Löwner partial ordering and space preordering of Hermitian non-negative definite matrices

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Abstract

If a matrix \( A \) is below a matrix \( B \) with respect to the Löwner partial ordering within the set of Hermitian non-negative definite matrices, then the columns of \( A \) are linear combinations of the columns of \( B \). In this note, the class of all such combinations providing a complete characterization of the Löwner partial ordering is presented. In addition, new results on partial ordering of squares of Hermitian non-negative definite matrices are given. © 2001 Elsevier Science Inc. All rights reserved.

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1. Introduction

Let \( \mathbb{C}_{m,n} \) denote the set of complex \( m \times n \) matrices and \( \mathbb{C}_m \) denote the set \( \mathbb{C}_{m,m} \). Further, let \( \mathbb{C}_{m}^\geq \) denote the subset of \( \mathbb{C}_m \) consisting of Hermitian non-negative definite matrices. The symbols \( A^*, A^+, A^- \) and \( \mathcal{R}(A) \) stand for the conjugate transpose, the Moore–Penrose inverse, any generalized inverse and the range of \( A \in \mathbb{C}_{m,n} \). The set of all eigenvalues of a matrix \( A \in \mathbb{C}_m \) is denoted by \( \sigma(A) \).

For matrices \( A, B \in \mathbb{C}_m^\geq \), the Löwner partial ordering \( \preceq \), the minus partial ordering \( \preceq^- \), and the star partial ordering \( \preceq^* \) are defined as

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\[ A \preceq B :\iff B - A = GG^* \quad \text{for some matrix } G, \]  
\[ A \sim B :\iff A^{-1} = A^{-}B \quad \text{for some generalized inverse } A^{-} \text{ of } A, \]  
\[ A \preceq B :\iff A^2 = AB, \]

whereas the space preordering \( s \) is defined as

\[ A \overset{s}{\preceq} B :\iff \mathcal{R}(A) \subseteq \mathcal{R}(B). \]

It is clear that

\[ A \overset{s}{\preceq} B \iff A = BK \quad \text{for some matrix } K. \]

Applying Theorem 4.1 of Baksalary and Mitra [3] to matrices \( A, B \in \mathbb{C}_{m}^{ \geq } \), we may characterize \( \preceq \) and \( \preceq^* \) by confining \( K \) to certain classes of matrices, namely,

\[ A \preceq B \iff A = BK \quad \text{for some idempotent matrix } K, \]

i.e., for some matrix \( K = K^2 \), and

\[ A \preceq B \iff A = BK \quad \text{for some Hermitian idempotent matrix } K. \]

The purpose of this paper is to identify the class of matrices \( K \) allowing an analogous characterization of \( \preceq \). In addition, the corresponding relations involving \( A^2 \) and \( B^2 \) are investigated.

As noted by Baksalary and Pukelsheim [4, p. 136], orderings (2) and (3) are restrictions to Hermitian matrices of the general definitions introduced by Hartwig [9] and Drazin [8], respectively. The ordering \( \preceq \) is due to Löwner [11] and can be defined as in (1) also for Hermitian (even for possibly non-Hermitian) matrices \( A, B \in \mathbb{C}_m \).

However, if \( A \) is Hermitian and \( B \) is Hermitian non-negative definite, then \( A \preceq B \) does not necessarily imply \( \mathcal{R}(A) \subseteq \mathcal{R}(B) \), as can be seen from

\[ A = \frac{1}{2} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \]

Hence, for our purpose the assumption \( A, B \in \mathbb{C}_m^{ \geq } \) is essential.

2. Löwner partial ordering

The following lemma is easily derived by using results of Stępiński [12, Theorem 1], Baksalary et al. [5, Eqs. (1.19) and (1.20)] and Baksalary et al. [7, Theorem 1, Eq. (5)].
**Lemma.** Let $A, B \in \mathbb{C}^{m}_{\geq}$. Then the following statements are equivalent:

(i) $A \leq L B$,

(ii) $A \prec L B$ and $\lambda_1(B^{-}A) \leq 1$,

(iii) $A \prec L B$ and $AB^{-}A \leq A$,

where $\lambda_1(\cdot)$ denotes the largest eigenvalue and $B^{-}$ is any generalized inverse of $B$.

From the above lemma it follows that the space preordering between two Hermitean non-negative definite matrices can be described in terms of the Löwner partial ordering, namely,

$$A \prec L B \iff AA^{+} \leq L BB^{+},$$

(8)

since $BB^{+}$ is a generalized inverse of itself and the eigenvalues of the product of two orthogonal projectors are known to lie in $[0, 1]$.

