$\zeta_q(k_1, k_2, \ldots, k_r) = \sum_{n_1 > n_2 > \cdots > n_r \geqslant 1} \frac{q^{n_1(k_1-1)+n_2(k_2-1)+\cdots+n_r(k_r-1)}}{[n_1]^{k_1}[n_2]^{k_2}\cdots[n_r]^{k_r}} \tag{3}$$

and

$$\zeta_q^\star(k_1, k_2, \ldots, k_r) = \sum_{n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r \geqslant 1} \frac{q^{n_1(k_1-1)+n_2(k_2-1)+\cdots+n_r(k_r-1)}}{[n_1]^{k_1}[n_2]^{k_2}\cdots[n_r]^{k_r}}, \tag{4}$$

where $[n] = [n]_q = (1-q^n)/(1-q)$ is a q-analogue of the positive integer n and conditions for the multi-index $\boldsymbol{k} = (k_1, \ldots, k_r)$ are exactly the same as in (1) and (2) (such multi-indices are called *admissible*). The corresponding q-analogues of the values of Riemann's zeta function are as follows:

$$\zeta_q(k) = \zeta_q^\star(k) = \sum_{n \geqslant 1} \frac{q^{n(k-1)}}{[n]^k}.$$

We add one more notation for our convenience:

$$\begin{aligned} \overline{\zeta_q^\star}(k_1, k_2, \ldots, k_r) &= (1-q)^{-(k_1+k_2+\cdots+k_r)} \zeta_q^\star(k_1, k_2, \ldots, k_r) \\ &= \sum_{n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r \geqslant 1} \frac{q^{n_1(k_1-1)+n_2(k_2-1)+\cdots+n_r(k_r-1)}}{(1-q^{n_1})^{k_1}(1-q^{n_2})^{k_2}\cdots(1-q^{n_r})^{k_r}}; \end{aligned} \tag{5}$$

the same convention is used for $\bar{\zeta}_q(k_1, k_2, \ldots, k_r)$.

A different version of q-analogues for the numbers (1) and (2) is given by the formulas

$$\mathfrak{z}_q(k_1, k_2, \ldots, k_r) = \sum_{n_1 > n_2 > \cdots > n_r \geqslant 1} \frac{q^{n_1}}{(1-q^{n_1})^{k_1}(1-q^{n_2})^{k_2}\cdots(1-q^{n_r})^{k_r}} \tag{6}$$

and

$$\mathfrak{z}_q^\star(k_1, k_2, \ldots, k_r) = \sum_{n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r \geqslant 1} \frac{q^{n_1}}{(1-q^{n_1})^{k_1}(1-q^{n_2})^{k_2}\cdots(1-q^{n_r})^{k_r}}; \tag{7}$$

this time we even do not require the condition $k_1 > 1$. Note however that, under the latter condition, the limits as $q \to 1$, $|q| < 1$, of

$$(1-q)^k \mathfrak{z}_q(\boldsymbol{k}) \quad \text{and} \quad (1-q)^k \mathfrak{z}_q(\boldsymbol{k}), \quad \text{where } k = \sum_{i=1}^r k_i,$$

exist and coincide with (1) and (2), respectively.

Several relations for the MZSVs have very simple q-analogues in terms of (7). The examples are

$$\mathfrak{z}_q^\star(2,1) = 2\mathfrak{z}_q^\star(3) - \mathfrak{z}_q^\star(2) \quad \left(= \sum_{n \geqslant 1} \frac{q^n(1+q^n)}{(1-q^n)^3}\right),$$

$$\mathfrak{z}_q^\star(2,1,1) = 3\mathfrak{z}_q^\star(4) - 2\mathfrak{z}_q^\star(3) \quad \left(= \sum_{n \geqslant 1} \frac{q^n(1+2q^n)}{(1-q^n)^4}\right),$$

$$\mathfrak{z}_q^\star(2,2,1) = 2\mathfrak{z}_q^\star(5) - \mathfrak{z}_q^\star(3) \quad \left(= \sum_{n \geqslant 1} \frac{q^n(1+2q^n-q^{2n})}{(1-q^n)^5}\right),$$

$$\mathfrak{z}_q^\star(2,1,1,1) = 4\mathfrak{z}_q^\star(5) - 3\mathfrak{z}_q^\star(4) \quad \left(= \sum_{n \geqslant 1} \frac{q^n(1+3q^n)}{(1-q^n)^5}\right),$$

