JOURNAL OF COMBINATORIAL THEORY, Series B 43, 95–115 (1987)

Nowhere-Zero Integral Chains and Flows in Bidirected Graphs

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Communicated by W. T. Tutte

Received June 8, 1984

General results on nowhere-zero integral chain groups are proved and then specialized to the case of flows in bidirected graphs. For instance, it is proved that every 4-connected (resp. 3-connected and balanced triangle free) bidirected graph which has at least an unbalanced circuit and a nowhere-zero flow can be provided with a nowhere-zero integral flow with absolute values less than 18 (resp. 30). This improves, for these classes of graphs, Bouchet's 216-flow theorem (*J. Combin. Theory Ser. B* **34** (1982), 279–292). We also approach his 6-flow conjecture by proving it for a class of 3-connected graphs. Our method is inspired by Seymour's proof of the 6-flow theorem (*J. Combin. Theory Ser. B* **30** (1981), 130–136), and makes use of new connectedness properties of signed graphs.

1. INTRODUCTION

The graphs we consider are finite and simple (i.e., without loops or multiple edges). We recall that a *cycle* in a graph $G=(V,E)$ is a regular connected subgraph of degree two. It wil be convenient to consider a cycle as a set of edges as well as a subgraph. Moreover, if F is a set of edges of G, the *subgraph of G induced by F* is the graph whose edge set is F and whose vertex set is $V(F)=\{v\in V \mid v \text{ is an end of some edge of } F\}$. This induced subgraph is also denoted by F. For other standard notions and basic properties of graphs, we refer to [1, 2, 5]. It is also assumed that the reader is familiar with the theory of chain groups and their related matroids. Otherwise he should refer to [8, 9].

Let $G=(V, E)$ be a graph, each edge of which is viewed as a pair of two half-edges (one incident to each end vertex). The set of half-edges of G is denoted by $H(G)$. If v is a vertex, $H(v)$ is the set of half-edges of G incident to v. If h is a half-edge, the edge containing it is denoted by e_h.

* The author is indebted to many valuable discussions with F. Jaeger and C. Payan during a stay at Laboratoire IMAG, Grenoble, France.

0095-8956/87 $3.00

582b/43/1-7

A *bidirected graph* is a graph $G = (V, E)$ together with a given signature $\tau\colon H(G) \to \{-1, +1\}$ (see [3] for motivations). If for every edge $e = \{h, h'\}$ we have $\tau(h)\,\tau(h') = -1$, then the bidirected graph can be considered as a directed graph in the usual sense: each edged will be directed from the vertex corresponding to the half-edge with positive sign to the other one.

An *(integer) flow in the bidirected graph* $G = (V, E)$ is an integer-valued function f on E such that for every vertex v of V

$$\sum_{h \in H(v)} \tau(h) \cdot f(e_h) = 0.$$

The set of flows in the bidirected graph G is a chain group, denoted by $N(G)$, whose related matroid is denoed by $M(N(G))$. An *m-isthmus* is a coloop of $M(N(G))$.

If $k \geqslant 1$ is an integer, a *k-flow* is a flow f such that for every element e of E, $|f(e)| < k$. A *nowhere-zero flow* is a flow f such that for every element e of E, $|f(e)| \neq 0$. Jaeger proved that every directed graph without isthmus has a nowhere-zero 8-flow [6]. This was the first result towards Tutte's conjecture that "Every directed graph without isthmus has a nowhere-zero 5-flow" (see [6] for references and a nice synthesis of known and related results). Bouchet uses the idea of Jaeger's proof to show

1. THEOREM [3, Theorem 4.3]. *Every bidirected graph without m-isthmus can be provided with a nowhere-zero* 216-*flow*.

However, he also proposes the following analog of Tutte's conjecture.

2. CONJECTURE (Bouchet [3]). *Every bidirected graph without m-isthmus can be provided with a nowhere-zero* 6-*flow*.

On the other hand, Seymour improved Jaeger's result by proving that "Every directed graph without isthmus has a nowhere-zero 6-flow" [7]. Using a method similar to Seymour's proof of the 6-flow theorem, we improve Bouchet's result. We proceed by

(i) generalizing the study of Seymour's "k-closure" to all chain groups, then

(ii) specializing to flows in bidirected graphs.

We introduce, for the second part (ii), a partition of the edge set of a bidirected graph G, associated with connectedness properties of the matroid $M(N(G))$.

3. DEFINITIONS. Let us recall that a *signed graph* is a graph $G = (V, E)$

together with a given signature $\sigma: E \to \{-1, +1\}$. A cycle of G is *balanced* if it has an even number of negatively signed edges and *unbalanced* otherwise. A subset $F \subseteq E$ is *balanced* if every cycle of the graph F is balanced.

There is a matroid $M(G)$ associated with the signed graph G, characterized as follows.

4. PROPOSITION [10, Theorem 5.1]. *If $G=(V, E)$ is a signed graph, a set of edges is a circuit of $M(G)$ iff it is either*

(i) *a balanced cycle, or*

(ii) *the union of two unbalanced cycles which have exactly one vertex in common, or*

(iii) *the union of two vertex-disjoint unbalanced cycles and a simple path meeting each of the two cycles at exactly one of its endpoints.*

(A special case is studied by Doob in [4].)

On the other hand, if $G=(V, E)$ is a graph, every signature $\tau: H(G) \to \{-1, +1\}$ gives rise to a signature $\sigma: E \to \{-1, +1\}$ defined by $\sigma(e) = -\tau(h) \cdot \tau(h')$ for every edge $e = \{h, h'\}$. Thus, with every bidirected graph is associated, in this manner, a signed graph. The reader may find in [3] and [10] more relationships between associated bidirected and signed graphs. The main result which we need is

5. THEOREM (Zaslavsky). *Let G be a bidirected graph and $N(G)$ its chain group of flows. If $M(G)$ is the matroid of the associated signed graphs, then $M(N(G)) = M(G)$.*

This result constitutes the link between connectedness properties in signed graphs as studied in Section 3 and their use for flows in bidirected graphs.

2. NOWHERE-ZERO INTEGRAL CHAINS

2.1. *Definitions and Basic Results*

Let N be an (integral) chain group on a finite set E. Let us recall that the supports of elementary chains of N are the circuits of a matroid denoted by $M(N)$. They are also called the circuits of N.

An element e of E is an *isthmus* (or coloop) of N if it belongs to no circuit of N. Equivalently, it is an element of E for which every chain takes the value 0. A *nowhere-zero chain* is a chain f such that $f(e) \neq 0$ for every element e of E.

1.1. [3, Proposition 3.1]. *There exists a nowhere-zero chain in N iff N has no isthmus.*

A *principal chain* of N is an elementary chain f of N which is a submultiple of all the elementary chains of N with the same support as f.