If for matrices $A, B \in \mathbb{C}^{m}_{\geq}$ the preordering $A \prec L B$ holds, then the set of non-zero eigenvalues of $B^{-}A$ coincides with the set of non-zero eigenvalues of $B^{+}A$, see e.g. Theorem 1 in [6]. Since the eigenvalues of $B^{+}A$ are real non-negative, statement (ii) in the lemma can be replaced by (ii\#) $A \prec L B$ and $\sigma(B^{-}A) \subset [0, 1]$.

It is clear from the implication ‘(i) $\Rightarrow$ (ii\#)’, that

$$A \leq L B \Rightarrow A = BK \quad \text{for some matrix } K \text{ with } \sigma(K) \subset [0, 1],$$

(9)

choose $K = B^{-}A$. The main result of this section is, that the implication in (9) can also be reversed.

**Theorem 1.** Let $A, B \in \mathbb{C}^{m}_{\geq}$. Then $A \leq L B$ if and only if $A = BK$ for some matrix $K$ with $\sigma(K) \subset [0, 1]$.

**Proof.** The ‘only if’ part is clear from the above considerations. To observe the ‘if’ part, let

$$B = U \begin{pmatrix} \Lambda & 0 \\ 0 & 0 \end{pmatrix} U^*,$$

where $U$ is unitary (i.e. $UU^* = U^*U = I_m$) and $\Lambda \in \mathbb{C}_r$ is a diagonal matrix containing the $r \leq m$ real positive eigenvalues of $B$ on its main diagonal. Then

$$S^*BS = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}, \quad \text{where } S = U \begin{pmatrix} \Lambda^{-1/2} & 0 \\ 0 & I_{m-r} \end{pmatrix}.$$

Since $S$ is non-singular, any matrix $K \in \mathbb{C}_m$ can be written as

$$K = S \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} S^{-1},$$
where \( K_{11} \in \mathbb{C}_r \). Now assume that \( A = BK \). Then \( BK = K^*B \), which may equivalently be written as

\[
\begin{pmatrix}
I_r & 0 \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
K_{11} & K_{12} \\
K_{21} & K_{22}
\end{pmatrix}
= \begin{pmatrix}
K^*_{11} & K^*_{12} \\
K^*_{21} & K^*_{22}
\end{pmatrix}
\begin{pmatrix}
I_r & 0 \\
0 & 0
\end{pmatrix}.
\]

This yields \( K_{12} = 0 \), which in turn gives \( \sigma(K) = \sigma(K_{11}) \cup \sigma(K_{22}) \). Therefore, \( \sigma(K_{11}) \subset [0, 1] \) since \( \sigma(K) \subset [0, 1] \). In view of \( \sigma(K_{11}) \subset [0, 1] \) and \( K_{11} = K^*_{11} \), the matrix difference

\[
S^*BKS - S^*K^*BKS = \begin{pmatrix}
K_{11} - K^*_{11} & 0 \\
0 & 0
\end{pmatrix}
\]

is Hermitian non-negative definite. Hence, also

\[
BK - K^*BK = BK - K^*BB^*BK = A - AB^*A
\]

is Hermitian non-negative definite, or, in other words, \( AB^*A \preceq A \). Eventually, \( A \preceq B \) follows from the lemma. \( \square \)

We note that the idea of the proof is partly similar to some idea in the proof of Theorem 3 in [2], the latter being concerned with admissible estimation in linear models. The usefulness for characterizing the relation \( A \preceq B \), however, is revealed here.

Since any idempotent matrix \( K \) has only eigenvalues in \( \{0, 1\} \), it is clear from (6), (7) and Theorem 1 that

\[
A \preceq B \Rightarrow A^* \preceq B \Rightarrow A \preceq B^L.
\]

which has been observed earlier by Baksalary et al. [1, p. 84], see also [4, Eq. (6)].

3. Ordering of squares

Baksalary and Pukelsheim [4] compare the relations \( A \preceq B, A^* \preceq B \) and \( A \preceq B \) with the corresponding relations involving \( A^2 \) and \( B^2 \). The latter specify further partial orderings, which is due to the fact that equality between \( A^2 \) and \( B^2 \) implies equality between \( A \) and \( B \) for \( A, B \in \mathbb{C}_m^+ \). From Theorem 3 in [4] it is seen that

\[
A^2 \preceq B^2 \Leftrightarrow A^* \preceq B.
\]

The inequality \( A^2 \preceq B^2 \) for matrices \( A, B \in \mathbb{C}_m^+ \) may be characterized as follows.

