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Algebraic deformations arising from orbifolds with discrete torsion

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Abstract

We develop methods for computing Hochschild cohomology groups and deformations of crossed product rings. We use these methods to find deformations of a ring associated to a particular orbifold with discrete torsion, and give a presentation of the center of the resulting deformed ring. This connects with earlier calculations by Vafa and Witten of chiral numbers and deformations of a similar orbifold.

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1. Introduction

Our motivation for this paper was a desire to provide a mathematical basis for physics statements in the paper [24] of Vafa and Witten. Specifically, we expect that the chiral numbers of an orbifold with discrete torsion come from the Hochschild cohomology groups of associated crossed product rings, and that geometric deformations of the orbifold correspond with algebraic deformations of these rings. In Section 2, we make these ideas more precise, and propose a mathematical definition of some of the chiral numbers. This gives the proper context for results in the remainder of the paper. We also describe in detail there the central example of [24], which consists of the quotient

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of the product of three elliptic curves by an action of the Klein four-group. A local version of this is our motivating example throughout the paper.

Specifically, let $R=\mathbb{C}[x,y,z]$, $X=\mathbb{C}^3=\operatorname{Spec} R$, and let $G=\mathbb{Z}/2\times\mathbb{Z}/2$ act on X by pairwise negation on two coordinates, leaving the third one fixed. There are three curves in X on which the action of G is not free (the coordinate axes), all meeting in the origin, which is fixed by G. Details are given in Example 3.1. We also include in Section 3 the definitions (more generally) of the crossed product ring $R\#_\alpha G$ (where α is a two-cocycle), and of the Hochschild cohomology groups and deformations of a ring.

Results of Ştefan [23] imply that (over $\mathbb{C}$) the Hochschild cohomology group $\mathrm{HH}^*(R\#_\alpha G)$ is the subspace of $\mathrm{HH}^*(R,R\#_\alpha G)$ invariant under an action of G. Letting $R=\mathbb{C}[x,y,z]$, we will use the Koszul resolution of R over $R\otimes R$ to find $\mathrm{HH}^*(R,R\#_\alpha G)$ explicitly, and we will want an action of G on the Koszul complex. This action is given in Lemma 4.2 in case G is a finite abelian group. We apply these ideas to find $\mathrm{HH}^*(R\#_\alpha G)$ when $G=\mathbb{Z}/2\times\mathbb{Z}/2$ in Example 4.3. For this example, unlike the compact example of Vafa and Witten, the Hochschild cohomology groups that we compute are infinite dimensional as vector spaces over $\mathbb{C}$.

In general, elements of $\mathrm{HH}^2(R\#_\alpha G)$ coincide with infinitesimal (or first-order) deformations of the multiplication of $R\#_\alpha G$. In order to find *explicit* formulas for the deformations in case $R=\mathbb{C}[x,y,z]$, we must relate the Koszul complex for R to the Hochschild complex for $R\#_\alpha G$. In Section 5, we develop a general method for doing so, and apply it to find the infinitesimal deformations of our example $A=\mathbb{C}[x,y,z]\#_\alpha(\mathbb{Z}/2\times\mathbb{Z}/2)$. In Section 6, we continue and show that for our example, in the presence of discrete torsion (α nontrivial), the infinitesimal deformations of $R\#_\alpha G$ not coming from R lift to formal deformations of $R\#_\alpha G$. This follows from the application of a universal deformation formula based on a particular bialgebra, using results of [13]. Our example is of independent interest in this context: it is the first example that we know of a formal deformation arising from a *noncocommutative* bialgebra.

In the last section of the paper we study the behavior of the center $Z(R\#_\alpha G)$, which is simply the G-invariant subring R^G, under deformation. In our example, the three curves of singularities of $\operatorname{Spec} R^G$ are smoothed out after formal deformation, and an isolated singularity is left at the origin. We give explicit equations for this singularity in terms of the polynomials arising in the first-order deformation.

We have now described briefly some of the calculations in this paper. They indicate that the behavior of our local picture closely mimics the picture described by Vafa and Witten in their example. This supports our belief that the mathematical study of an orbifold with discrete torsion should begin with the study of the Hochschild cohomology groups and deformations of associated crossed product rings, followed by an understanding of the behavior of the centers of these rings under deformation.

2. Physical and mathematical context

We begin with some motivating ideas from physics and describe, in particular, an example of Vafa and Witten.

2.1. Chiral numbers for Calabi–Yau manifolds and orbifolds

Given an action of a finite group G on a space X satisfying certain properties, one can construct several physics theories which represent the G-equivariant physics of X. There is more than one such theory because their construction takes as additional input a two-cocycle $\alpha : G \times G \to \mathbb{C}^\times$, called the *discrete torsion* of the theory. We denote (in a loose sense) the G-equivariant physics theory on X, with discrete torsion α, by $\mathrm{Physics}^{G,\alpha}(X)$. A similar construction exists in the nonequivariant setting (without any contribution from discrete torsion), assigning to a Calabi–Yau threefold X a physics theory $\mathrm{Physics}(X)$.

Every such theory has associated to it *chiral numbers* h^{ij} $(i, j = 0, \ldots, 3)$ which are defined as certain physical quantities. We are interested in giving a mathematical definition of these numbers in situations of interest to physicists.

For a nonequivariant theory arising from an ordinary Calabi–Yau threefold X it is well known that

$$h^{ij}(\mathrm{Physics}(X)) = h^{ij}(X),$$

the ordinary Hodge numbers of X. An important observation is that $h^{12}(X)$ is the dimension of the space of *complex deformations* of X, which is also the dimension of the space of deformations of the multiplication of the structure sheaf $\mathcal{O}_X$ of X.

Chiral numbers are also well understood for a theory arising from a group action G on a space X, but with trivial discrete torsion. Physics predicts that the chiral numbers of such a theory should be the Hodge numbers of a crepant resolution of the singularities of X/G.[2] This follows from the general principle of Kontsevich that (at least some of) the data contained in the physics model can be extracted from a certain derived category. For a theory built from an ordinary Calabi–Yau threefold X this category is $\mathbf{D}^b_{\mathrm{coh}}(X)$, the derived category of coherent sheaves on X. In the equivariant setting this should be replaced by the derived category $\mathbf{D}^b_{\mathrm{coh}}([X/G])$ of *G-equivariant* sheaves on X. Results of Bridgeland et al. [6], combined with further results of Bridgeland [5] show that in the cases of interest in physics there is an equivalence

$$\mathbf{D}^b_{\mathrm{coh}}([X/G]) \cong \mathbf{D}^b_{\mathrm{coh}}(Y)$$

for any crepant resolution Y of the singularities of X/G. Since one should be able to extract $h^{ij}(\mathrm{Physics}^G(X))$ from $\mathbf{D}^b_{\mathrm{coh}}([X/G])$, the above isomorphism would yield

$$\begin{aligned} h^{ij}(\mathrm{Physics}^G(X)) &= h^{ij}(\mathbf{D}^b_{\mathrm{coh}}([X/G])) = h^{ij}(\mathbf{D}^b_{\mathrm{coh}}(Y)) = h^{ij}(\mathrm{Physics}(Y)) \\ &= h^{ij}(Y). \end{aligned}$$

This argument is unsatisfactory for two reasons: first, there is no good notion of Hodge numbers for an arbitrary derived category, so the above equalities should be thought of only as general principles, not as mathematical statements. Second, there is

[2] A resolution $Y \to X/G$ of the singularities of X/G is *crepant* if the relative canonical bundle is trivial. In situations of interest in physics, this is equivalent to Y being Calabi–Yau.

no generalization of the above argument regarding chiral numbers in the presence of discrete torsion.

It should be mentioned here that there is another, topological approach to chiral numbers for orbifolds, called *orbifold cohomology*. See [9,18] for details.

2.2. Hochschild cohomology

Noncommutative geometry provides a different approach. Some of the following ideas have been around for a while, see for example Connes' book [10].

If R is a commutative ring with an action of a finite group G, we can construct the crossed product ring $R\#G$ (see Section 3 for a definition). It is an associative, non-commutative ring with the property that $R\#G$-modules correspond with G-equivariant R-modules. When the finite group G acts on a scheme X, this construction yields a coherent sheaf $\mathcal{O}_X\#G$ of noncommutative algebras on X, with the property that coherent sheaves of $\mathcal{O}_X\#G$-modules are identified with G-equivariant coherent sheaves of $\mathcal{O}_X$-modules on X. More generally if $\alpha \in \mathrm{H}^2(G, \mathbb{C}^\times)$, we may construct a crossed product ring $R\#_\alpha G$ twisted by α. Again this construction globalizes to schemes.