1.2. [3, Proposition 3.2]. *For every elementary chain of N, there exists a principal chain of N with the same support.*

The *length* of a chain f is the integer $l(f)=\max\{|f(e)| \mid e\in E\}$. The length of the chain group N is the integer

$$l(N)=\max\{l(f) \mid f \text{ is a principal chain of } N\}.$$

Note that our length is the width of [3], minus one.

1.3. [3, Corollary 2.3]. *Let $N(G)$ be the chain group of flows in a bidirected graph G. Then $l(N(G))\leqslant 2$ and the following are equivalent statements*:

(i) *$M(G)$ has a type* (iii) *circuit* (as defined in Proposition 4, Sect. 1).

(ii) $l(N(G))=2$.

Let us recall that if $k\geqslant 2$ is an integer, two chains f and f' are *k-equivalent* if $f(e)\equiv f'(e) \pmod k$ for every element e of E.

1.4. [4, Proposition 3.5]. *Let $k\geqslant 2$ be an integer. Every chain f of N is* k-equivalent to a chain f' of N such that $l(f')<k\cdot l(N)$.

2.2. *Seymour's Closure and Nowhere-Zero Chains*

We now extend the results of [7] to general chain groups.

2.1. *Let M be a matroid on the finite set E and $k\geqslant 1$ an integer. If $X\subseteq E$, the k*-closure of X (*relative to M*) *is the smallest set $Y\subseteq E$ with the following properties*:

(i) $X\subseteq Y$,

(ii) *there is no circuit C of M with $0<|C-Y|\leqslant k$.*

Note that the set E satisfies (i) and (ii) and if Y_1 and Y_2 both satisfy (i) and (ii) then so does $Y_1\cap Y_2$. Therefore, the k-closure of any subset $X\subseteq E$, denoted by $\langle X\rangle_k$, always exists. It is easy to see that the correspondence $X\to\langle X\rangle_k$ is indeed a closure operator, that is, it satisfies

$$X\subseteq\langle X\rangle_k, \qquad \langle\langle X\rangle_k\rangle_k=\langle X\rangle_k, \qquad X\subseteq Y \text{ implies } \langle X\rangle_k\subseteq\langle Y\rangle_k.$$

We have also $\langle X\rangle_k \subseteq \langle X\rangle_{k+1}$.

From now on, the k-closure is relative to the matroid associated with the chain group N on E.

2.2. PROPOSITION. *Let N be a chain group on the finite set E and $k>1$ an integer such that* g.c.d.$(k, g(e))=1$ *for every principal chain g of N and every element e of E such that $g(e)\neq 0$. For every subset X of E such that $\langle X\rangle_{k-1}=E$, there exists a chain f of N such that $f(e)\not\equiv 0 \pmod k$ for every element e of $E-X$.*

Proof. By induction on $|E-X|$. If $|E-X|=0$, then $X=E$ and we may take f equal to the zero chain. If $|E-X|\neq 0$, since $X\neq E=\langle X\rangle_{k-1}$, there is a circuit C of N such that $0<|C-X|\leqslant k-1$. If we write $X'=X\cup C$, we have $\langle X'\rangle_{k-1}=E$. By the induction hypothesis, there is a chain g' of N such that $g'(e)\not\equiv 0 \pmod k$, $\forall e\in E-X'$. Let g be a principal chain of N with support C (which exists by 1.2).

LEMMA. *There exists an integer n, $0\leqslant n\leqslant k-1$, such that*

$$ng(e)+g'(e)\not\equiv 0 \pmod k, \qquad \forall e\in C-X.$$

Proof. Just note that for two integers n_1, n_2, $0\leqslant n_i\leqslant k-1$ $(i=1,2)$ such that $n_i g(e)+g'(e)\equiv 0 \pmod k$ we also have (n_1-n_2), $g(e)\equiv 0 \pmod k$. Therefore, g.c.d.$(k, g(e))=1$ implies that $n_1=n_2$. Hence, each element of $C-X$ excludes at most one value of n. It follows that the total number of forbidden values of n is at most $|C-X|\leqslant k-1$. This completes the proof of the lemma.

End of the Proof of 2.2. Take $f=ng+g'$. The chain f is in N and

$$f(e)=g'(e)\not\equiv 0 \pmod k, \qquad \forall e\in E-(X\cup C)$$

$$f(e)=ng(e)+g'(e)\not\equiv 0 \pmod k, \qquad \forall e\in C-X.$$

2.3. COROLLARY. *Under the hypotheses of Proposition* 2.2, *there exists a chain f' of N such that*

(i) $f'(e)\not\equiv 0 \pmod k$, $\forall e\in E-X$;

(ii) $|f'(e)|<k\cdot l(N)$, $\forall e\in E$.

Proof. This is clear from 1.4 applied to the chain f found in 2.2.

The following theorem is the general result on nowhere-zero chains which we need.

2.4. THEOREM. *Let N be a chain group on the finite set E. Let $m = l(N)$. If there exist an integer $k > 1$ and disjoint circuits $C_1, \ldots, C_r$ of N such that:*

(i) g.c.d.$(k, g(e)) = 1$ *for every principal chain g of N and every element e of E such that $g(e) \neq 0$,*

(ii) $\langle C_1 \cup \cdots \cup C_r \rangle_{k-1} = E$,

then N has a chain f such that, for every element e of E

$$0 < |f(e)| < km(m+1).$$

Proof. By 2.3, there exists a chain f' of N such that $|f'(e)| < km$ for every element e of E, and $f'(e) \not\equiv 0 \pmod{k}$ for every element e of $E - (C_1 \cup \cdots \cup C_r)$. For $i = 1, \ldots, r$, let g_i be a principal chain of N with support C_i and let $g = \sum_{i=1}^{r} g_i$. The support of g is the union of the C_i, and if e belongs to C_i, $g(e) = g_i(e)$. Let

$$f = f' + km \cdot g.$$

The chain f is in N and $f(e) \neq 0$ for every element e of E. Indeed, if e does not belong to the union of the C_i's, $f(e) = f'(e) \neq 0$, and if e belongs to the union of the C_i's,

$$\begin{aligned} |f'(e) &< km \\ &\leqslant |km \cdot g(e)|, \end{aligned}$$

and hence

$$f(e) = f'(e) + km \cdot g(e) \neq 0.$$

On the other hand,

$$\begin{aligned} |f(e)| &\leqslant |f'(e)| + km \cdot |g(e)| \\ &\leqslant km + km \cdot m = km(m+1). \end{aligned}$$

2.5. *Remark.* The integer $k > 1$ fulfils hypothesis (i) of 2.4, in particular, in the following cases:

if $l(N) = 1$,	every k,
if $l(N) = 2$,	every odd k.