**Theorem 2.** Let \( A, B \in \mathbb{C}_m^+ \). Then the following statements are equivalent:

(i) \( A^2 \preceq B^2 \).

(ii) \( A = BK \) for some matrix \( K \) with \( \sigma(K) \cup \sigma(KK^*) \subset [0, 1] \).

(iii) \( A = BK \) for some matrix \( K \) with \( \sigma(KK^*) \subset [0, 1] \).
Proof. That (i) implies (ii) follows from the lemma, by noting that $A^2 \leq B^2$ implies $A = BK$ and $\lambda_1(KK^*) \leq 1$ for $K = B^+A$, see also [7, Eq. (7)]. Obviously this matrix $K$ has only real non-negative eigenvalues such that $\lambda_1(K) \leq 1$ due to $\lambda_1(KK^*) \leq 1$. Since any matrix $KK^*$ has only real non-negative eigenvalues, it follows that $A^2 \leq B^2$ implies $A = BK$ with $\sigma(K) \cup \sigma(KK^*) \subset [0, 1]$ for $K = B^+A$. That (ii) implies (iii) is clear. To see that (iii) implies (i), let $A = BK$ for a matrix $K$ such that $\sigma(KK^*) \subset [0, 1]$. Then the inequality $KK^* \leq I_m$ implies $BKK^*B \leq BB$, i.e., $A^2 \leq B^2$. □

Baksalary et al. [7, p. 121] state that

$$A_1 A_1^* L \leq B_1 B_1^* \iff A_1 = B_1 K_0 \quad \text{for some contraction } K_0,$$

i.e., for some matrix $K_0$ satisfying $\sigma(K_0 K_0^*) \subset [0, 1]$. Theorem 2 shows that for the special case $A_1 \in \mathbb{C}^m_n$ and $B_1 \in \mathbb{C}^m_n$, it is possible to sharpen the right-hand part of such statement in a specific way. From Theorem 1 and the equivalence between (i) and (ii) in Theorem 2 it is immediate that

$$A^2 \leq B^2 \implies A \leq B \quad (12)$$

for matrices $A, B \in \mathbb{C}^m_n$. This implication is well known in the literature, see e.g. [4, p. 137]. The authors note that the implication in (12) can be reversed when $A, B \in \mathbb{C}^m_n$ commute. Such a statement follows immediately from our Theorems 1–3, where Theorem 3 is as follows.

Theorem 3. Let $A, B \in \mathbb{C}^m_n$. Then the following two statements hold:

(i) $A \prec B$ and $AB = BA$ hold together if and only if $A = BK$ for some Hermitian matrix $K$.

(ii) $A \leq B$ and $AB = BA$ hold together if and only if $A = BK$ for some Hermitian matrix $K$ with $\sigma(K) \subset [0, 1]$.

Proof. If $A \prec B$ and $AB = BA$, then $A = BB^+A$ and $B^+A = B^+ABB^+ = B^+BAB^+ = AB^+$, showing that $A = BK$, where $K = B^+A$ is Hermitian. Conversely, if $A = BK$ for some Hermitian matrix $K$, then $AB = BKB = BA$, showing (i). To observe (ii) note that if $A \leq B$ and $AB = BA$, then $A = BK$ for $K = B^+A$, where $K = K^*$ and $\sigma(K) \subset [0, 1]$. Conversely, if $A = BK$ for some Hermitian matrix $K$ with $\sigma(K) \subset [0, 1]$, then $A \leq B$ and $AB = BA$ follow from Theorem 1 and statement (i). □
Considering orderings of squares of Hermitian non-negative definite matrices, it remains to investigate $A^2 \preceq B^2$.

**Theorem 4.** Let $A, B \in \mathbb{C}^m$. Then $A^2 \preceq B^2$ if and only if $A = BK$ for some matrix $K$ with $KK^*K = K$ and $KK^*BB^+ = BB^+KK^*$.