The proper choice of cohomology theory for noncommutative rings is Hochschild cohomology, which is defined for sheaves of algebras in [11]. Hochschild *ho*mology is defined for schemes in [26], and even more generally, Hochschild homology is defined for exact categories [15,20]. Standard formalism gives as well Hochschild *co*homology for exact categories [19]. Thus one can consider the Hochschild cohomology of the exact category of sheaves of $\mathcal{O}_X\#_\alpha G$-modules on a scheme X. Hochschild homology has been shown to be invariant with respect to derived equivalences coming from localization of pairs [15], and the same techniques can be applied for cohomology as well. In fact, Hochschild cohomology for a scheme X can be shown to be invariant under Fourier–Mukai transforms [7] (following ideas in [16], where the affine case is studied). In particular, the Bridgeland–King–Reid equivalence

$$\mathbf{D}^b_{\mathrm{coh}}([X/G]) \cong \mathbf{D}^b_{\mathrm{coh}}(Y),$$

to which we referred earlier, yields an isomorphism on Hochschild cohomology

$$\mathrm{HH}^i(\mathcal{O}_X\#G) \cong \mathrm{HH}^i(\mathcal{O}_Y)$$

for every i. For a smooth quasi-projective scheme Y we have [11,17]

$$\mathrm{HH}^i(\mathcal{O}_Y) \cong \bigoplus_{p+q=i} \mathrm{H}^p\left(Y, \bigwedge^q T_Y\right),$$

where T_Y is the tangent bundle of Y. Thus for a smooth Calabi–Yau threefold Y, for which we have

$$\bigwedge^q T_Y \cong \bigwedge^{3-q} \Omega_Y,$$

the dimensions of $\mathrm{HH}^i(\mathcal{O}_Y)$ are given by

$$1, 0, h^{12}(Y), 2\cdot h^{11}(Y)+2, h^{12}(Y), 0, 1.$$

In the case of a Calabi–Yau threefold then, the physical statement that

$$h^{ij}(\mathrm{Physics}^G(X)) = h^{ij}(\mathrm{Physics}(Y))$$

can be viewed as a consequence of the mathematical statement that

$$\mathrm{HH}^i(\mathscr{O}_X \# G) \cong \mathrm{HH}^i(\mathscr{O}_Y)$$

for all i. (Note that knowing the dimensions of the Hochschild cohomology groups for a Calabi–Yau threefold allows us to recover its Hodge numbers h^{11} and h^{12}.) In higher dimensions we do not expect to recover all the chiral numbers but rather only the sums

$$\dim \mathrm{HH}^i = \sum_{n+p-q=i} h^{pq},$$

where n is the dimension of the underlying space.

In the case of a Calabi–Yau threefold X, we could define the chiral numbers h^{11} and h^{12} as

$$h^{11} = \tfrac{1}{2} \dim \mathrm{HH}^3(\mathscr{O}_X) - 1 \quad \text{and} \quad h^{12} = \dim \mathrm{HH}^2(\mathscr{O}_X).$$

The same definition makes sense for an orbifold $[X/G]$ (possibly with discrete torsion α), in which case we replace $\mathscr{O}_X$ by $\mathscr{O}_X \#_\alpha G$. This allows us to talk about chiral numbers in the generalized setting of orbifolds.

A further argument that this definition is the right one is the following observation: for an arbitrary algebra A, $\mathrm{HH}^2(A)$ is the space of infinitesimal (first-order) deformations of A (see Section 3). In the case Y is a Calabi–Yau threefold, we know that these deformations are measured by $h^{12}(X)$. Thus defining $h^{12} = \dim \mathrm{HH}^2$ for Calabi–Yau-like spaces agrees with the interpretation of these numbers as dimensions of first-order deformation spaces.

2.3. The example of Vafa and Witten [24]

Let $X = E_1 \times E_2 \times E_3$ be the product of three elliptic curves, let $G = \mathbb{Z}/2 \times \mathbb{Z}/2$, and consider the action of G on X in which every nonidentity element of G acts by negation on two of the E_i's and leaves the third one fixed. Note that the nonzero holomorphic 3-form on X is invariant with respect to the action of G, so that X/G is in a natural sense a (singular) Calabi–Yau 3-space. There are 48 curves in X where the action is not free, and these curves intersect in the 64 points that are fixed by all of G.

The discrete torsion group (or Schur multiplier) $\mathrm{H}^2(G, \mathbb{C}^\times)$ is equal to $\mathbb{Z}/2$. Thus there are precisely two physical theories that we can build on the orbifold $[X/G]$ corresponding to the two choices of discrete torsion $\alpha \in \mathrm{H}^2(G, \mathbb{C}^\times)$: no discrete torsion ($\alpha = 1$) and nontrivial discrete torsion ($\alpha \neq 1$).

Vafa and Witten computed the chiral numbers of these physical theories [24]

$$h^{11} = 51, \quad h^{12} = 3 \quad \text{when } \alpha = 1,$$

$$h^{11} = 3, \quad h^{12} = 51 \quad \text{when } \alpha \neq 1.$$

The case $\alpha = 1$ is again perfectly understood: essentially $h^{12} = 3$ means that every deformation of $\mathcal{O}_X \# G$ must arise from a deformation of X (note that $h^{12}(X) = 3$ corresponding to the possible changes of complex structure on each elliptic curve). The fact that $h^{11} = 51$ corresponds to the fact that to obtain a crepant resolution of X/G one needs to blow-up the 48 curves of singularities in X/G, thus obtaining 48 new Kähler deformations of the resolution Y. (Again, $h^{11}(X) = 3$, corresponding to a choice of volume for each elliptic curve.)

The surprise lies in the value $h^{12} = 51$ in the presence of discrete torsion. In general, given a singular theory whose singularities are resolved by blow-ups (as in the resolution $Y \to X/G$), physics predicts the existence of another theory which removes the singularities by deforming them. Thus one *does* expect to have more deformations. The problem is that the total number of deformations of X/G is 115, which exceeds h^{12}, and so some deformations of X/G are not allowed in the physical model. Vafa and Witten guessed that this means the allowed deformations of X/G only partially smooth X/G, that is they are required (for some mysterious reason) still to have 64 ordinary double points. An explanation for the appearance of these 64 ordinary double points has since been given by the first author [8], following ideas of Aspinwall et al. [1].

From the perspective of noncommutative geometry, this situation can be described as follows: the data of $\mathrm{Physics}^{G,\alpha}(X)$ (both with and without discrete torsion) is encoded in the derived category $\mathbf{D}^b_{\mathrm{coh}}(\mathcal{O}_X \#_\alpha G)$. Deformations of the ring $\mathcal{O}_X \#_\alpha G$ are measured by the chiral number h^{12}, and each deformation of this ring yields a deformation of its center, which coincides with the ring of invariants $(\mathcal{O}_X)^G = \mathcal{O}_{X/G}$. In each case we get an allowable set of deformations of $\mathcal{O}_{X/G}$. In the absence of discrete torsion, there is only a 3-dimensional space of allowable deformations (that all come from deformations of X), while in the presence of discrete torsion, there is a 51-dimensional space of deformations, all of which deform the center $\mathcal{O}_{X/G}$ to the structure sheaf of a space which has at least 64 ordinary double points.

2.4. Our example

For the purposes of our calculations we simplify Vafa and Witten's example by replacing the product of three elliptic curves by affine 3-space. One important difference is that in our case, deformations of arbitrary degree are allowed. (The situation is analogous to the difference between affine and projective geometry: for example if X is the affine scheme $\mathrm{Spec}\, k[x]$, then $\mathrm{H}^0(X, \mathcal{O}_X)$ consists of all polynomials in $k[x]$ of arbitrary degree, while if X' is the projective scheme $\mathrm{Proj}\, k[x, y]$, the global sections $\mathrm{H}^0(X', \mathcal{O}_{X'}(n))$ of a line bundle $\mathcal{O}_{X'}(n)$ will correspond only to homogeneous polynomials of fixed degree n.) In our example then, the Hochschild cohomology groups $\mathrm{HH}^*(R\#_\alpha G)$ that we will compute will be infinite dimensional. When $\alpha = 1$, there is another difference between our local picture and the one described by Vafa and Witten: $\mathrm{HH}^2(R\#_\alpha G)$ picks up many first-order deformations of A that do *not* arise from deformations of R (see Example 4.3).