3. CONNECTEDNESS IN SIGNED GRAPHS

Let us recall that if M is a matroid on the finite set E, the binary relation R on E defined for every x and y belonging to E by xRy iff $x = y$ or there exists a circuit C of M such that $\{x, y\} \subseteq C$, is an equivalence relation [9, Chap. 5]. The equivalence classes of the relation R are the connected components of M.

3.1

1.1. DEFINITIONS. If $G = (V, E)$ is a signed graph, each connected component of the matroid $M(G)$ induces a subgraph of G which we call an *m-connected component of* G (m stands here for matroid).

The *signed graph* $G = (V, E)$ *is m-connected* if the set E is an m-connected component, i.e., if $M(G)$ is connected and G has no isolated vertices. It is clear that an m-connected component is a connected and m-connected graph. A *subset* F *of* E *is m-connected* if the subgraph of G induced by F is an m-connected signed graph. It is clear that if F and F' are both m-connected and $F \cap F' \neq \emptyset$, then $F \cup F'$ is also m-connected. Since G has no loops an element e of E is a coloop of $M(G)$ iff the set $\{e\}$ is an m-connected component of G. Such an edge we call an *m-isthmus*. The signed graph is *m-balanced* if every circuit of $M(G)$ is of type (i), and *m-unbalanced* otherwise.

1.2. *Remarks.* (i) A balanced graph is m-balanced but the converse is not always true, even if the graph is m-connected. For instance, the complete graph on four vertices where every edge is negative is an m-balanced and m-connected graph but not a balanced one.

(ii) An m-connected graph which is not an m-isthmus has at least three edges and thus at least three vertices.

(iii) If G is a signed m-connected graph with no isolated vertices and which is not an m-isthmus, for every pair $\{x, y\}$ of the vertices of G there is a circuit C of $M(G)$ such that $\{x, y\} \subseteq V(C)$.

(iv) It should be noted that some of the results of this Section 3 appear in [10], though in a rather different form.

3.2. *Basic Results*

Let us recall that a *theta-graph* is a graph which is the union of a cycle and a simple path which are edge-disjoint and meet exactly at the endpoints of te path.

2.1. LEMMA. *Let* $G = (V, E)$ *be a connected graph and let* C *and* C' *be two distinct cycles of* G. *Then there exists a subgraph* C'' *of* G *which is either*

(a) *the union* $C \cup C'$ *if* $|V(C) \cap V(C')| = 1$;

(b) *the union* $C \cup C' \cup P$, *where* P *is a simple path of* G *meeting each of the two cycles at exactly one of its endpoints if* $|V(C) \cap V(C')| = 0$;

(c) *a theta-graph* $C' \cup P$ *where* P *is a subpath of* C *if* $|V(C) \cap V(C')| \geqslant 2$. *In this case, for every edge* e *belonging to* $C - C'$ *such a subpath* P *of* C *containing* e *exists and is unique.*

Proof. If $|V(C) \cap V(C')| \leqslant 1$, the result is trivial. If $|V(C) \cap V(C')| \geqslant 2$, let e be an edge of $C - C'$ with ends u and v. Let u' (resp. v') be the first vertex of C' encountered when one walks along the path $C - \{e\}$ from u (resp. v) to v (resp. u). It is clear that $u' \neq v'$. Let P be the subpath of C with ends u' and v' and containing e. Clearly $C' \cup P$ is a theta-graph and P is the unique subpath of C containing e such that $C' \cup P$ is a theta-graph.

As a consequence of Lemma 2.1, one obtains the following result.

2.2. Proposition. *If* C *and* C' *are two distinct unbalanced cycles of a connected signed graph* G, *then*

(i) *If* $|V(C) \cap V(C')| \leqslant 1$, *there exists a circuit* C_1 *of* $M(G)$ *such that* $C \cup C' \subseteq C_1$.

(ii) *If* $|V(C) \cap V(C')| \geqslant 2$, *for every edge* e *of* $C - C'$ *there exists a subpath* P *of* C *containing* e *and a subpath* P' *of* C' *such that* $P \cup P'$ *is a balanced cycle of* G.

Proof. If $|V(C) \cap V(C')| \leqslant 1$, by 2.1(a) or (b), the subgraph C'' we have chosen is a circuit C_1 of $M(G)$ such that $C \cup C' \subseteq C_1$.

If $|V(C) \cap V(C')| \geqslant 2$, by 2.1(c) there exists a subpath P of C containing e such that $C'' = C' \cup P$ is a theta-graph. Since $C' \subseteq C''$ is unbalanced, one (and only one) of the cycles of C'' must be balanced, and this balanced cycle contains P and a subpath P' of C'.

2.3. Corollary. *If* G *is a connected signed graph, then* G *has at most one unbalanced* m-*connected component.*

Proof. If C and C' are two unbalanced cycles of G, they cannot be contained in distinct m-connected components of G.

So, from now on we shall speak of THE unbalanced m-connected component of a connected signed graph, if such a component exists.

2.4. *Remark.* It should be pointed out that an unbalanced connected signed graph needs not have an unbalanced m-connected component (see Lemma 3.1).

3.3. *The Graph of the m-Connected Components*

We are now in position to define for signed graphs the analog of the block cutpoint tree (see [5, pp. 35–36). If $G=(V, E)$ is a signed graph, let $C(G)$ be the set of m-connected components of G and V_c the set of vertices of G belonging to at least two m-connected components of G. The set V_c is empty iff each connected component of G is m-connected.

The *graph of the m-connected components of* G is the bipartite graph whose vertex set is $C(G) \cup V_c$ and where the vertex v belonging to V_c is adjacent to the component B belonging to $C(G)$ iff v belongs to $V(B)$. This graph is denoted by $\Gamma(G)$.

The graph $\Gamma(G)$ has no edge iff V_c is empty. Moreover, $\Gamma(G)$ is connected iff G is connected and more generally, the connected components of $\Gamma(G)$ are the $\Gamma(G_i)$'s where the G_i's are the connected components of G.

By [10, Corollary 3.3], if G is balanced $M(G)$ is the usual cycle matroid of G. Therefore, $\Gamma(G)$ coincides with the block cutpoint tree if G is balanced.

3.1. LEMMA. *Let $G=(V, E)$ be a connected signed graph and $u_1 B_1 \cdots u_p B_p u_{p+1}$ a cycle of $\Gamma(G)$, where, for every $i=1,\ldots, p$, u_i is a vertex of G and B_i is an m-connected component ($u_{p+1}=u_1$). If for every $i=1,\ldots, p$, P_i is a path in B_i linking u_i to u_{i+1}, then*

(i) *the union $R=P_1 \cup \cdots \cup P_p$ is an unbalanced cycle of G;*

(ii) *G has no unbalanced m-connected component.*

Proof. (i) It is clear that R is the union of mutually edge-disjoint cycles of G, each of which meets at least two components in $\{B_1,\ldots, B_p\}$ and is unbalanced. If R contains distinct unbalanced cycles C and C', then by Proposition 2.2, Section 2, we can find a circuit meeting simultaneously two distinct m-connected components in $\{B_1,\ldots, B_p\}$, which is impossible.