$$\mathfrak{z}_q^\star(2,1,2,1) + \mathfrak{z}_q^\star(2,2,1,1) = 5\mathfrak{z}_q^\star(6) - 3\mathfrak{z}_q^\star(4) \quad \left(= \sum \frac{q^n(2+6q^n-3q^{2n})}{(1-q^n)^6}\right),$$

$$\mathfrak{z}_q^\star(2,2,2,1) = 2\mathfrak{z}_q^\star(7) - \mathfrak{z}_q^\star(4) \quad \left(= \sum_{n \geqslant 1} \frac{q^n(1+3q^n-3q^{2n}+q^{3n})}{(1-q^n)^7}\right).$$

One of the natural questions is finding a general formula for these simple relations. The answer on this original question is given in Section 1. Briefly speaking the key is the so-called *cyclic sum formula* for the MZSVs discovered in [14] and its q-version for (4) given in [13]. Surprisingly, the q-model (7) admits a much simpler formula and the examples above are just its particular cases.

1. Cyclic sum formula and sum formula of q-MZSVs

To present our main result, we define for any function f depending on r positive integer parameters, the cyclic sum ${}_{\mathrm{cycl}} f$ by

$${}_{\mathrm{cycl}} f(k_1, \ldots, k_r) = \sum_{i=1}^r \sum_{j=0}^{k_i-2} f(k_i - j, k_{i+1}, \ldots, k_r, k_1, \ldots, k_{i-1}, j+1),$$

where the empty sums (for $k_i = 1$) are interpreted as zero. Under this notation, the result is as follows.

Main theorem. *For any positive integers $r \geqslant 1$ and $k_1, k_2, \dots, k_r$ with $k = \sum_{i=1}^{r} k_i > r$, we have*

$$_{\mathrm{cycl}}\mathfrak{z}_q^\star(k_1, \dots, k_r) = k\mathfrak{z}_q^\star(k+1) - r\mathfrak{z}_q^\star(r+1).$$

Also, as an easy consequence of our Main theorem, we newly get the sum formula of the q-MZSVs. We denote by $I_0(k,r)$ a set of indices

$$I_0(k,r) = \left\{ (k_1, \dots, k_r) \in \mathbb{Z}^r \;\middle|\; k = \sum_{i=1}^{r} k_i,\ k_1 > 1,\ k_2, \dots, k_r \geqslant 1 \right\}$$

for $k > r \geqslant 1$.

Theorem 1 *(Sum formula of $\mathfrak{z}_q^\star$). For any positive integers $k > r \geqslant 1$, we have*

$$\sum_{\boldsymbol{k} \in I_0(k+1, r+1)} \mathfrak{z}_q^\star(\boldsymbol{k}) = \binom{k}{r} \mathfrak{z}_q^\star(k+1) - \binom{k-1}{r-1} \mathfrak{z}_q^\star(r+1).$$

Both expressions in the above formulas are close to those formulas of $\zeta^\star$ compared with the formulas of $\zeta_q^\star$ (see, e.g., Theorem 3 below). As a by-product, following Hoffman's argument in [7], we obtain a version of Theorem 1.

Theorem 2 *(Sum formula of $\mathfrak{z}_q$). For any positive integers $k > r \geqslant 1$, we have*

$$\sum_{\boldsymbol{k} \in I_0(k+1, r+1)} \mathfrak{z}_q(\boldsymbol{k}) = \mathfrak{z}_q(k+1) - \sum_{j=1}^{r} (-1)^{r-j} \binom{k-1-j}{r-j} \binom{k-1}{j-1} \mathfrak{z}_q(j+1).$$

For the non-q-versions of Theorems 1 and 2, cf. [14,13,10] and [6,21], respectively. It is interesting that the sum formula for (3) has exactly the same expression as for (1) (cf. [1,17]), while the sum formula for (4) is quite involved (cf. [13]).

2. Proof of the Main theorem

To prove the required identity we rewrite the cyclic sum formula of (4) in terms of (7). Note that the equivalence of the formula for (3) and (4) is shown in [10, §4]. The cyclic sum formula of q-MZSVs in [13] is as follows.