The deformations of $Z(R\#_\alpha G) = R^G$, corresponding to those of $R\#_\alpha G$ that we will calculate, exhibit different behaviors for different degrees involved. Similar to the example of Vafa and Witten, in all cases the three curves of singularities of $\operatorname{Spec} R^G$ are smoothed out after a formal deformation and an isolated singularity is left at the origin. However, a surprising consequence of our calculations is that in order to see this, calculations to first-order do not suffice: first-order deformations of $R\#_\alpha G$ do *not* give rise to nontrivial *first*-order deformations of R^G (see Section 7).

This could be described geometrically as follows: if there were a geometric germ of a moduli space $(\mathrm{Def}_{R\#_\alpha G}, 0)$ of formal deformations of $R\#_\alpha G$ such that maps from $\operatorname{Spec} k[[t]]$ to $\mathrm{Def}_{R\#_\alpha G}$ that map the closed point of $\operatorname{Spec} k[[t]]$ to 0 correspond to formal deformations of $R\#_\alpha G$, then the tangent space to $\mathrm{Def}_{R\#_\alpha G}$ at 0 would be naturally isomorphic to $\mathrm{HH}^2(R\#_\alpha G)$, the space of first-order deformations of $R\#_\alpha G$. The same picture can be imagined for R^G with a germ of a moduli space $(\mathrm{Def}_{R^G}, 0)$. The operation of taking a ring to its center would give a (partially defined) map $\mathrm{Def}_{R\#_\alpha G} \to \mathrm{Def}_{R^G}$, which in turn would induce a partially defined map on tangent spaces at the origin $\mathrm{HH}^2(R\#_\alpha G) \to \mathrm{HH}^2(R^G)$. (The reason this map is only partially defined is that there are flat deformations of $R\#_\alpha G$ over $\operatorname{Spec} k[[t]]$ whose center is *not* flat over $\operatorname{Spec} k[[t]]$.) Our statement in the previous paragraph amounts to the statement that this map on tangent spaces is zero whenever it is defined. In other words, the map $\mathrm{Def}_{R\#_\alpha G} \to \mathrm{Def}_{R^G}$ is completely ramified at 0. However, when we move from the map on tangent spaces to the map on deformation spaces the map stops being constant, and this yields the change in the type of singularity.

3. Definitions

In this section, we recall some needed ideas regarding algebraic deformations (see [12] for the details), Hochschild cohomology (see [3]), and crossed products (see [22]). A *formal deformation* of an associative $\mathbb{C}$-algebra A is an algebra $A[[t]]$ over the formal power series ring $\mathbb{C}[[t]]$ in one variable, with multiplication defined by

$$u * v = uv + \mu_1(u \otimes v)t + \mu_2(u \otimes v)t^2 + \cdots$$

$(u, v \in A)$, where the $\mu_i : A \otimes A \to A$ are linear maps. Associativity of $A[[t]]$ imposes constraints on the μ_i. In particular, the *infinitesimal* (or *first-order*) *deformation* μ_1 must satisfy

$$\mu_1(u \otimes v)w + \mu_1(uv \otimes w) = \mu_1(u \otimes vw) + u\mu_1(v \otimes w) \tag{3.1}$$

for all $u, v, w \in A$, that is μ_1 is a Hochschild two-cocycle (a representative of an element in $\mathrm{HH}^2(A)$). Here, as A is an algebra over a field, its *Hochschild cohomology* may be defined as

$$\mathrm{HH}^*(A) := \mathrm{Ext}^*_{A^e}(A, A),$$

where $A^e := A \otimes A^{\mathrm{op}}$ acts on A by left and right multiplication. Often this is expressed in terms of the (*acyclic*) *Hochschild complex*, that is the A^e-free resolution of A given by

$$\cdots \xrightarrow{\delta_3} A^{\otimes 4} \xrightarrow{\delta_2} A^{\otimes 3} \xrightarrow{\delta_1} A^e \xrightarrow{m} A \to 0, \tag{3.2}$$

where m is the multiplication map and

$$\delta_n(a_0 \otimes a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=0}^{n} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1}.$$

Removing the term A from complex (3.2) and applying $\mathrm{Hom}_{A^e}(-,A)$, we obtain the *Hochschild* (*cochain*) *complex*

$$0 \to \mathrm{Hom}_{A^e}(A^e,A) \xrightarrow{\delta_1^*} \mathrm{Hom}_{A^e}(A^{\otimes 3},A) \xrightarrow{\delta_2^*} \mathrm{Hom}_{A^e}(A^{\otimes 4},A) \xrightarrow{\delta_3^*} \cdots \tag{3.3}$$

Thus $\mathrm{HH}^n(A) = \mathrm{Ker}(\delta_{n+1}^*)/\mathrm{Im}(\delta_n^*)$. Since $\mathrm{Hom}_{A^e}(A^{\otimes(n+2)},A) \cong \mathrm{Hom}_{\mathbb{C}}(A^{\otimes n},A)$, we may identify $\mathrm{HH}^2(A)$ with a subquotient of $\mathrm{Hom}_{\mathbb{C}}(A^{\otimes 2},A)$, the space of infinitesimal deformations of A mentioned earlier (see (3.1)). Obstructions to lifting an infinitesimal deformation μ_1 to a formal deformation $A[[t]]$ of A lie in $\mathrm{HH}^3(A)$ [12].

More generally, if M is any A-bimodule (equivalently, A^e-module), then

$$\mathrm{HH}^*(A,M) := \mathrm{Ext}^*_{A^e}(A,M).$$

Next we recall the definition of a crossed product ring. Let G be a finite group acting by automorphisms on a $\mathbb{C}$-algebra R. Let $\alpha: G \times G \to R^\times$ be a two-cocycle (where $R^\times$ is the group of units of R), that is

$$(\rho \cdot \alpha(\sigma,\tau))\alpha(\rho,\sigma\tau) = \alpha(\rho,\sigma)\alpha(\rho\sigma,\tau)$$

for all $\rho,\sigma,\tau \in G$. We assume that the image of α is *central* in R. (More generally, the image of α need not be central, and the action of G is twisted by α. See for example [22].) In fact we will mainly be interested in two-cocycles α with image in $\mathbb{C}^\times$. Sometimes we will identify α with its cohomology class in $\mathrm{H}^2(G,R^\times)$. Let

$$A := R\#_\alpha G$$

(or $R\#_\alpha \mathbb{C}G$) be the corresponding *crossed product* ring. That is, as a vector space, $A = R \otimes_{\mathbb{C}} \mathbb{C}G$, where $\mathbb{C}G$ is the group algebra, and the multiplication is given by

$$(p \otimes \sigma)(q \otimes \tau) = p(\sigma \cdot q)\alpha(\sigma,\tau) \otimes \sigma\tau$$

for all $p,q \in R$ and $\sigma,\tau \in G$. To shorten notation, we will write $\bar{\sigma} := 1 \otimes \sigma$, so $p\bar{\sigma} := p \otimes \sigma$. (Note that R is subalgebra of A, but $\mathbb{C}G$ is a subalgebra only if α is a coboundary.) The action of G on R becomes an inner action on A, as

$$\sigma \cdot p = \bar{\sigma} p (\bar{\sigma})^{-1} \quad (\text{where}(\bar{\sigma})^{-1} = \alpha^{-1}(\sigma,\sigma^{-1})\overline{\sigma^{-1}})$$

for all $\sigma \in G$, $p \in R$.

We will obtain specific information about Hochschild cohomology and deformations of A when $R = \mathbb{C}[x,y,z]$ and G is abelian. The following algebra will be the main example of this paper.

Example 3.1. Let $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ be the Klein four-group whose elements will be denoted $1,a,b,c$. We define an action of G as automorphisms on $R = \mathbb{C}[x,y,z]$ by

$$a \cdot x = -x, \quad a \cdot y = y, \quad a \cdot z = -z,$$

$$b \cdot x = -x, \quad b \cdot y = -y, \quad b \cdot z = z.$$

Up to coboundaries, there is exactly one nontrivial two-cocycle $\alpha : G\times G\to\mathbb{C}^\times$, which we take to be given by $\alpha(1,\sigma)=1=\alpha(\sigma,1)=\alpha(\sigma,\sigma)$ for all $\sigma\in G$, and

$$\alpha(a,b)=\mathtt{i}=-\alpha(b,a),\quad \alpha(b,c)=\mathtt{i}=-\alpha(c,b),\quad \alpha(c,a)=\mathtt{i}=-\alpha(a,c),$$

where $\mathtt{i}=\sqrt{-1}$. Then $A:=R\#_\alpha G$ is a crossed product algebra, as defined above.