(ii) If B is an unbalanced m-connected component of G, let R' be an unbalanced cycle of B. As in the proof of (i), applying Proposition 2.2 to R and R', we can find a circuit meeting simultaneously two distinct m-connected components in $\{B_1,\ldots, B_p\}$, which is contradiction.

As a consequence of Lemma 3.1, we obtain

3.2. THEOREM. *Let G be a signed graph. If B and B' are two distinct m-connected components of G, then $|V(B) \cap V(B')| \leqslant 2$.*

Proof. If u and v are distinct vertices of $V(B) \cap V(B')$, by 3.1(ii) all the simple paths joining u to v have the same parity, which we denote by

$p_B(u, v)$. We define similarly $p_{B'}(u, v)$ and by 3.1(i) $p_B(u, v) \neq p_{B'}(u, v)$. On the other hand, if w is a third vertex in $V(B) \cap V(B')$, we have also

$$p_B(u, w) = p_B(u, v) + p_B(v, w).$$

But $p_B(u, v) \neq p_{B'}(u, v)$ and $p_B(v, w) \neq p_{B'}(v, w)$ imply $p_B(u, w) = p_{B'}(u, w)$, and hence a contradiction.

The following result gives the structure of $\Gamma(G)$ in general.

3.3. THEOREM. *Let $G = (V, E)$ be a connected signed graph. Then*

(i) *$\Gamma(G)$ has at most one cycle.*

(ii) *If $\Gamma(G)$ has a cycle, then each m-connected component of G is balanced.*

Moreover, exactly one of the following statements holds:

(a) *There exist two m-connected components B and B' of G such that $V(B) \cap V(B') = \{u, u'\} \subseteq V$ and the cycle of $\Gamma(G)$ is $uBu'B'u$, and for every pair of m-connected components $\{B_1, B_2\} \neq \{B, B'\}$, $|V(B_1) \cap V(B_2)| \leqslant 1$.*

(b) *For every pair $\{B, B'\}$ of m-connected components of G, $|V(B) \cap V(B')| \leqslant 1$ and the cycle of $\Gamma(G)$ has length greater than or equal to* 6.

(iii) *If $\Gamma(G)$ has no cycle, then G has at most one unbalanced m-connected component.*

Proof. (i) Let us assume that $\Gamma(G)$ has two distinct cycles $\Gamma = u_1 B_1 \cdots u_p B_p u_{p+1}$ $(u_{p+1} = u_1)$ and $\Gamma' = u'_1 B'_1 \cdots u'_q B'_q u'_{q+1}$ $(u'_{q+1} = u'_1)$, $p \geqslant 2$ and $q \geqslant 2$. Let R (resp. R') be an unbalanced cycle of G obtained as a union of paths $P_i \subseteq B_i$ (resp. $P'_j \subseteq B'_j$) linking u_i to u_{i+1} (resp. u'_j to u'_{j+1}) for every $i = 1, \ldots, p$ (resp. $j = 1, \ldots, q$), as in Lemma 3.1(i).

(a) By Proposition 2.2 applied to R and R', it is easy to show that $\{B_1, \ldots, B_p\} = \{B'_1, \ldots, B'_q\}$ and in particular $p = q$.

(b) If $p = q = 2$, the length of both Γ and Γ' is 4 and, by Theorem 3.2, $V(B_1) \cap V(B_2) = \{u_1, u_2\} = \{u'_1, u'_2\}$, hence $\Gamma = \Gamma'$.

(c) If $p = q \geqslant 3$, it is easy to show that Γ is an induced subgraph of $\Gamma(G)$. Indeed, if Γ is not an induced subgraph of $\Gamma(G)$, we may suppose without loss of generality that there exists an index j, $2 < j \leqslant p$, such that B_1 is adjacent to u_j in $\Gamma(G)$. But the cycle given by $\Gamma'' = u_j B_1 \cdots B_{j-1} u_j$ is a cycle of $\Gamma(G)$ whose set of vertices in $C(G)$ is different from $\{B_1, \ldots, B_p\}$. This is a contradiction with (a) above.

As a consequence of (a) and (c), if $\Gamma \neq \Gamma'$ then $\{u_1, \ldots, u_p\} \neq \{u'_1, \ldots, u'_p\}$. Therefore, there exists an index j in $\{1, \ldots, p\}$ such that u'_j does not belong to $\{u_1,..., u_p\}$. Let i and k be the indices such that $B'_j = B_i$ and $B'_{j+1} = B_k$, that is u'_j is adjacent in $\Gamma(G)$ to B_i and B_k. We may suppose that $B_i = B_1$. If $k = p$, then Γ and the cycle $u'_j B_1 u_1 B_p u'_j$ of $\Gamma(G)$ would contradict (a). If $2 \leqslant k < p$, then Γ and the cycle $u'_j B_1 u_2 \cdots B_k u'_j$ of $\Gamma(G)$ would also contradict (a), and this completes the proof of (i).

To prove (ii), the arguments are similar to those used before, and (iii) is just a recall of the result of Corollary 2.3.

3.4. *Remarks.* (i) The graph $\Gamma(G)$ is a tree if G has an unbalanced m-connected component.

(ii) *A* pendant vertex (i.e., a vertex of degree one) in $\Gamma(G)$ must belong to $C(G)$.

We can also write the following generalization of [7, 4].

3.5. *Let G be a signed graph. If B is a pendant vertex of $\Gamma(G)$ then $B \in C(G)$ (see Remark* 3.4(ii)). *Moreover, if $|B| > 1$ we have*

(i) *B is m-connected;*

(ii) *B has at least three vertices;*

(iii) *B has at most one vertex adjacent in G to vertices not in B.*

3.6. EXAMPLES. (i) If G is a 2-connected signed graph, then $\Gamma(G)$ is a cycle or is reduced to a single vertex.

(ii) A 2-connected signed graph which has a circuit of type (ii) or (iii) is m-connected.

(iii) A 3-connected signed graph with at least three vertices is m-connected iff it has no m-isthmus.

These are easy consequences of the properties of the structure of $\Gamma(G)$, as given in Theorem 3.3.