Theorem 3 *(Cyclic sum formula). (See [13].) For any positive integers $r \geqslant 1$ and $k_1, k_2, \dots, k_r$ with $k = \sum_{i=1}^{r} k_i > r$,*

$$_{\mathrm{cycl}}\zeta_q^\star(k_1, k_2, \dots, k_r) = \sum_{l=0}^{r} (k-l) \binom{r}{l} (1-q)^l \zeta_q(k-l+1). \tag{8}$$

For any non-negative integer b and positive integers $r, a_1, a_2, \dots, a_r$, we define the index set J as follows:

$$J(a_1, \dots, a_r; b) = \left\{ (b_1, \dots, b_r) \in \mathbb{Z} \;\middle|\; a_i > b_i \geqslant 0,\ b_1 + \dots + b_r = b \right\}.$$

In this notation, we get the following identity to rewrite the left-hand side of formula (8).

Proposition 1. *For any positive integers $r \geqslant 1$ and $k_1, k_2, \ldots, k_r$ with $k = \sum_{i=1}^{r} k_i > r$,*

$$
{}_{\mathrm{cycl}}\overline{\zeta_q^{\star}}(k_1, k_2, \ldots, k_r) = \sum_{b=1}^{k-r} \sum_{(b_1,\ldots,b_r)\in J(k_1,\ldots,k_r;b)} (-1)^{k-r-b} \prod_{i=1}^{r} \binom{k_i-1}{b_i} {}_{\mathrm{cycl}}\mathfrak{z}_q^{\star}(b_1+1, \ldots, b_r+1). \tag{9}
$$

Proof. The left-hand side of the above identity is

$$
\sum_{i=1}^{r} \sum_{j=0}^{k_i-2} \overline{\zeta_q^{\star}}(k_i - j, k_{i+1}, \ldots, k_r, k_1, \ldots, k_{i-1}, j+1)
= \sum_{i=1}^{r} \sum_{j=0}^{k_i-2} \sum_{n_1 \geqslant n_2 \geqslant \cdots \geqslant n_r \geqslant n_{r+1} \geqslant 1} \frac{q^{n_1(k_i-j-1)+n_2(k_{i+1}-1)+\cdots+n_r(k_{i-1}-1)+n_{r-1}j}}{(1-q^{n_1})^{k_i-j}(1-q^{n_2})^{k_{i+1}}\cdots(1-q^{n_r})^{k_{i-1}}(1-q^{n_{r+1}})^{j+1}},
$$

where the inner sum with respect to r is

$$
\begin{aligned}
&\sum_{j=0}^{k_i-2} \frac{q^{n_1(k_i-j-1)+n_2(k_{i+1}-1)+\cdots+n_r(k_{i-1}-1)+n_{r-1}j}}{(1-q^{n_1})^{k_i-j}(1-q^{n_2})^{k_{i+1}}\cdots(1-q^{n_r})^{k_{i-1}}(1-q^{n_{r+1}})^{j+1}} \\
&= \sum_{j=0}^{k_i-2} \frac{q^{n_1}(1-(1-q^{n_1}))^{k_i-j-2}(1-(1-q^{n_2}))^{k_{i+1}-1}\cdots}{(1-q^{n_1})^{k_i-j}(1-q^{n_2})^{k_{i+1}}\cdots} \\
&\quad \times \frac{(1-(1-q^{n_r}))^{k_{i-1}-1}(1-(1-q^{n_{r+1}}))^{j}}{(1-q^{n_r})^{k_{i-1}}(1-q^{n_{r+1}})^{j+1}} \\
&= \sum_{b_{i+1},\ldots,b_{i-1}} \sum_{j=0}^{k_i-2} \sum_{\epsilon_i=2}^{k_i-j} \sum_{j_0=0}^{j} \frac{(-1)^{k-e} q^{n_1} \binom{k_i-j-2}{\epsilon_i-2}\binom{k_{i+1}-1}{b_{i+1}}\binom{k_{i+2}-1}{b_{i+2}}\cdots\binom{k_{i-1}-1}{b_{i-1}}\binom{j}{j_0}}{(1-q^{n_1})^{\epsilon_i}(1-q^{n_2})^{b_{i+1}+1}\cdots(1-q^{n_r})^{b_{i-1}+1}(1-q^{n_{r+1}})^{j_0+1}} \\
&= \sum_{b_{i+1},\ldots,b_{i-1}} \sum_{\epsilon_i=2}^{k_i} \sum_{j_0=0}^{k_i-\epsilon_i} \sum_{j=j_0}^{k_i-\epsilon_i} \frac{(-1)^{k-e} q^{n_1} \binom{k_i-j-2}{\epsilon_i-2}\binom{k_{i+1}-1}{b_{i+1}}\binom{k_{i+2}-1}{b_{i+2}}\cdots\binom{k_{i-1}-1}{b_{i-1}}\binom{j}{j_0}}{(1-q^{n_1})^{\epsilon_i}(1-q^{n_2})^{b_{i+1}+1}\cdots(1-q^{n_r})^{b_{i-1}+1}(1-q^{n_{r+1}})^{j_0+1}}.
\end{aligned}
$$