4. Hochschild cohomology

We will first state a general result about Hochschild cohomology of the rings $A=R\#_\alpha G$, that is an immediate consequence of a result of Ştefan on the Hochschild cohomology of a Hopf Galois extension A/R [23]. Alternatively, we provide a more constructive (for our purposes) proof in Section 5 under additional hypotheses. Note that A is an R-bimodule under left and right multiplication. The superscript G in the statement of the following proposition denotes the subspace of G-invariant elements, that is all elements left unchanged by the action of any $g\in G$.

Proposition 4.1. *Let $A=R\#_\alpha G$. For each $n\geqslant 0$, there is an action of G on $\mathrm{HH}^n(R,A)$ such that*

$$\mathrm{HH}^n(A)\cong\mathrm{HH}^n(R,A)^G.$$

Proof. By [23, Corollary 3.4] (see also [23, Propositions 2.3 and 2.4 aand Theorem 3.3]), there is an action of G on $\mathrm{HH}^n(R,A)$ and a spectral sequence with

$$E_2^{m,n}=\mathrm{H}^m(G,\mathrm{HH}^n(R,A))\Rightarrow\mathrm{HH}^{m+n}(A).$$

As we are working in characteristic 0, the cohomology of G is concentrated in degree 0, that is

$$\mathrm{H}^*(G,\mathrm{HH}^n(R,A))=\mathrm{H}^0(G,\mathrm{HH}^n(R,A))\cong\mathrm{HH}^n(R,A)^G.$$

Therefore, $E_2^{m,n}=E_\infty^{m,n}$ and $\mathrm{HH}^n(A)\cong\mathrm{HH}^n(R,A)^G$. $\square$

It is again a result of Ştefan that a G-action on $\mathrm{HH}^n(R,A)$ extending the action inherited from A on

$$\mathrm{HH}^0(R,A)\cong A^R:=\{u\in A\mid up=pu \text{ for all } p\in R\}$$

exists and is unique [23, Proposition 2.4]. Therefore, if we can find such a group action on $\mathrm{HH}^n(R,A)$, it is necessarily the action to which Proposition 4.1 refers.

Assuming that the image of α is in $\mathbb{C}^\times$, the group action in the proposition arises from an action of G on the Hochschild complex (3.3) for R,A:

$$(\sigma\cdot f)(p_1\otimes\cdots\otimes p_n):=\bar\sigma f((\sigma^{-1}\cdot p_1)\otimes\cdots\otimes(\sigma^{-1}\cdot p_n))(\bar\sigma)^{-1} \tag{4.1}$$

for all $\sigma\in G$, $f\in\mathrm{Hom}_{R^e}(R^{\otimes n},A)$. However in case R is a polynomial algebra, we would like to have a group action on the Koszul complex, as this is the complex with which we will compute cohomology. We will describe such an action explicitly in case $R=\mathbb{C}[x,y,z]$ and G is abelian.

Let

$$f := x \otimes 1 - 1 \otimes x, \quad g := y \otimes 1 - 1 \otimes y, \quad h := z \otimes 1 - 1 \otimes z \in R^e.$$

The *Koszul complex* (a free R^e-resolution of R) is a complex in which the terms are exterior powers of R^e [25], and it is equivalent to

$$0 \to R^e \xrightarrow{\delta_3} (R^e)^{\oplus 3} \xrightarrow{\delta_2} (R^e)^{\oplus 3} \xrightarrow{\delta_1} R^e \xrightarrow{m} R \to 0, \tag{4.2}$$

where m is multiplication, and

$$\delta_1 = (f \;\; g \;\; h), \quad \delta_2 = \begin{pmatrix} -h & 0 & -g \\ 0 & -h & f \\ f & g & 0 \end{pmatrix} \quad \text{and} \quad \delta_3 = \begin{pmatrix} -g \\ f \\ h \end{pmatrix}.$$

Dropping the last term R from complex (4.2), mapping into A, and identifying $\mathrm{Hom}_{R^e}((R^e)^{\oplus n}, A)$ with $A^{\oplus n}$, we obtain the complex

$$0 \to A \xrightarrow{\delta_1^*} A^{\oplus 3} \xrightarrow{\delta_2^*} A^{\oplus 3} \xrightarrow{\delta_3^*} A \to 0. \tag{4.3}$$

Therefore δ_n^* is just the transpose of δ_n in each case, that is

$$\delta_1^* = \begin{pmatrix} f \\ g \\ h \end{pmatrix}, \quad \delta_2^* = \begin{pmatrix} -h & 0 & f \\ 0 & -h & g \\ -g & f & 0 \end{pmatrix} \quad \text{and} \quad \delta_3^* = (-g \;\; f \;\; h).$$

We have $\mathrm{HH}^n(R, A) = \mathrm{Ker}(\delta_{n+1}^*)/\mathrm{Im}(\delta_n^*)$.

We will need to compute the action of G on complex (4.3) induced by the diagonal action on R^e and its exterior powers. In the following lemma, we assume that G is abelian, and further that the action is diagonalized so that G acts by scalar multiplication on each monomial. We define the symbol $p(\sigma)$ by

$$\sigma \cdot p = p(\sigma) p \quad (\sigma \in G, p \text{ a monomial}) \tag{4.4}$$

that is $p(\sigma)$ denotes the scalar by which σ acts on p. The lemma below may be verified by direct computation.

Lemma 4.2. *Let $R = \mathbb{C}[x, y, z]$, let G be an abelian group acting by scalar multiplication on each of x, y, z, and let $\alpha \in \mathrm{H}^2(G, \mathbb{C}^\times)$. The action of G on R induces the following action on the cochain complex* (4.4). *Letting $\sigma \in G$ and $u, v, w \in A = R\#_\alpha G$, the action is given by*

(i) (*degree* 0) $\sigma \cdot u = \bar{\sigma} u \bar{\sigma}^{-1}$,

(ii) (*degree* 1)

$$\sigma \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} x(\sigma^{-1})\bar{\sigma}u\bar{\sigma}^{-1} \\ y(\sigma^{-1})\bar{\sigma}v\bar{\sigma}^{-1} \\ z(\sigma^{-1})\bar{\sigma}w\bar{\sigma}^{-1} \end{pmatrix},$$

(iii) (*degree* 2)

$$\sigma \cdot \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} xz(\sigma^{-1})\bar{\sigma}u\bar{\sigma}^{-1} \\ yz(\sigma^{-1})\bar{\sigma}v\bar{\sigma}^{-1} \\ xy(\sigma^{-1})\bar{\sigma}w\bar{\sigma}^{-1} \end{pmatrix},$$

(iv) (*degree* 3) $\sigma \cdot u = xyz(\sigma^{-1})\bar{\sigma}u\bar{\sigma}^{-1}$.

We now have the following algorithm for computing $\mathrm{HH}^*(A)$ under our hypotheses: Find $\mathrm{HH}^*(R,A)$ by direct calculation using cochain complex (4.3). Then compute the elements of $\mathrm{HH}^*(R,A)$ invariant under the G-action given in Lemma 4.2. By Proposition 4.1 and the remarks immediately following its proof, $\mathrm{HH}^*(A) \cong \mathrm{HH}^*(R,A)^G$.

We make one general remark before giving an example: as an R-bimodule, A decomposes into a direct sum $\oplus_{\sigma\in G} R\bar{\sigma}$, and so

$$\mathrm{HH}^*(R,A) \cong \bigoplus_{\sigma\in G} \mathrm{HH}^*(R,R\bar{\sigma}).$$

If G is abelian, the action of G on cohomology preserves this decomposition, and by Proposition 4.1 we have an additive decomposition

$$\mathrm{HH}^*(A) \cong \bigoplus_{\sigma\in G} \mathrm{HH}^*(R,R\bar{\sigma})^G.$$

This decomposition of Hochschild cohomology is illustrated in the following example.

Example 4.3. Let $A = R\#_\alpha(\mathbb{Z}/2 \times \mathbb{Z}/2)$ as in Example 3.1, where α may be either trivial or nontrivial. Note that the R-bimodule structure of A is the same in either of the cases that α is trivial or nontrivial, so $\mathrm{HH}^*(R,A)$ is the same in either case. We find $\mathrm{HH}^0(R,A) \cong A^R = R \cong \mathrm{HH}^0(R)$, $\mathrm{HH}^n(R,A) = 0$ $(n > 3)$, and

(i) $\mathrm{HH}^1(R,A) \cong R^{\oplus 3} \cong \mathrm{HH}^1(R)$.