3.4. *Seymour's Closure and m-Connectedness*

4.1. PROPOSITION. *Let M be a matroid on the finite set E and $p \geqslant 2$ an integer such that every circuit of M has cardinality $\geqslant p$. If $X \subseteq E$ is connected then so is $\langle X \rangle_k$ for $1 \leqslant k \leqslant p-1$.*

Proof. Let us consider a connected $X' \subseteq \langle X \rangle_k$ which contains X and is maximal for these properties. If $X' \neq \langle X \rangle_k$, there exists a circuit C of M such that $0 < |C - X'| \leqslant k$. Then $C \subseteq \langle X \rangle_k$ and $X' \cap C \neq \varnothing$ because $|C - X'| \leqslant k < p \leqslant |C|$. Therefore, $X'' = X' \cup C$ is connected and $X'' \subseteq \langle X \rangle_k$. This is a contradiction.

4.2. COROLLARY. *Let G be a signed graph. If $X \subseteq E$ is m-connected then so is the k-closure $\langle X \rangle_k$ relative to $M(G)$ for $k = 1, 2$.*

4.3. THEOREM. *Let $G = (V, E)$ be a connected signed graph with no m-isthmus. If $E' \subseteq E$ is such that the graph $G' = (V, E')$ is connected, then $\langle E' \rangle_2 = E$.*

Let us start by proving the following:

LEMMA. *If $T \subseteq E$ is a spanning tree of G, then $|E - \langle T \rangle_2| \leqslant 1$.*

Proof of the Lemma. Let $S = \langle T \rangle_2$,

$$E_b = \{e \in E \mid T \cup \{e\} \textit{ has a balanced cycle}\},$$

$$E_u = \{e \in E \mid T \cup \{e\} \textit{ has an unbalanced cycle}\}.$$

If $e \in E_b - S$, the balanced cycle C of $T \cup \{e\}$ is a circuit of $M(G)$ such that $C - S = \{e\}$. This contradicts the definition of S. Therefore, $E_b \subseteq S$ and thus $E - S = E_u - S$. If e and e' are distinct elements of $E_u - S$, let C (resp. C') be the unbalanced cycle of $T \cup \{e\}$ (resp. $T \cup \{e'\}$). By Proposition 2.2 used with the graph $T \cup \{e, e'\}$, there exists a circuit $C_1 \subseteq T \cup \{e, e'\}$ such that $\{e, e'\} \cap C_1 \neq \varnothing$, (the result is trivial when $|V(C) \cap V(C')| \leqslant 1$, and if $|V(C) \cap V(C')| \geqslant 2$, choose as C_1 a balanced cycle of $T \cup \{e, e'\}$ containing the edge e, for example). Therefore $0 < |C_1 - S| \leqslant 2$, which is a contradiction with the definition of $S = \langle T \rangle_2$.

Proof of 4.3. If $T \subseteq E'$ is a spanning tree of G', then $|E - \langle E' \rangle_2| \leqslant |E - \langle T \rangle_2| \leqslant 1$. If $E - \langle E' \rangle_2 = \{e\}$, there exists a circuit C_2 of $M(G)$ such that $e \in C_2$. Thus $0 < |C_2 - \langle E' > 2| = 1$. This is a contradiction.

4. NOWHERE-ZERO FLOWS IN BIDIRECTED GRAPHS

We establish here the main results of this paper.

4.1. *Preliminaries*

By [10, Corollary 3.3], every bidirected and balanced graph can be switched to a graph with no negative edge which can be viewed as a directed graph. Therefore, Seymour's 6-flow theorem can be rephrased as

1.1. THEOREM (Seymour [7]). *Every bidirected and balanced graph without isthmus can be provided with a nowhere-zero 6-flow.*

On the other hand, we note that as in the case of usual flows we have

1.2. PROPOSITION. *Let $k \geqslant 2$ be an integer and G a bidirected graph. If every m-connected component of G has a nowhere-zero k-flow, then so does G.*

Hence Seymour's Theorem 1.1 may be extended as follows:

1.3. THEOREM. *Every bidirected graph with neither m-isthmus nor unbalanced m-connected component can be provided with a nowhere-zero 6-flow.*

The graphs for which this result extends Seymour's theorem are those unbalanced graphs whose graphs of m-connected components do have a cycle (see Theorem 3.3 of Sect. 3.3).

4.2. *The Main Result*

THEOREM. *Every bidirected and m-unbalanced graph G with no m-isthmus can be provided with a nowhere-zero q-flow with*

(2.1) $q=6$ *if G is 3-connected and without type* (iii) *circuit;*

(2.2) $q=18$ *if G is 4-connected;*

(2.3) $q=30$ *if G is 3-connected and has no balanced triangle.*

Note that if $G=(V, E)$ has no balanced triangle, any circuit of $M(G)$ has cardinality $\geqslant 4$. Hence, for every m-connected subset X of E, $\langle X \rangle_3$ is also m-connected by Proposition 4.1 of Section 3.4.

Proofs of 2.1, 2.2, *and* 2.3

We proceed by applying the general theorem on nowhere-zero chains (Theorem 2.4 of Sect. 2.2) with the following values of the parameters (see 1.3, Sect. 2.1 for the values of m)

	(2.1)	(2.2)	(2.3)
k	3	3	5
m	1	2	2
$km(m+1)$	6	18	30

More precisely,

2.4. LEMMA. *If the graph G fulfils the hypotheses of either* (2.1) *or* (2.2) (*resp.* (2.3)), *then there exist mutually disjoint circuits $C_1,\dots, C_r$ of $M(G)$ such that $\langle C_1 \cup \cdots \cup C_r \rangle_2 = E$ (resp. $\langle C_1 \cup \cdots \cup C_r \rangle_3 = E$).*

Note that this lemma completes the proofs in cases (2.1) and (2.2). But in case (2.3), $k=4$ would not do because g.c.d$(4, 2) \neq 1$. However, $\langle C_1 \cup \cdots \cup C_r \rangle_3 = E$ implies $\langle C_1 \cup \cdots \cup C_r \rangle_4 = E$, so we can take $k=5$, and hence the result.

Proof of Lemma 2.4. Let $h=2$ in cases (2.1) and (2.2) and $h=3$ in case (2.3). Let C_1 be a circuit of type (ii) or (iii) of $M(G)$. By Proposition 4.1, Sect. 3.4, $\langle C_1 \rangle_h$ is m-connected, so we can choose $r \geqslant 1$ maximum such that there exist mutually disjoint circuits $C_1, \dots, C_r$ of $M(G)$ such that $\langle C_1 \cup \cdots \cup C_r \rangle_h$ is m-connected and C_1 is of type (ii) or (iii). Let

$$X = C_1 \cup \cdots \cup C_r, \qquad E' = \langle X \rangle_h, \qquad \text{and} \qquad V' = V(E').$$

If $V' = V$, then $\langle E' \rangle_2 = E$ by Theorem 4.3, Section 3.4. On the other hand,

$$\langle E' \rangle_2 \subseteq \langle E' \rangle_h = E',$$

so $E' = E$ in all cases, and the proofs are over.