The first sum on the left-hand side runs over all indices in $\mathbb{Z}^{r-1}$ subject to the conditions

$$
0 \leqslant b_{i+1} < k_{i+1}, \quad 0 \leqslant b_{i+2} < k_{i+2}, \quad \ldots, \quad 0 \leqslant b_{i-1} < k_{i-1}.
$$

By using a variant of Vandermonde's identity for binomial sums (cf., e.g., [20, p. 9]) we have

$$
\sum_{j=j_0}^{k_i-\epsilon_i} \binom{k_i-j-2}{\epsilon_i-2}\binom{j}{j_0} = \binom{k_i-1}{\epsilon_i+j_0-1},
$$

hence the right-hand side of the above equality equals

$$
\sum_{b_{i+1},\ldots,b_{i-1}} \sum_{\epsilon_i=2}^{k_i} \sum_{j_0=0}^{k_i-\epsilon_i} \frac{(-1)^{k-r-b} q^{n_1} \binom{k_i-1}{\epsilon_i+j_0-1}\binom{k_{i+1}-1}{b_{i+1}}\binom{k_{i+2}-1}{b_{i+2}}\cdots\binom{k_{i-1}-1}{b_{i-1}}}{(1-q^{n_1})^{\epsilon_i}(1-q^{n_2})^{b_{i+1}+1}\cdots(1-q^{n_r})^{b_{i-1}+1}(1-q^{n_{r+1}})^{j_0+1}}
$$

$$= \sum_{b_{i+1},\dots,b_{i-1}} \sum_{b_i=1}^{k_i-1} \sum_{j_0=0}^{b_i-1} \frac{(-1)^{k-r-b} q^{n_1} \binom{k_i-1}{b_i} \binom{k_{i+1}-1}{b_{i+1}} \binom{k_{i+2}-1}{b_{i+2}} \cdots \binom{k_{i-1}-1}{b_{i-1}}}{(1-q^{n_1})^{b_i-j_0+1}(1-q^{n_2})^{b_{i+1}+1}\cdots(1-q^{n_r})^{b_{i-1}+1}(1-q^{n_{r+1}})^{j_0+1}}.$$

Thus we obtain the desired identity (9). □

To rewrite the right-hand side in (8) we use the following proposition.

Proposition 2. *For any positive integers n, r, t, we have*

$$\sum_{l=0}^{r} (t+l) \binom{r}{l} \frac{q^{n(t+l)}}{(1-q^n)^{t+l+1}} = \sum_{j=0}^{t} (-1)^{t-j} (r+j) \binom{t}{j} \frac{q^n}{(1-q^n)^{r+j+1}}, \tag{10}$$

hence

$$\sum_{l=0}^{r} (t+l) \binom{r}{l} \overline{\zeta_q^\star}(t+l+1) = \sum_{j=0}^{t} (-1)^{t-j} (r+j) \binom{t}{j} \mathfrak{z}_q^\star(r+j+1).$$

Proof. For the function $f(x) = x^t(1+x)^r = ((1+x)-1)^t(1+x)^r$ we have the expansions

$$f(x) = \sum_{l=0}^{r} \binom{r}{l} x^{t+l} \quad \text{and} \quad f(x) = \sum_{j=0}^{t} (-1)^{t-j} \binom{t}{j} (1+x)^{r+j},$$

hence

$$f'(x) = \sum_{l=0}^{r} (t+l) \binom{r}{l} x^{t+l-1} = \sum_{j=0}^{t} (-1)^{t-j} (r+j) \binom{t}{j} (1+x)^{r+j}.$$

It remains to use these two representations for

$$\frac{q^n}{(1-q^n)^2} f'\left(\frac{q^n}{1-q^n}\right)$$

to arrive at identity (10). □

By using Propositions 1 and 2 we can now write the cyclic sum formula (8) in terms of $\mathfrak{z}_q^\star$ as follows.