(ii)

$$\mathrm{HH}^2(R,A) = \left\{ \begin{pmatrix} p_1 + q_1\bar{a} \\ p_2 + q_2\bar{c} \\ p_3 + q_3\bar{b} \end{pmatrix} \middle|\, p_i \in R, q_1 \in \mathbb{C}[y], q_2 \in \mathbb{C}[x], q_3 \in \mathbb{C}[z] \right\}$$
$$= R^{\oplus 3} \oplus \mathbb{C}[y]\bar{a} \oplus \mathbb{C}[z]\bar{b} \oplus \mathbb{C}[x]\bar{c}.$$

(iii)

$$\mathrm{HH}^3(R,A) = \{ p_1 + p_2\bar{a} + p_3\bar{b} + p_4\bar{c} \mid p_1 \in R,\ p_2 \in \mathbb{C}[y],\ p_3 \in \mathbb{C}[z],\ p_4 \in \mathbb{C}[x] \}$$
$$= R \oplus \mathbb{C}[y]\bar{a} \oplus \mathbb{C}[z]\bar{b} \oplus \mathbb{C}[x]\bar{c}.$$

Applying Lemma 4.2, we have $\mathrm{HH}^0(A) \cong \mathbb{C}[x^2, y^2, z^2, xyz] = Z(A)$, the center of A (as expected), $\mathrm{HH}^n(A) = 0$ $(n > 3)$, and

(iv) In either the case that α is trivial or that α is nontrivial,

$$\mathrm{HH}^1(A) = \left\{ \begin{pmatrix} p_1 \\ p_2 \\ p_3 \end{pmatrix} \middle| \begin{array}{l} p_1 \in x\mathbb{C}[x^2, y^2, z^2] + yz\mathbb{C}[x^2, y^2, z^2] \\ p_2 \in y\mathbb{C}[x^2, y^2, z^2] + xz\mathbb{C}[x^2, y^2, z^2] \\ p_3 \in z\mathbb{C}[x^2, y^2, z^2] + xy\mathbb{C}[x^2, y^2, z^2] \end{array} \right\}.$$

(v)

$$\mathrm{HH}^2(A) = \left\{ \begin{pmatrix} p_1 + q_1\bar{a} \\ p_2 + q_2\bar{c} \\ p_3 + q_3\bar{b} \end{pmatrix} \right\},$$

where $p_1 \in y\mathbb{C}[x^2, y^2, z^2] + xz\mathbb{C}[x^2, y^2, z^2]$, $p_2 \in x\mathbb{C}[x^2, y^2, z^2] + yz\mathbb{C}[x^2, y^2, z^2]$, and $p_3 \in z\mathbb{C}[x^2, y^2, z^2] + xy\mathbb{C}[x^2, y^2, z^2]$. If α is trivial, we take $q_1 \in y\mathbb{C}[y^2]$, $q_2 \in x\mathbb{C}[x^2]$, and $q_3 \in z\mathbb{C}[z^2]$. If α is nontrivial, we take $q_1 \in \mathbb{C}[y^2]$, $q_2 \in \mathbb{C}[x^2]$, and $q_3 \in \mathbb{C}[z^2]$.

(vi) $\mathrm{HH}^3(A) = \{p_1 + p_2\bar{a} + p_3\bar{b} + p_4\bar{c}\}$, where $p_1 \in Z(A)$. If α is trivial, we take $p_2 \in \mathbb{C}[y^2]$, $p_3 \in \mathbb{C}[z^2]$, and $p_4 \in \mathbb{C}[x^2]$. If α is nontrivial, we take $p_2 \in y\mathbb{C}[y^2]$, $p_3 \in z\mathbb{C}[z^2]$, and $p_4 \in x\mathbb{C}[x^2]$.

5. Infinitesimal deformations

Under some hypotheses on $A = R\#_\alpha G$, we find in this section an explicit formula for the infinitesimal deformation $\mu_1 : A \otimes A \to A$ corresponding to a given element of $\mathrm{HH}^2(A)$. We assume that $\mathrm{HH}^2(A)$ has been computed via Proposition 4.1, using an R^e-projective resolution of R that itself carries an action of G. (See for example Lemma 4.2.) Since our elements of $\mathrm{HH}^2(A)$ are given as G-invariant elements of $\mathrm{HH}^2(R, A)$, we will therefore need to relate the Hochschild complex for A to this resolution of R. With this in mind, we will now give a more constructive (for our purposes) proof of Proposition 4.1, using the action (4.1) of G on the Hochschild complex.

We assume now that the image of α is contained in $\mathbb{C}^\times$. Let

$$\Delta := \bigoplus_{\sigma \in G} R\bar{\sigma} \otimes R\bar{\sigma}^{-1},$$

a subalgebra of $A^e = A \otimes A^{\mathrm{op}}$ containing R^e. Then R is a Δ-module and $A \cong (A^e) \otimes_\Delta R$ as an A^e-module (see [4, Lemma 3.3]). Note that $A^e = \oplus_{\sigma \in G} \Delta(\bar{\sigma} \otimes 1)$ is a free Δ-module. Therefore by the Eckmann–Shapiro Lemma [2, Corollary 2.8.4],

$$\mathrm{Ext}^n_{A^e}(A, A) \cong \mathrm{Ext}^n_{A^e}((A^e) \otimes_\Delta R, A) \cong \mathrm{Ext}^n_\Delta(R, A).$$

Once we show that $\mathrm{Ext}^n_\Delta(R, A) \cong (\mathrm{Ext}^n_{R^e}(R, A))^G$, we will have proved Proposition 4.1.

Let $\cdots \xrightarrow{\delta_3} P_2 \xrightarrow{\delta_2} P_1 \xrightarrow{\delta_1} P_0 \xrightarrow{\delta_0} R \to 0$ be a Δ-projective resolution of R. By restriction, it is an R^e-projective resolution of R, and there is an inclusion $\mathrm{Hom}_\Delta(P_n, A) \subset \mathrm{Hom}_{R^e}(P_n, A)$ for each n. Moreover, there is an action of G on the complex given by $\sigma \cdot p = \bar{\sigma} p \bar{\sigma}^{-1}$ (as the image of α is in $\mathbb{C}^\times$), and a corresponding action on the cochain complex given by $(\sigma \cdot f)(p) = \bar{\sigma} f((\bar{\sigma})^{-1} p \bar{\sigma})(\bar{\sigma})^{-1}$ for all $\sigma \in G$, $p \in P_n$, and $f \in \mathrm{Hom}_{R^e}(P_n, A)$. Therefore, $\mathrm{Hom}_\Delta(P_n, A) = \mathrm{Hom}_{R^e}(P_n, A)^G$. As $|G|$ is invertible in $\mathbb{C}$, the subspace of G-invariant elements is just the image of the trace map $(1/|G|)\sum_{\sigma\in G} \sigma\cdot$, and so $(\mathrm{Ker}(\delta_{n+1}^*)/\mathrm{Im}(\delta_n^*))^G \cong \mathrm{Ker}(\delta_{n+1}^*)^G/\mathrm{Im}(\delta_n^*)^G$. Further, if $f = \delta_n^*(f')$ is G-invariant, then $f = (1/|G|)\sum_{\sigma\in G} \sigma \cdot \delta_n^*(f') = \delta_n^*((1/|G|)\sum_{\sigma\in G} \sigma \cdot f')$ is in the image of δ_n^* restricted to G-invariant homomorphisms, that is Δ-homomorphisms. Therefore $\mathrm{Ext}^n_\Delta(R, A) \cong (\mathrm{Ext}^n_{R^e}(R, A))^G$, which proves Proposition 4.1 under the assumption that $\alpha \in \mathrm{H}^2(G, \mathbb{C}^\times)$.

We will find useful the following Δ-projective resolution of R. For each $n \geqslant 0$, let

$$\Delta_n := \left\{ \sum p_0\overline{\sigma_0} \otimes \cdots \otimes p_{n+1}\overline{\sigma_{n+1}} \mid p_i \in R, \sigma_i \in G \text{ and } \sigma_0 \cdots \sigma_{n+1} = 1 \right\},$$

a Δ-submodule of $A^{\otimes(n+2)}$. Thus $\Delta_0 = \Delta$, and each Δ_n is a projective Δ-module. Consider the complex

$$\cdots \xrightarrow{\delta_3} \Delta_2 \xrightarrow{\delta_2} \Delta_1 \xrightarrow{\delta_1} \Delta_0 \xrightarrow{m} R \to 0, \tag{5.1}$$

where m denotes multiplication and the maps δ_n are restrictions of the standard maps from Hochschild complex (3.2). The above complex (5.1) is exact, as there is a chain contraction $s_n : \Delta_{n-1} \to \Delta_n$ given by

$$s_n(u_0 \otimes \cdots \otimes u_n) = u_0 \otimes \cdots \otimes u_n \otimes 1.$$

Therefore (5.1) is a Δ-projective resolution of R.