If $V' \neq V$, let $V'' = V - V'$ and E'' be the subgraph of G induced by V'' and let us find a contradiction. We proceed by examining the structure of the graph $\Gamma(E'')$. In the following figures, we will draw a balanced cycle as a lozenge and an unbalanced one as a triangle.

Let us start with the following properties.

2.4.1. (i) *If F is a subset of E and C'' a circuit of $M(G)$ such that $|C'' - F| \leqslant h$, hen $C'' \subseteq \langle F \rangle_h$.*

(ii) *If F and F' are two m-connected subsets of E and C is a circuit of $M(G)$ such that $F \cap C \neq \varnothing \neq F' \cap C$, then $F \cup C \cup F'$ is m-connected.*

The proofs are straightforward.

2.4.2. Lemma. *Let E_1 be an unbalanced and m-connected subset of E. For every path P of G linking distinct vertices u and v of $V(E_1)$ and with no common edge with E_1, there exists a circuit C'' of $M(G)$ such that $P \subseteq C'' \subseteq E_1 \cup P$. Moreover, $E_1 \cup P = E_1 \cup C''$ is m-connected.*

Proof. Let P_1 be a path in E_1 linking u to v and let us form the cycle $C = P \cup P_1$.

If C is balanced, we may take $C'' = C$.

If C is unbalanced, let us consider an unbalanced cycle C' of E_1. It is easy at this stage to complete the proof by applying Proposition 2.2, Section 3.2, to the connected graph $E_1 \cup P$ and unbalanced cycles C and C'.

2.4.3. LEMMA. *For every integer $h \geq 1$ and for every subsets A and B of E*

$$\langle \langle A \rangle_h \cup B \rangle_h = \langle A \cup B \rangle_h.$$

Proof. Clearly $A \cup B \subseteq \langle A \rangle_h \cup B$, hence $\langle A \cup B \rangle_h \subseteq \langle \langle A \rangle_h \cup B \rangle_h$. On the other hand, $\langle A \rangle_h \subseteq \langle A \cup B \rangle_h$ and $B \subseteq \langle A \cup B \rangle_h$, and hence

$$\langle \langle A \rangle_h \cup B \rangle_h \subseteq \langle \langle A \cup B \rangle_h \rangle_h = \langle A \cup B \rangle_h.$$

We are now in position to settle the structure of E''. In what follows, h is as defined in the beginning of the proof of Lemma 3.4.

2.4.4. *Excluded configurations*

(i) For every vertex v of V'' there is at most one edge of G linking v to V'.

(ii) For every circuit C of type (ii) or (iii) in E'' there is no vertex of C at distance less than or equal to h from V'.

(iii) For every balanced cycle C of E'' every pair of edges linking C to V' have a common end.

Proof. (i) Let $e' = vv'$ and $e'' = vv''$ be two edges linking v to vertices v' and v'' of V', and let P be the path $v'e've''v''$. By Lemma 2.4.2, there exists a circuit C'' such that $P \subseteq C'' \subseteq E' \cup P$. We obtain

$$0 < |C'' - E'| = |\{e', e''\}| = 2 \leq h,$$

which contradicts the definition of $E' = \langle X \rangle_h$.

(ii) By contradiction with the maximality of r. Let $C_{r+1} = C$ and P be a path of length less than or equal to h linking a vertex v of C to a vertex v' of V'. Let P' be a path of E' linking v' to an unbalanced cycle C' of E' and meeting C' only at an endpoint; let P'' be a subpath of C_{r+1} linking v to an unbalanced cycle $C''' \subseteq C_{r+1}$ and meeting C''' only at an endpoint (P' or P'' may be empty). It is clear that $C'' = C' \cup P' \cup P \cup P'' \cup C''$ is a circuit of type (iii) and $E' \cup P \cup C_{r+1} = E' \cup C'' \cup C_{r+1}$. Since $E' \cap C'' \neq \emptyset \neq C_{r+1} \cap C''$, $E' \cup C'' \cup C_{r+1}$ is m-connected (by 2.4.1(ii)). Moreover

$$|C'' - (E' \cup C_{r+1})| = |P| \leq h,$$

hence, by 2.4.1(i), $C'' \subseteq \langle E' \cup C_{r+1} \rangle_h$. Therefore, by Lemma 2.4.3,

$$\langle (E' \cup C_{r+1}) \cup C'' \rangle_h = \langle \langle E' \cup C_{r+1} \rangle_h \cup C'' \rangle_h = \langle \langle E' \cup C_{r+1} \rangle_h \rangle_h$$
$$= \langle E' \cup C_{r+1} \rangle_h,$$

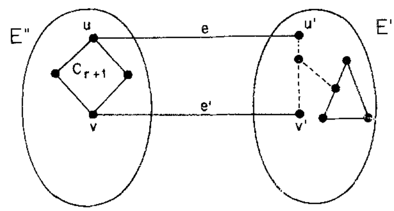

FIGURE 1

and since $E' = \langle C_1 \cup \cdots \cup C_r \rangle_h$,

$$\langle \langle E' \cup C_{r+1} \rangle_h \cup C'' \rangle_h = \langle C_1 \cup \cdots \cup C_r \cup C_{r+1} \rangle_h.$$

On the other hand, since $E' \cup C_{r+1} \cup C''$ is m-connected, we obtain that $\langle C_1 \cup \cdots \cup C_r \cup C_{r+1} \rangle_h$ is m-connected. This is a contradiction since $C_1, \ldots, C_r, C_{r+1}$ are mutually disjoint circuits of $M(G)$.

(iii) By contradiction with the maximality of r. Let $C_{r+1} = C$ and suppose that $e = uu'$ and $e' = vv'$ are two non-adjacent edges linking the vertices u and v of C_{r+1} to the vertices u' and v' of V' (Fig. 1). Let $C_{r+1}(u, v)$ be a subpath of C_{r+1} linking u to v and let us denote by P the path $\{e, e'\} \cup C_{r+1}(u, v)$. By Lemma 2.4.2, there exists a circuit C'' such that $P \subseteq C'' \subseteq E' \cup P$ and $E' \cup P = E' \cup C''$ is m-connected. Since $(E' \cup C'') \cap C_{r+1} \neq \varnothing$, $E' \cup C'' \cup C_{r+1}$ is also m-connected. On the other hand

$$|C'' - (E' \cup C_{r+1})| = |\{e, e'\}| \leqslant h.$$

At this stage, the proof may be completed as above in the proof of (ii).

2.4.5. LEMMA. *If G is 3-connected, then*

(i) *the minimum degree in E'' is $\geqslant 2$;*

(ii) *for every connected component A of E'', there exist three mutually non-adjacent edges linking A to V'.*

Proof. (i) This is true by 2.4.4(i) because the minimum degree in G is $\geqslant 3$.