Proposition 3. *For any positive integers r and $k_1, k_2, \dots, k_r$ with $k = \sum_{i=1}^r k_i > r$,*

$$\begin{aligned} &\sum_{b=1}^{k-r} \sum_{(b_1,\dots,b_r)\in J(k_1,\dots,k_r;b)} (-1)^b \prod_{i=1}^{r} \binom{k_i-1}{b_i} \operatorname{cycl} \mathfrak{z}_q^\star(b_1+1,\dots,b_r+1) \\ &\quad = \sum_{j=0}^{k-r} (-1)^j (r+j) \binom{k-r}{j} \mathfrak{z}_q^\star(r+j+1), \end{aligned} \tag{11}$$

where the set $J(a_1,\dots,a_r;b)$ is as above.

To apply the inverse relation of binomial coefficients to our computation, we introduce notation

$$F(n_1,\dots,n_r)=\sum_{j=0}^{n}(-1)^j(r+j)\binom{n}{j}\mathfrak{z}_q^\star(r+j+1),\quad \text{where } n=n_1+\dots+n_r,$$

and

$$G(n_1,\dots,n_r)=\begin{cases} {}_{\mathrm{cycl}}\mathfrak{z}_q^\star(n_1+1,\dots,n_r+1)+F(0,\dots,0) & \text{if } n>0,\\ F(0,\dots,0) & \text{if } n=0.\end{cases}$$

Note that $F(0,\dots,0)=r\mathfrak{z}_q^\star(r+1)$ is a correction term required to start the summation on the left-hand side in (11) from $b=0$: Since

$$\sum_{b=0}^{k-r}\sum_{(b_1,\dots,b_r)\in J(k_1,\dots,k_r;b)}(-1)^b\prod_{i=1}^{r}\binom{k_i-1}{b_i}F(0,\dots,0)=\begin{cases}0 & \text{if } k-r>0,\\ F(0,\dots,0) & \text{if } k-r=0,\end{cases}$$

the relation (11) can be translated as

$$F(k_1-1,\dots,k_r-1)=\sum_{b=0}^{k-r}\sum_{(b_1,\dots,b_r)\in J(k_1,\dots,k_r;b)}(-1)^b\prod_{i=1}^{r}\binom{k_i-1}{b_i}\cdot G(b_1,\dots,b_r).$$

We now recall an inverse relation of binomial coefficients.

Proposition 4. *(See L.C. Hsu [9].) The equality*

$$F(n_1,\dots,n_r)=\sum_{\substack{0\leqslant m_i\leqslant n_i\\ i=1,\dots,r}}(-1)^{m_1+\dots+m_r}\prod_{i=1}^{r}\binom{n_i}{m_i}\cdot G(m_1,\dots,m_r)$$

implies

$$G(n_1,\dots,n_r)=\sum_{\substack{0\leqslant m_i\leqslant n_i\\ i=1,\dots,r}}(-1)^{m_1+\dots+m_r}\prod_{i=1}^{r}\binom{n_i}{m_i}\cdot F(m_1,\dots,m_r).$$

Using the inverse relation we obtain

$$\begin{aligned}
&G(k_1-1,\dots,k_r-1)\\
&\quad={}_{\mathrm{cycl}}\mathfrak{z}_q^\star(k_1,\dots,k_r)+r\mathfrak{z}_q^\star(r+1)\\
&\quad=\sum_{b=0}^{k-r}\sum_{(b_1,\dots,b_r)\in J(k_1,\dots,k_r;b)}\prod_{i=1}^{r}\binom{k_i-1}{b_i}\sum_{j=0}^{b}(-1)^{b-j}\binom{b}{j}(r+j)\mathfrak{z}_q^\star(r+j+1)\\
&\quad=\sum_{j=0}^{k-r}\left\{\sum_{b=j}^{k-r}\sum_{(b_1,\dots,b_r)\in J(k_1,\dots,k_r;b)}(-1)^{b-j}\binom{b}{j}\prod_{i=1}^{r}\binom{k_i-1}{b_i}\right\}(r+j)\mathfrak{z}_q^\star(r+j+1).
\end{aligned}\tag{12}$$

To deduce the desired formula in the Main theorem it remains to use one more proposition.