The Hochschild complex (3.2) is just the complex (5.1) induced from Δ to A^e. Thus the isomorphism $\mathrm{Ext}^n_{A^e}(A, A) \xrightarrow{\sim} \mathrm{Ext}^n_\Delta(R, A)$ is given at the chain level simply by restricting maps from $\mathrm{Hom}_{A^e}(A^{\otimes(n+2)}, A)$ to $\mathrm{Hom}_\Delta(\Delta_n, A)$. As discussed above, the isomorphism $\mathrm{Ext}^n_\Delta(R, A) \xrightarrow{\sim} \mathrm{Ext}^n_{R^e}(R, A)^G$ is given at the chain level by inclusion of $\mathrm{Hom}_\Delta(\Delta_n, A)$ into $\mathrm{Hom}_{R^e}(\Delta_n, A)$. Now there is a map from (5.1) to the Hochschild complex for R, as they are both R^e-projective resolutions of R. Under the assumption that $\alpha \in \mathrm{H}^2(G, \mathbb{C}^\times)$, this is given by

$$\begin{aligned} & p_0\overline{\sigma_0} \otimes p_1\overline{\sigma_1} \otimes p_2\overline{\sigma_2} \otimes \cdots \otimes p_{n+1}\overline{\sigma_{n+1}} \mapsto \\ & \quad \left(\prod_{i=0}^{n} \alpha(\sigma_0 \cdots \sigma_i, \sigma_{i+1}) \right) p_0 \otimes \sigma_0 \cdot p_1 \otimes (\sigma_0\sigma_1) \cdot p_2 \\ & \qquad \otimes \cdots \otimes (\sigma_0\sigma_1 \cdots \sigma_n) \cdot p_{n+1}. \end{aligned} \tag{5.2}$$

In particular, this map is the identity on the submodule $R^{\otimes(n+2)}$ of Δ_n.

Now suppose that

$$\cdots P_2 \to P_1 \to P_0 \to R \to 0 \tag{5.3}$$

is any other R^e-projective resolution of R carrying an action of G. Let

$$\psi_n : R^{\otimes(n+2)} \to P_n \quad (n \geqslant 0)$$

be R^e-homomorphisms giving a map of chain complexes from the Hochschild complex (3.2) for R to the above complex (5.3). The next result is an explicit formula for the infinitesimal deformation $\mu_1 : A \otimes A \to A$ corresponding to an element of $\mathrm{HH}^2(A) \cong \mathrm{HH}^2(R,A)^G$ expressed as a cochain in terms of (5.3).

Theorem 5.1. *Let $A = R\#_\alpha G$ with $\alpha \in \mathrm{H}^2(G,\mathbb{C}^\times)$. Let $f : P_n \to A$ be a function representing an element of $\mathrm{HH}^n(R,A)^G \cong \mathrm{HH}^n(A)$ expressed in terms of complex* (5.3). *The corresponding function $\tilde{f} \in \mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, A) \cong \mathrm{Hom}_{A^e}(A^{\otimes(n+2)}, A)$ of Hochschild complex* (3.3) *is given by*

$$\tilde{f}(p_1\overline{\sigma_1} \otimes \cdots \otimes p_n\overline{\sigma_n}) = ((f \circ \psi_n)(1 \otimes p_1 \otimes \sigma_1 \cdot p_2 \otimes \cdots \otimes (\sigma_1 \cdots \sigma_{n-1}) \cdot p_n \otimes 1))\overline{\sigma_1} \cdots \overline{\sigma_n}.$$

In particular, if $n = 2$, we obtain the infinitesimal deformation $\mu_1 : A \otimes A \to A$,

$$\mu_1(p\bar{\sigma} \otimes q\bar{\tau}) = ((f \circ \psi_2)(1 \otimes p \otimes \sigma \cdot q \otimes 1))\bar{\sigma} \cdot \bar{\tau}.$$

Proof. Identifying $\mathrm{Hom}_{\mathbb{C}}(A^{\otimes n}, A)$ with $\mathrm{Hom}_{A^e}(A^{\otimes(n+2)}, A)$ and applying the map (5.2), we have

$$\begin{aligned}
&\tilde{f}(1 \otimes p_1\overline{\sigma_1} \otimes \cdots \otimes p_n\overline{\sigma_n} \otimes 1)\\
&\quad= \alpha^{-1}(\sigma_1 \cdots \sigma_n, \sigma_n^{-1} \cdots \sigma_1^{-1})\tilde{f}(1 \otimes p_1\overline{\sigma_1} \otimes \cdots \otimes p_n\overline{\sigma_n} \otimes \overline{\sigma_n^{-1} \cdots \sigma_1^{-1}})\overline{\sigma_1 \cdots \sigma_n}\\
&\quad= \prod_{i=1}^{n-1} \alpha(\sigma_1 \cdots \sigma_i, \sigma_{i+1})((f \circ \psi_n)(1 \otimes p_1 \otimes \sigma_1 \cdot p_2\\
&\qquad \otimes \cdots \otimes (\sigma_1 \cdots \sigma_{n-1}) \cdot p_n) \otimes 1))\overline{\sigma_1 \cdots \sigma_n}\\
&\quad= ((f \circ \psi_n)(1 \otimes p_1 \otimes \sigma_1 \cdot p_2 \otimes \cdots \otimes (\sigma_1 \cdots \sigma_{n-1}) \cdot p_n \otimes 1)\overline{\sigma_1} \cdots \overline{\sigma_n}. \qquad \square
\end{aligned}$$

We will use Theorem 5.1 to calculate the infinitesimal deformations for Example 3.1, that is for $A = \mathbb{C}[x,y,z]\#_\alpha(\mathbb{Z}/2 \times \mathbb{Z}/2)$ (whose Hochschild cohomology was given in Example 4.3). Let $R = \mathbb{C}[x,y,z]$ for the rest of this section. There are R^e-homomorphisms giving a map from the Hochschild complex (3.2) to the Koszul complex (4.2) for R,

$$\begin{array}{ccccccccccc}
\cdots \xrightarrow{\delta_4} & R^{\otimes 5} & \xrightarrow{\delta_3} & R^{\otimes 4} & \xrightarrow{\delta_2} & R^{\otimes 3} & \xrightarrow{\delta_1} & R^{\otimes 2} & \xrightarrow{m} & R & \longrightarrow 0\\
& \downarrow{\scriptstyle \psi_3} & & \downarrow{\scriptstyle \psi_2} & & \downarrow{\scriptstyle \psi_1} & & \| & & \| & \\
0 \longrightarrow & R^e & \xrightarrow{\delta_3} & (R^e)^{\oplus 3} & \xrightarrow{\delta_2} & (R^e)^{\oplus 3} & \xrightarrow{\delta_1} & R^e & \xrightarrow{m} & R & \longrightarrow 0.
\end{array}$$

A computation shows that such maps may be defined as follows (cf. [14]):

$$\psi_1(1\otimes x^iy^jz^k\otimes 1)=\begin{pmatrix}\displaystyle\sum_{\ell=1}^{i}x^{i-\ell}y^jz^k\otimes x^{\ell-1}\\ \displaystyle\sum_{\ell=1}^{j}y^{j-\ell}z^k\otimes y^{\ell-1}x^i\\ \displaystyle\sum_{\ell=1}^{k}z^{k-\ell}\otimes x^iy^jz^{\ell-1}\end{pmatrix},$$

$$\psi_2(1\otimes x^iy^jz^k\otimes x^ry^sz^t\otimes 1)$$

$$=\begin{pmatrix}\displaystyle\sum_{m=1}^{t}\sum_{\ell=1}^{i}x^{i-\ell}y^{j+s}z^{k+t-m}\otimes x^{r+\ell-1}z^{m-1}\\ \displaystyle\sum_{m=1}^{t}\sum_{\ell=1}^{j+s}y^{j+s-\ell}z^{k+t-m}\otimes x^{i+r}y^{\ell-1}z^{m-1}-\sum_{m=1}^{t}\sum_{\ell=1}^{s}x^iy^{j+s-\ell}z^{k+t-m}\otimes x^ry^{\ell-1}z^{m-1}\\ \displaystyle\sum_{m=1}^{s}\sum_{\ell=1}^{i}x^{i-\ell}y^{j+s-m}z^k\otimes x^{r+\ell-1}y^{m-1}z^t\end{pmatrix},$$

$$\psi_3(1\otimes x^iy^jz^k\otimes x^ry^sz^t\otimes x^uy^vz^w\otimes 1)$$
$$=-\sum_{n=1}^{v}\sum_{m=1}^{t}\sum_{\ell=1}^{i}x^{i-\ell}y^{j+s+n-1}z^{k+t-m}\otimes x^{r+u+\ell-1}y^{v-n}z^{w+m-1}.$$