**Proof.** To observe the ‘only if’ part, note that $A^2 \preceq B^2$ if and only if $R(A) \subseteq R(B)$ and $AA(BB)^+AA = AA$, (13)

see e.g. Eq. (9) in [4]. Using $(BB)^+ = B^+B^+$, and multiplying the equality in (13) from the left by $B^+$ and from the right by $A^+$ yields

$$B^+AAB^+B^+A = B^+A,$$

showing $KK^*K = K$ and $KK^*BB^+ = BB^+KK^*$ for $K = B^+A$. Since $R(A) \subseteq R(B)$ is equivalent to $A = BK$, the ‘only if’ part is shown. For the ‘if’ part, let

$$B = U \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} U^*,$$

where $U$ is unitary and $A \in \mathbb{C}_r$ is a diagonal matrix containing the $r \leq m$ real positive eigenvalues of $B$ on its main diagonal. Any matrix $K \in \mathbb{C}_m$ can be written as

$$K = U^* \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix} U,$$

where $K_{11} \in \mathbb{C}_r$. Assume $A = BK$. Then $BK = K^*B$ yields $K_{12} = 0$, and $U^*AU$ can be written as

$$U^*AU = \begin{pmatrix} AK_{11} & 0 \\ 0 & 0 \end{pmatrix}.$$

Further, assume that $K$ satisfies $KK^*K = K$ and $KK^*BB^+ = BB^+KK^*$. Using $K_{12} = 0$, the latter gives $K_{11}K_{21}^* = 0$, which in turn shows that the former condition yields $K_{11}K_{11}^*K_{11} = K_{11}$, i.e., $K_{11}^* = K_{11}^+$. Hence, from the identity

$$U^*B^2U - U^*A^2U = \begin{pmatrix} A^2 - AK_{11}K_{11}^+A & 0 \\ 0 & 0 \end{pmatrix},$$

we obtain

$$\text{rk}(B^2 - A^2) = \text{rk}(I_r - K_{11}K_{11}^+) = \text{rk}(B) - \text{rk}(K_{11}) = \text{rk}(B^2) - \text{rk}(A^2).$$

This is satisfied if and only if $A^2 \preceq B^2$, see e.g. Eq. (4) in [4]. □

We note that a matrix $K = KK^*$ is called a partial isometry. It is seen that $A = BK$ for some $K = KK^*$ alone does not give $A^2 \preceq B^2$, consider
A = \frac{1}{2} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad K = \frac{1}{2} \begin{pmatrix} \sqrt{3} & 0 \\ 1 & 0 \end{pmatrix}.

Baksalary and Pukelsheim [4] noticed that neither of the relations $A \preceq B$ and $A^2 \preceq B^2$ implies the other. We may strengthen this observation by noting that for matrices $A, B \in \mathbb{C}^m$ the relations $A \preceq B$ and $A^2 \preceq B^2$ cannot hold together unless $A \preceq B$.

**Theorem 5.** Let $A, B \in \mathbb{C}^m$. Then $A \preceq B$ and $A^2 \preceq B^2$ hold together if and only if $A^* \preceq B$.

**Proof.** It is clear from (10) and (11) that $A^* \preceq B$ implies $A \preceq B$ and $A^2 \preceq B^2$. Conversely, if $A \preceq B$, then

$$A = BK = AK$$

for $K = B^+ A$,

see e.g. Eq. (5) in [4]. Multiplying $A = AK$ from the left by $B^+$ yields $K = K^2$. In addition, the proof of Theorem 4 shows that $A^2 \preceq B^2$ implies $KK^*K = K$. But $K$ can satisfy both properties only if $K$ is Hermitian and idempotent. Hence, $A \preceq B$ follows in view of (7). □

Alternatively, the ‘only if’ part of Theorem 5 can be deduced from known results. As a matter of fact, from Theorem 2 in [4] we know that the two inequalities $A \preceq B$ and $A^2 \preceq B^2$ imply $AB = BA$. In addition, from Corollary 1 in [10] it follows that $A \preceq B$ and $AB = BA$ imply $A^* \preceq B$ for Hermitian matrices $A, B \in \mathbb{C}_m$.

4. Conclusion

We summarize the characterization of certain inequalities between matrices $A, B \in \mathbb{C}^m$ via the space preordering in the following table.

The table reads: two matrices $A, B \in \mathbb{C}^m$ are related as described in some row of the first column if and only if $A = BK$ for some matrix $K$ satisfying the property described in the same row of the second column.

The following scheme summarizes the relationships between the orderings $A \preceq B$, $A \preceq B$, $A \preceq B$ and the orderings $A^2 \preceq B^2$, $A^2 \preceq B^2$, $A^2 \preceq B^2$ involving squares of matrices $A, B \in \mathbb{C}_m$. 
\[ A \preceq B \quad \text{and} \quad A^* \preceq B^* \quad \Rightarrow \quad A^2 \preceq B^2 \]