Proposition 5. *For any positive integer r and non-negative integers $a_1, a_2, \ldots, a_r$ and $c \leqslant a = a_1 + \cdots + a_r$, we have*

$$\sum_{b=c}^{a} \sum_{(b_1,\ldots,b_r)\in J(a_1+1,\ldots,a_r+1;b)} (-1)^{b-c}\binom{b}{c}\prod_{i=1}^{r}\binom{a_i}{b_i} = \begin{cases} 1 & \text{if } c = a, \\ 0 & \text{if } c < a. \end{cases}$$

Proof. We use the following expansion:

$$\begin{aligned}
(x-y+z)^a &= \prod_{i=1}^{r}\bigl(x-(y-z)\bigr)^{a_i} = \prod_{i=1}^{r}\sum_{b_i=0}^{a_i}(-1)^{b_i}\binom{a_i}{b_i}x^{a_i-b_i}(y-z)^{b_i} \\
&= \sum_{b=0}^{a}\Biggl(\sum_{(b_1,\ldots,b_r)\in J(a_1+1,\ldots,a_r+1;b)}(-1)^b\prod_{i=1}^{r}\binom{a_i}{b_i}\Biggr)x^{a-b}(y-z)^b \\
&= \sum_{b=0}^{a}\Biggl(\sum_{(b_1,\ldots,b_r)\in J(a_1+1,\ldots,a_r+1;b)}(-1)^b\prod_{i=1}^{r}\binom{a_i}{b_i}\Biggr)x^{a-b}\sum_{c=0}^{b}(-1)^c\binom{b}{c}y^{b-c}z^c \\
&= \sum_{c=0}^{a}\sum_{b=c}^{a}\Biggl(\sum_{(b_1,\ldots,b_r)\in J(a_1+1,\ldots,a_r+1;b)}(-1)^{b-c}\binom{b}{c}\prod_{i=1}^{r}\binom{a_i}{b_i}\Biggr)x^{a-b}y^{b-c}z^c.
\end{aligned}$$

Putting $x = y = 1$ for the both sides of this computation we deduce

$$z^a = \sum_{c=0}^{a}\Biggl(\sum_{b=c}^{a}\sum_{(b_1,\ldots,b_r)\in J(a_1+1,\ldots,a_r+1;b)}(-1)^{b-c}\binom{b}{c}\prod_{i=1}^{r}\binom{a_i}{b_i}\Biggr)z^c.$$

It remains to compare the coefficients of z^c on the both sides. □

Putting $a_i = k_i - 1$, $a = k - r$, $c = j$ in Proposition 5, we get our Main theorem immediately from the right-hand side of equality (12). Furthermore, we deduce Theorem 1 by the argument similar to [14].

3. *q*-Shuffle relations

It looks quite sophisticated that identities for the multiple zeta (star) values (1) and (2) have so different complexity of the corresponding q-analogues. Although our examples here (Main theorem and Theorem 1) demonstrate an advantage of the q-model (7) compared with (5), there are many identities for (3) or (4) (hence, for $\overline{\zeta_q}(\boldsymbol{k})$ and $\overline{\zeta_q^\star}(\boldsymbol{k})$) having the same or almost the same form as their prototypes for (1) or (2); see [1] and [17]. On the other hand, there are several examples when q-analogues involve certain series not all expressible in terms of the q-MZVs (see, e.g., [2]), or when q-analogues are not known at all, like for the two-one (conjectured) formula and the weighted sum theorem in [16]. What is a reason for all this?

Without presenting here a deep but standard algebraic setup for the multiple zeta values (1) (or (2)), recall that the presumable structure of algebraic relations for (1) is given by the so-called *double shuffle relations*, the relations that come out of identifying the model (1) with a certain algebra on words, with two products (see, e.g., [4,11,8,22]). One of these products, *harmonic* (or stuffle), originated from the product formula for series, has a very natural q-generalization for the model (3) or (4) (hence, for $\overline{\zeta_q}(\boldsymbol{k})$ and $\overline{\zeta_q^\star}(\boldsymbol{k})$), but the corresponding form for (6) and (7) is rather awkward.