Example 5.2. Let $A=\mathbb{C}[x,y,z]\#_\alpha(\mathbb{Z}/2\times\mathbb{Z}/2)$ as in Example 3.1, where α may be either trivial or nontrivial. For any integer i, let $\bar{i}$ be its reduction modulo 2, that is $\bar{i}=0$ if i is even and $\bar{i}=1$ if i is odd. Direct computation yields the following: the infinitesimal deformation $\mu_1: A\otimes A\to A$ given by Theorem 5.1 and map ψ_2 above, and corresponding to the element $(p_1,p_2,p_3;q_1\bar{a};q_2\bar{c};q_3\bar{b})$ of Example 4.3(v) is

$$\begin{aligned}&\mu_1(x^iy^jz^k\bar{\sigma}\otimes x^\ell y^mz^n\bar{\tau})\\ &\quad=x^\ell y^mz^n(\sigma)\{(x^{i+\ell-1}y^{j+m}z^{k+n-1})(in\,p_1+\overline{in}(-1)^\ell q_1\bar{a})\\ &\qquad+(x^{i+\ell}y^{j+m-1}z^{k+n-1})(jn\,p_2+\overline{jn}(-1)^m q_2\bar{c})\\ &\qquad+(x^{i+\ell-1}y^{j+m-1}z^{k+n})(im\,p_3+\overline{im}(-1)^\ell q_3\bar{b})\}\bar{\sigma}\cdot\bar{\tau}.\end{aligned}$$

Here the scalar $x^\ell y^mz^n(\sigma)$ is defined by (4.4), and a negative power of x,y or z indicates that the term is 0.

6. Formal deformations

We will give a universal deformation formula (definition below) that will apply in particular to $A=\mathbb{C}[x,y,z]\#_\alpha(\mathbb{Z}/2\times\mathbb{Z}/2)$, producing a formal deformation that lifts some of its infinitesimal deformations in case α is nontrivial. Our formula was inspired by an unpublished result of the second author and Zhang, and uses results in their paper [13].

A *universal deformation formula* based on a bialgebra B is an element $F\in(B\otimes B)[[t]]$ satisfying the equations

$$(\varepsilon\otimes\mathrm{id})(F)=1\otimes 1=(\mathrm{id}\otimes\varepsilon)(F) \tag{6.1}$$

and

$$[(\Delta\otimes\mathrm{id})(F)]\cdot(F\otimes 1)=[(\mathrm{id}\otimes\Delta)(F)]\cdot(1\otimes F),$$

where id is the identity map. The virtue of such a formula is that if S is any B-module algebra, then F provides a formal deformation of S [13]. Specifically, if $x,y\in S$, then the deformed product is given by $x*y=(m\circ F)(x\otimes y)$, where $m:S\otimes S\to S$ is the ordinary multiplication.

Let H_1 be the associative $\mathbb{C}$-algebra generated by the elements D_1, D_1', and β_1, subject to the relations

$$D_1^2=0=(D_1')^2,\quad D_1D_1'=D_1'D_1,\quad D_1\beta_1=-\beta_1D_1,\quad D_1'\beta_1=-\beta_1D_1',$$
$$\text{and}\quad \beta_1^2=1.$$

Then H_1 is a bialgebra with comultiplication determined by

$$\Delta(D_1)=D_1\otimes\beta_1+1\otimes D_1,\quad \Delta(D_1')=D_1'\otimes 1+\beta_1\otimes D_1',\quad \Delta(\beta_1)=\beta_1\otimes\beta_1$$

and counit $\varepsilon(D_1)=0=\varepsilon(D_1')$, $\varepsilon(\beta_1)=1$. In fact, H_1 is a quotient of the Drinfel'd double of a Taft algebra (see [21, Lemma 4.4]). Let $H_1[[t]]=H_1\otimes_{\mathbb{C}}\mathbb{C}[[t]]$ be the Hopf algebra H_1 with coefficients extended to the formal power series ring $\mathbb{C}[[t]]$. Take H_2 and H_3 to be two more copies of the Hopf algebra H_1 (with appropriately changed indices), and $H=H_1\otimes H_2\otimes H_3$, the tensor product Hopf algebra.

Lemma 6.1. *The element* $F_i=1\otimes 1+tD_i\otimes D_i'$ *of* $H_i[[t]]\otimes H_i[[t]]$ $(i=1,2,3)$ *is a universal deformation formula. Consequently, the product* $F=F_1F_2F_3$ *is a universal deformation formula, based on* $H[[t]]\otimes H[[t]]$.

Proof. It is straightforward to check Eq. (6.1) for each F_i. The second statement of the lemma follows, as the product of universal deformation formulas is again a universal deformation formula, based on the tensor product of the bialgebras. □

Example 6.2. Let $A=\mathbb{C}[x,y,z]\#_\alpha(\mathbb{Z}/2\times\mathbb{Z}/2)$, as in Example 3.1. We assume the cocycle $\alpha\in\mathrm{H}^2(\mathbb{Z}/2\times\mathbb{Z}/2,\mathbb{C}^\times)$ is *nontrivial*. We claim that A is an H-module algebra, that is the D_i and D_i' act as skew derivations, and the β_i act as automorphisms. Let

$q_1 \in \mathbb{C}[y^2]$, $q_2 \in \mathbb{C}[x^2]$, and $q_3 \in \mathbb{C}[z^2]$ (see Example 4.3(v)). Define, for all $i,j,k \in \mathbb{Z}^{\geqslant 0}$ and $\sigma \in \mathbb{Z}/2 \times \mathbb{Z}/2$,

$$D_1(x^i y^j z^k \bar{\sigma}) = \bar{i}x(\sigma)x^{i-1}y^j z^k \bar{\sigma}, \quad D_1'(x^i y^j z^k \bar{\sigma}) = (-1)^i \bar{k} x^i y^j z^{k-1} q_1 \bar{a} \cdot \bar{\sigma},$$

$$D_2(x^i y^j z^k \bar{\sigma}) = \bar{j}z(\sigma)x^i y^{j-1} z^k q_2 \bar{c} \cdot \bar{\sigma}, \quad D_2'(x^i y^j z^k \bar{\sigma}) = \bar{k} x^i y^j z^{k-1} \bar{\sigma},$$

$$D_3(x^i y^j z^k \bar{\sigma}) = \bar{i}y(\sigma)x^{i-1}y^j z^k q_3 \bar{b} \cdot \bar{\sigma}, \quad D_3'(x^i y^j z^k \bar{\sigma}) = \bar{j} x^i y^{j-1} z^k \bar{\sigma},$$

$$\beta_1(x^i y^j z^k \bar{\sigma}) = (-1)^i x(\sigma) x^i y^j z^k \bar{\sigma},$$

$$\beta_2(x^i y^j z^k \bar{\sigma}) = (-1)^k z(\sigma) x^i y^j z^k \bar{\sigma},$$

$$\beta_3(x^i y^j z^k \bar{\sigma}) = (-1)^j y(\sigma) x^i y^j z^k \bar{\sigma},$$

where the scalars $x(\sigma)$, $y(\sigma)$, and $z(\sigma)$ are defined in (4.4). Then A is a left H-module algebra under the above operations: from their definitions, it is clear that β_1, β_2, and β_3 are automorphisms on A, and that they commute with each other. It is straightforward to verify that the D_i, D_i' are skew derivations on A with respect to the automorphisms β_i, in accordance with their coproducts in $H \otimes H$. From their definitions we have $D_i^2 = 0 = (D_i')^2$ and $\beta_i^2 = \mathrm{id}$. Again it may be verified directly that $D_i\beta_i = -\beta_i D_i$, and $D_i'\beta_i = -\beta_i D_i'$ as operators on A $(i = 1,2,3)$, and that all other pairs of these operators commute.