(ii) This part follows also from 2.4.4(i) if we remark that a connected component of E'' has at least three vertices, and the set S_A of vertices of V' adjacent to some vertices of A verifies $|S_A| \geqslant 3$ ($|S_A| \leqslant 2$ would contradict the 3-connectedness of G).

Now we may prove the main result describing $\Gamma(E'')$.

2.4.6. LEMMA. *If G is 3-connected, then for every connected component A of E'' the graph $\Gamma(A)$ is a cycle of length $\geqslant 6$.*

Proof. By Theorem 3.3 of Section 3.3, this statement is a consequence of the following results (i), (ii), and (iii).

(i) *A is not m-connected,* otherwise there exist two vertices x and y of A adjacent to distinct vertices of V' (by 2.45(ii)) and a circuit C of $M(G)$ such that $\{x, y\} \subseteq V(C)$ because A has no isolated vertex and is not an m-ishtmus. If C is unbalanced (resp. balanced) we have a configuration excluded by 2.4.4(ii) (resp. 2.4.4(iii)).

(ii) *$\Gamma(A)$ has no pendant vertex.* A pendant vertex of $\Gamma(A)$ is an m-connected component B of A which is not an m-isthmus (by 2.45(i)). By definition B has just one vertex (which we denote by v) in common with $A - B$. Therefore, the set S_B of all vertices of V' with a neighbor in $V(B) - \{v\}$ satisfies $|S_B| \geqslant 2$ ($|S_B| \leqslant 1$ would contradict the 3-connectedness of G). As a consequence (and by 2.4.4(i)), B has two distinct vertices x and y adjacent to distinct vertices of S_B and there exists a circuit C in B such that $\{x, y\} \subseteq V(C)$. This is a configuration excluded by 2.4.4(ii) or 2.4.4(iii).

(iii) *A is not the union of two m-connected components B and B' of E'' with $|V(B) \cap V(B')| = 2$* (i.e., $\Gamma(A)$ is not a cycle of length 4). Otherwise by Lemma 3.1(ii),Section 3.3, B and B' are both balanced. By 2.45(ii), at least one, say B, is not an m-isthmus and has two distinct vertices adjacent to distinct vertices of V'. Therefore we have a configuration excluded by 2.4.4(iii).

2.4.7. As a first consequence of the previous results, we can now conclude in case (2.1): we have obtained a contradiction because $\Gamma(E'')$ has no cycle (otherwise we would find a circuit of type (iii) in G since E'' contains an unbalanced cycle (see Fig. 2)).

2.4.8. In case (2.2) (i.e., when G is 4-connected), let A be a connected component of E'' for which $\Gamma(A)$ is the cycle $u_1 B_1 \cdots u_p B_p u_{p+1}$ ($u_{p+1} = u_1$), $p \geqslant 3$. There exists an index i belonging to $\{1,..., p\}$ such that

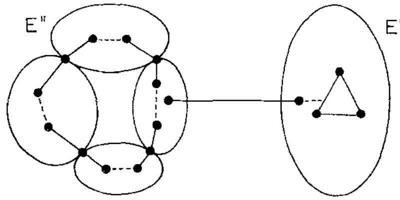

FIGURE 2

B_i is not an m-isthmus (otherwise by the excluded configuration 2.4.4(i), for every i belonging to $\{1,..., p\}$ $d_G(u_i) \leqslant 3$, which contradicts the 4-connectedness of G). If B_i is an m-connected component of A which is not an m-isthmus, the set S_i of vertices of V' adjacent to some vertices of $V(B_i) - \{u_i, u_{i+1}\}$ satisfies $|S_i| \geqslant 2$ ($|S_i| \leqslant 1$ would contradict the 4-connectedness of G). Therefore, by 2.4.4(i) B_i has two distinct vertices adjacent to two distinct vertices of V'. Since B_i is balanced (by Lemma 3.1(ii), Sect. 3.3) we obtain a configuration excluded by 2.4.4(iii).

It is now clear that 2.4.8 completes the proof in case (2.2).

2.4.9. In case (2.3), G is 3-connected and has no balanced triangle. Therefore every circuit of G has cardinality $\geqslant 4$, hence $\langle X \rangle_3$ is m-connected if X is m-connected (Proposition 4.1, Sect. 3.4) (this is the only use of the absence of balanced triangles). Let A be a connected component of E'' for which $\Gamma(A)$ is the cycle $u_1 B_1 \cdots u_p B_p u_{p+1}$ $(u_{p+1} = u_1)$, $p \geqslant 3$ and let S_A denotes the set of vertices of V' adjacent to some vertices of A. By 2.4.5(ii), $|S_A| \geqslant 3$.

2.4.9.1. Every m-connected component B_i which is not an m-isthmus has a vertex v_i such that $u_i \neq v_i \neq u_{i+1}$ and v_i is adjacent to a vertex of S_A (otherwise $G - \{u_i, u_{i+1}\}$ is not connected since $|V(B_i)| \geqslant 3$). Similarly, if for some index j belonging to $\{1,..., p\}$ each of the components B_{j-1} and B_j is an m-isthmus, then their common vertex u_j is adjacent to a vertex in S_A (otherwise $d_G(u_j) = 2$, thus contradicting the 3-connectedness of G).

2.4.9.2. Let us consider the subset $\Theta(A)$ of the set $\{u_1, B_1,..., u_p, B_p\}$ of vertices of $\Gamma(A)$ made up by the components B_i not reduced to m-isthmi of A and the vertices u_j for which B_{j-1} and B_j are both m-isthmi of A. It is clear that $\Theta(A) \neq \emptyset$. On the other hand, the cyclic order defined by the cycle $\Gamma(A)$ on the set $\{u_1, B_1,..., u_p, B_p\}$ induces (by restriction) a cyclic order on the set $\Theta(A)$; we denote also by $\Theta(A)$ the cycle thus obtained. By 2.4.9.1, each element of $\Theta(A)$ is linked by at least one edge to S_A and by definition, each vertex of S_A is linked to at least one element of $\Theta(A)$. Since $|S_A| \geqslant 3$, there exist at least two adjacent elements of the cycle $\Theta(A)$ which are linked to two distinct vertices (which we denote by u' and v') of S_A. We examine below the three possible cases for such two elements of $\Theta(A)$, and we will show that contradictions happen.

First Case. The two elements of $\Theta(A)$ are two vertices u_i and u_{i+1} adjacent to u' and v' respectively (Fig. 3).