The main difficulty arises when we look for a reasonable q-generalization of the shuffle product of (1), the product originated from the differential equations for the multiple polylogarithms

$$\mathrm{Li}_{k_1,\ldots,k_r}(z)=\sum_{n_1>\cdots>n_r\geqslant 1}\frac{z^{n_1}}{n_1^{k_1}\cdots n_r^{k_r}}. \tag{13}$$

Namely, one has

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{Li}_{k_1,k_2,\ldots,k_r}(z)=\begin{cases}\frac{1}{z}\mathrm{Li}_{k_1-1,k_2,\ldots,k_r}(z) & \text{if } k_1\geqslant 2,\\ \frac{1}{1-z}\mathrm{Li}_{k_2,\ldots,k_r}(z) & \text{if } k_1=1,\end{cases} \tag{14}$$

and this comes from the "fundamental theorem of calculus",

$$\frac{\mathrm{d}}{\mathrm{d}z}\big(f(z)g(z)\big)=\frac{\mathrm{d}}{\mathrm{d}z}f(z)\cdot g(z)+f(z)\cdot\frac{\mathrm{d}}{\mathrm{d}z}g(z). \tag{15}$$

The differential equations (14) give rise to an integral representation of the polylogarithms (13) (hence, of the multiple zeta values (1)), where the participating differential forms $\mathrm{d}z/z$ and $\mathrm{d}z/(1-z)$ are assigned as two non-commutative letters, so that the integrals themselves are interpreted as words on these letters.

The q-analogue of (15) reads as

$$D_q\big(f(z)g(z)\big)=D_qf(z)\cdot g(z)+f(z)\cdot D_qg(z)-(1-q)z\cdot D_qf(z)\cdot D_qg(z), \tag{16}$$

where

$$D_qf(z)=\frac{f(z)-f(qz)}{(1-q)z}.$$

Defining a q-analogue of the multiple polylogarithms (13) as

$$\mathrm{Li}_{k_1,\ldots,k_r}(z;q)=\sum_{n_1>\cdots>n_r\geqslant 1}\frac{z^{n_1}}{[n_1]^{k_1}\cdots[n_r]^{k_r}}, \tag{17}$$

from (16) we deduce the following analogue of (14):

$$D_q\,\mathrm{Li}_{k_1,k_2,\ldots,k_r}(z;q)=\begin{cases}\frac{1}{z}\mathrm{Li}_{k_1-1,k_2,\ldots,k_r}(z;q) & \text{if } k_1\geqslant 2,\\ \frac{1}{1-z}\mathrm{Li}_{k_2,\ldots,k_r}(z;q) & \text{if } k_1=1.\end{cases}$$

This q-model of the multiple polylogarithms, together with classical formulas in the theory of basic hypergeometric series [5], were used in [17] to derive a q-analogue of the main result in [15]. This is a reason to believe that the q-multiple polylogarithms (17) are 'motivated' q-analogues of (13). Note also that the q-MZVs in (6) come as the values of (17),

$$\mathfrak{z}_q(k_1,\ldots,k_r)=(1-q)^{-(k_1+\cdots+k_r)}\,\mathrm{Li}_{k_1,\ldots,k_r}(q;q).$$

Although the rule (16) might be interpreted as a shuffle product of a suitable functional q-model of the multiple polylogarithms and the corresponding q-MZVs, these models are different from and even 'incompatible' with already given models. For example, the q-analogue of the formula

$$\mathrm{Li}_1(z)^r=r!\,\mathrm{Li}_{\{1\}_r}(z) \tag{18}$$

in terms of (17) involves certain undesired 'parasites': if $r=2$, from

$$D_q\big(\mathrm{Li}_1(z;q)\,\mathrm{Li}_1(z;q)\big) = \frac{1}{1-z}\,\mathrm{Li}_1(z;q) + \mathrm{Li}_1(z;q)\,\frac{1}{1-z} - (1-q)\frac{z}{(1-z)^2}$$

we have

$$\mathrm{Li}_1(z;q)^2 = 2\,\mathrm{Li}_{1,1}(z;q) - (1-q)\sum_{n=1}^{\infty}\frac{(n-1)z^n}{[n]},$$

where the latter series cannot be expressed by means of (17). On the other hand, the identity (18) has a different q-generalization in [23], free of 'parasites'.

A related problem is a q-generalization of Euler's decomposition formula [3]

$$\zeta(s)\zeta(t) = \sum_{i=0}^{s-1}\binom{t-1+i}{i}\zeta(t+i,s-i) + \sum_{i=0}^{t-1}\binom{s-1+i}{i}\zeta(s+i,t-i), \tag{19}$$

since the known proofs make use (explicitly or not) of the shuffle relations. It seems that a way to overcome this difficulty is to extend the algebra of q-MZVs *differentially*, i.e., to consider a differential algebra of q-MZVs and all their δ-derivatives of arbitrary order, where $\delta = q\frac{\mathrm{d}}{\mathrm{d}q}$. Although it is hard to 'justify' this claim, let us demonstrate how problems may be fixed on the example of a q-analogue of (19) when $t = s = 2$,

$$\zeta(2)^2 = 2\zeta(2,2) + 4\zeta(3,1), \tag{20}$$

by means of (6). (Even this particular case in [2] involves something, which does not belong to q-MZVs.)