Now set $p_1 = p_2 = p_3 = 0$ in Example 5.2. We claim that the corresponding infinitesimal deformation μ_1 of A lifts to a formal deformation of A over the power series ring $\mathbb{C}[[t]]$: by [13, Theorem 1.3, Definition 1.11] and Lemma 6.1, the universal deformation formula F gives a formal deformation of A. From their definitions, as operators on A, we see that

$$D_1 D_3|_A = 0, \quad D_2 D_3'|_A = 0 \quad \text{and} \quad D_1' D_2'|_A = 0.$$

Therefore, we may write the universal deformation formula F, as an operator on $A \otimes A$, as

$$F|_{A\otimes A} = 1 \otimes 1 + t(D_1 \otimes D_1' + D_2 \otimes D_2' + D_3 \otimes D_3') + t^2 D_2 D_3 \otimes D_2' D_3'. \tag{6.2}$$

It may be verified that $D_1 \otimes D_1' + D_2 \otimes D_2' + D_3 \otimes D_3'$, composed with multiplication in A, yields the infinitesimal deformation μ_1 given in Example 5.2.

In case the functions p_i are *not* all 0, we recover the infinitesimal deformation μ_1 of Example 5.2 from the following skew derivations d_i, d_i' and automorphisms γ_i:

$$d_1(x^i y^j z^k \sigma) = iz(\sigma)zx^{i-1}y^j z^k p_1 \sigma, \quad d_1'(x^i y^j z^k \sigma) = kx^i y^j z^{k-1}\sigma,$$

$$d_2(x^i y^j z^k) = jz(\sigma)zx^i y^{j-1} z^k p_2 \sigma, \quad d_2' = d_1',$$

$$d_3(x^i y^j z^k \sigma) = iy(\sigma)x^{i-1}y^j z^k p_3 \sigma, \quad d_3'(x^i y^j z^k \sigma) = jx^i y^{j-1} z^k \sigma,$$

$$\gamma_1(x^i y^j z^k \sigma) = z(\sigma)x^i y^j z^k \sigma, \quad \gamma_2 = \gamma_1, \quad \gamma_3(x^i y^j z^k \sigma) = y(\sigma)x^i y^j z^k \sigma.$$

The functions $d_1 \otimes d_1'$, $d_2 \otimes d_2'$, and $d_3 \otimes d_3'$ on $A \otimes A$ produce the infinitesimal deformations μ_1 of Example 5.2 corresponding to p_1, p_2, and p_3, respectively. However, it may be checked for example that $d_1 d_1' \neq d_1' d_1$ in general, so A is not an H_1-module algebra under the actions of d_1, d_1' and γ_1. Therefore Lemma 6.1 does not produce a formal deformation of A lifting the infinitesimal μ_1 in this case.

Remark 6.3. In case $A = \mathbb{C}[x,y,z]\#_\alpha(\mathbb{Z}/2 \times \mathbb{Z}/2)$ with α *trivial*, A is an H_i-module algebra for each i, under the operators D_i, D_i' and β_i defined above, but with $q_1 \in y\mathbb{C}[y^2]$, $q_2 \in x\mathbb{C}[x^2]$ and $q_3 \in z\mathbb{C}[z^2]$ (see Example 4.3(v)). So for example, if $q_1 \in y\mathbb{C}[y^2]$ and $q_2 = q_3 = p_1 = p_2 = p_3 = 0$, the universal deformation formula $F = 1 \otimes 1 + tD_1 \otimes D_1'$ of Lemma 6.1 yields a formal deformation of A lifting the infinitesimal deformation μ_1 of Example 5.2. However, $D_1 D_2 \neq D_2 D_1$ in general in this case, and so A is not an H-module algebra, and the universal deformation formula of Lemma 6.1 based on $H[[t]]$ does not apply to A in case $q_1 q_2 \neq 0$.

7. The center of the deformed algebra

In this section, we consider only the example $A = \mathbb{C}[x,y,z]\#_\alpha(\mathbb{Z}/2 \times \mathbb{Z}/2)$, where α is *nontrivial*. Let A_F denote the deformed algebra over $\mathbb{C}[[t]]$ given in Example 6.2, in the case $p_1 = p_2 = p_3 = 0$. The center of the original algebra A is generated by x^2, y^2, z^2, and xyz. Inspection of the universal deformation formula F given in (6.2) shows that the images of x^2, y^2, and z^2 remain central in A_F, but that this is not necessarily the case for xyz. In the next lemma, we adjust the element xyz so that it is central in A_F. The resulting elements generate the new center.

Lemma 7.1. *Let F be the universal deformation formula given in* (6.2) *above. Let Z be the $\mathbb{C}[[t]]$-subalgebra of A_F generated by x^2, y^2, z^2, and*

$$w = xyz + \tfrac{1}{2}t(yq_1\bar{a} + xq_2\bar{c} + zq_3\bar{b}).$$

The center of A_F is the completion $\hat{Z}$ of Z with respect to the ideal (t).

Proof. The element w will be central in A_F once we see that $w * x^i y^j z^k \bar{\sigma} = x^i y^j z^k \bar{\sigma} * w$ for any $\sigma \in G$ and nonnegative integers i,j,k. In either product, the resulting term in A is $x^{i+1} y^{j+1} z^{k+1} \bar{\sigma}$. Lengthy calculations show that the coefficients of t, t^2, and t^3 in the products $w * x^i y^j z^k \bar{\sigma}$ and $x^i y^j z^k \bar{\sigma} * w$ are equal. Clearly this implies that $\hat{Z}$ is contained in the center of A_F.

Now suppose that

$$v = \sum_{i=0}^{\infty} t^i v_i \tag{7.1}$$

is in the center of A_F. Then v_0 must be in the center of A, and so is equal to a polynomial in x^2, y^2, z^2, and xyz, say $v_0 = P_0(x^2, y^2, z^2, xyz)$. Let $v_0' = P_0(x^2, y^2, z^2, w)$ as an element of A_F, known to be central by our previous argument. Note that $v_0' = v_0 \bmod (t)$. Write

$$v = v_0' + \left((v_0 - v_0') + \sum_{i=1}^{\infty} t^i v_i \right),$$

a sum of an element in Z and an element with constant term 0. Again, the coefficient of t in the expression $(v_0 - v_0') + \sum_{i=1}^{\infty} t^i v_i$ must be central in A, and so must be a polynomial in x^2, y^2, z^2, and xyz, say $P_1(x^2, y^2, z^2, xyz)$. Let $v_1' = P_1(x^2, y^2, z^2, w)$, and note that v_1' is equivalent to the coefficient of t in $(v_0 - v_0') + \sum_{i=1}^{\infty} t^i v_i$, modulo (t^2). By induction, for any positive integer n, we may find $v_0', v_1', \ldots, v_{n-1}' \in \mathbb{Z}$ with

$$v = v_0' + tv_1' + \cdots + t^{n-1} v_{n-1}' \bmod (t^n).$$

As $\hat{Z} := \varprojlim Z/(t^n)$, we may identify v with an element of $\hat{Z}$. $\square$

Finally, for homogeneous q_1, q_2 and q_3, we will find a presentation of the central subalgebra Z of A_F generated by x^2, y^2, z^2, and w. To this end, we calculate

$$w * w = x^2 y^2 z^2 + \tfrac{1}{4} t^2 (y^2 q_1^2 + x^2 q_2^2 + z^2 q_3^2) - \tfrac{1}{4} \mathtt{i} t^3 q_1 q_2 q_3,$$

where $\mathtt{i} = \sqrt{-1}$. Letting $\hat{x} = x^2$, $\hat{y} = y^2$, $\hat{z} = z^2$, and $\hat{w} = w$ in A_F, and choosing $q_1 = \hat{y}^{\,j}/2$, $q_2 = \hat{x}^i/2$, and $q_3 = \hat{z}^k/2$ for nonnegative integers i, j, k, we have a single relation in Z,

$$\hat{w}^2 = \hat{x}\hat{y}\hat{z} + t^2(\hat{x}^{2i+1} + \hat{y}^{\,2j+1} + \hat{z}^{2k+1}) - 2\mathtt{i} t^3 \hat{x}^i \hat{y}^{\,j} \hat{z}^k. \qquad (7.2)$$

Thus Z is generated by $\hat{x}$, $\hat{y}$, $\hat{z}$, and $\hat{w}$, subject to the relation (7.2).

A few comments are in order here:

(1) The deformed defining equation (7.2) of the center Z has no terms in t^1. Therefore, to first-order Z does not deform. This explains our statement in the introduction that the map $\mathrm{Def}_{R\#_\alpha G} \to \mathrm{Def}_{R^G}$ is totally ramified at 0.

(2) Unlike the compact situation of Vafa and Witten [24], we never obtain ordinary double points as our singularities. In this respect the local picture is more similar to the second example studied in [24].

(3) The deformation smooths out the initial three curves of singularities, leaving only a singularity at the origin, as can be verified by direct calculation.

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