Let P be the path $u'u_iu_{i+1}v'$. Since E' is unbalanced and m-connected, by Lemma 2.4.2 there exists a circuit C'' such that $P \subseteq C'' \subseteq E' \cup P =$

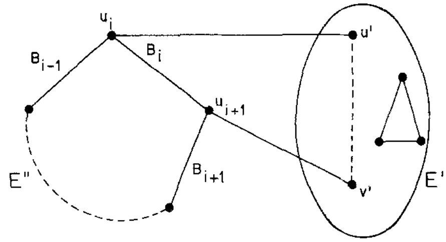

FIGURE 3

$E' \cup C''$; hence $0 < |C'' - E'| = |P| = 3$, which contradicts the definition of $E' = \langle X \rangle_3$.

Second Case. The two elements of $\Theta(A)$ are a vertex u_{i-1}(u_{i-1} is common to two isthmi of A) and a component B_i(Fig. 4).

We may assume that u' is adjacent to u_{i-1} and v' is adjacent to a vertex v_i of B_i (see Fig. 4).

If $v_i = u_i$, this case is similar to the first case, hence a contradiction. Therefore we may assume that $v_i \neq u_i$ and we will find a contradiction with the maximality of r. Since B_i is a balanced m-connected component which is not an m-isthmus of A, there exists a balanced cycle C_{r+1} of B_i such that $\{u_i, v_i\} \subseteq V(C_{r+1})$. Let $C_{r+1}(u_i, v_i)$ be a subpath of C_{r+1} linking u_i to v_i and let us consider the path $P = \{u'u_{i-1}, u_{i-1}u_i, v_iv'\} \cup C_{r+1}(u_i, v_i)$. At this stage, by applying Lemmas 2.4.1, 2.4.2, and 2.4.3 (e.g., as in he proof of 2.4.4(ii), it is possible to show that $\langle C_1 \cup \cdots \cup C_r \cup C_{r+1} \rangle_3$ is m-connected, which is impossible since $C_1, \ldots, C_r, C_{r+1}$ are mutually disjoint circuits of $M(G)$.

Third Case. The elements of $\Theta(A)$ are two components B_i and B_j (each of which is not an m-isthmus) with $j > i$.

By definition of $\Theta(A)$, we must consider two subcases:

(i) $j = i + 2$ and B_{i+1} is an m-isthmus (Fig. 5);

(ii) $j = i + 1$ (Fig. 6).

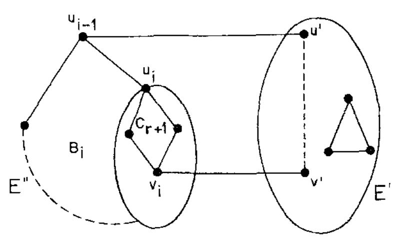

FIGURE 4

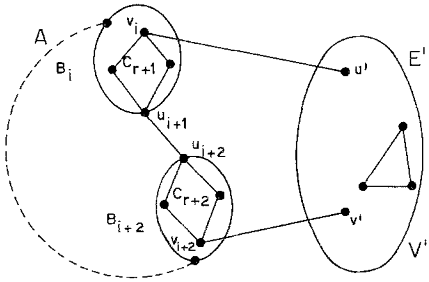

FIGURE 5

In subcase (i), we may assume that u' is adjacent to a vertex v_i of B_i and v' is adjacent to a vertex v_{i+2} of B_{i+2}. We note that if $v_i = u_{i+1}$ or $v_{i+2} = u_{i+2}$, we can argue as in the second case. Therefore we have $v_i \neq u_{i+1}$ and $v_{i+2} \neq u_{i+2}$ (see Fig. 5). Let C_{r+1} (resp. C_{r+2}) be a balanced cycle of B_i (resp. B_{i+2}) such that $\{v_i, u_{i+1}\} \subseteq V(C_{r+1})$ (resp. $\{v_{i+2}, u_{i+2}\} \subseteq V(C_{r+2})$). Let $C_{r+1}(v_i, u_{i+1})$ (resp. $C_{r+2}(v_{i+2}, u_{i+2})$) denotes a subpath of C_{r+1} (resp. C_{r+2}) linking v_i to u_{i+1} (resp. v_{i+2} to u_{i+2}) and P be the path given by

$$P = \{u'v_i, u_{i+1}u_{i+2}, v_{i+2}v'\} \cup C_{r+1}(v_1, u_{i+1}) \cup C_{r+2}(v_{i+2}, u_{i+2}).$$

Since E' is unbalanced and m-connected, by Lemma 2.4.2 above there exists a circuit C'' such that $P \subseteq C'' \subseteq E' \cup P = E' \cup C''$. A double application of Lemma 2.4.1(i) gives that $E' \cup C'' \cup C_{r+1} \cup C_{r+2}$ is m-connected, and hence $\langle E' \cup C'' \cup C_{r+1} \cup C_{r+2} \rangle_3$ is also m-connected. On the other hand

$$|C'' - (E' \cup C_{r+1} \cup C_{r+2})| = |\{u'v_i, u_{i+1}u_{i+2}, v_{i+2}v'\}| = 3,$$

and hence by 2.4.1(i) $C'' \subseteq \langle E' \cup C_{r+1} \cup C_{r+2} \rangle_3$. By a double use of Lemma 2.4.3 above we find

$$\begin{aligned}\langle E' \cup C'' \cup C_{r+1} \cup C_{r+2} \rangle_3 &= \langle E' \cup C_{r+1} \cup C_{r+2} \rangle_3 \\ &= \langle C_1 \cup \cdots \cup C_r \cup C_{r+1} \cup C_{r+2} \rangle_3,\end{aligned}$$

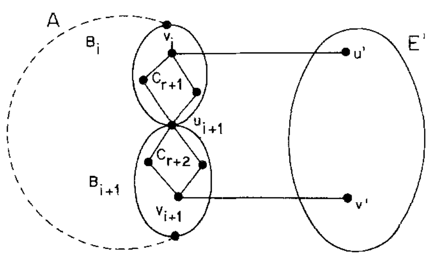

FIGURE 6

and hence a contradiction with the maximality of r, since $\langle C_1 \cup \cdots \cup C_r \cup C_{r+1} \cup C_{r+2} \rangle_3$ is m-connected and $C_1, \ldots, C_r, C_{r+1}, C_{r+2}$ are mutually disjoint circuits of $M(G)$. The proof in subcase (ii) (see Fig. 6) is similar to that in subcase (i). This completes the proof of Lemma 2.4.

Final Remarks

(1) It is not difficult to see that if we allow parallel edges, but no parallel edges with the same sign in G, then Theorems 2.1, 2.2, and 2.3 remain valid. The verifications of the details of the intermediary lemmas are left to the reader.

(2) I have recently received the announcement of the following result:

THEOREM (Ondrej Zyka, Praha). *Every bidirected graph without m-isthmus has a nowhere-zero* 30-*flow*.

Acknowledgments

The author would like to express his gratitute to the referees for their stimulating comments and suggestions.

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