We use the strategy of [2] but start with the identity

$$\frac{1}{(1-x)(1-y)} = \frac{1}{2}\big(f(x,y)+f(y,x)\big), \quad \text{where } f(x,y) = \frac{1+x}{(1-x)(1-xy)}, \tag{21}$$

which is the particular case of Lemma 2 in [23]. Differentiate both sides of (21) with respect to x and y,

$$\frac{\partial f(x,y)}{\partial x\partial y} = \frac{2}{(1-x)^2(1-xy)^2} + \frac{4}{(1-x)(1-xy)^3} - \frac{4}{(1-x)(1-xy)^2} - \frac{1+xy}{(1-xy)^3};$$

multiply the result by xy; substitute $x = q^n$ and $y = q^m$; use

$$\begin{aligned}\sum_{n,m=1}^{\infty}\frac{xy(1+xy)}{(1-xy)^3}\bigg|_{x=q^n,\,y=q^m} &= \sum_{l=1}^{\infty}(l-1)\frac{q^l(1+q^l)}{(1-q^l)^3}\\ &= \delta\sum_{l=1}^{\infty}\frac{q^l}{(1-q^l)^2} - \sum_{l=1}^{\infty}\frac{q^l(1+q^l)}{(1-q^l)^3} = \delta\mathfrak{z}_q(2) - 2\mathfrak{z}_q(3) + \mathfrak{z}_q(2).\end{aligned}$$

All this finally results in

$$\mathfrak{z}_q(2)^2 + \delta\mathfrak{z}_q(2) = 2\mathfrak{z}_q(2,2) + 4\mathfrak{z}_q(3,1) - 4\mathfrak{z}_q(2,1) + 2\mathfrak{z}_q(3) - \mathfrak{z}_q(2),$$

which is the desired q-analogue of (20).

One can also use Ramanujan's system of differential equations satisfied by the Eisenstein series [19] to get rid of the term $\delta\mathfrak{z}_q(2)$. Namely, using

$$\delta\mathfrak{z}_q(2) = \mathfrak{z}_q(2) - 5\mathfrak{z}_q(3) + 5\mathfrak{z}_q(4) - 2\mathfrak{z}_q(2)^2$$

we obtain

$$\mathfrak{z}_q(2)^2 = -2\mathfrak{z}_q(2,2) - 4\mathfrak{z}_q(3,1) + 4\mathfrak{z}_q(2,1) + 5\mathfrak{z}_q(4) - 7\mathfrak{z}_q(3) + 2\mathfrak{z}_q(2),$$

which is also a q-analogue of (20). But for a general q-analogue of (19) we do expect terms involving $\delta\mathfrak{z}_q(s)$ and $\delta\mathfrak{z}_q(t)$, hence working in the δ-*differential* algebra generated by the multiple q-zeta values (6) (or (7)). We wonder if there exists a nice form of double shuffle relations in this differential algebra.

Another related problem, which is worth being investigated in its own but goes far beyond the aim of this paper, is the comparison of the limiting $q \to 1$ behavior of q-analogues versus the regularized MZVs and MZSVs themselves for non-admissible indices $\boldsymbol{k} = (k_1, \dots, k_n)$ with $k_1 = 1$. By [12, formula (2.5)],

$$\lim_{\substack{q\to 1\\ |q|<1}} \left((1-q)\mathfrak{z}_q(1) - \log\frac{1}{1-q}\right) = \lim_{\substack{q\to 1\\ |q|>1}} \left(\zeta_q(1) - \log\frac{1}{1-q^{-1}}\right) = \gamma,$$

Euler's constant, suggesting that at least one of the two q-versions has a good chance to be related to one of the standard regularizations Z^* or $Z^{\sqcup\!\sqcup}$ given in [11, Proposition 1]. Note however that for the multiple q-zeta values (3) we need to work with $|q| > 1$, whereas for our q-MZVs (6) we can stay with the more natural domain $|q| < 1$.

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