Almost periodic factorization of block triangular matrix functions revisited

Yuri I. Karlovich a,1,2, Ilya M. Spitkovsky b,*,1,3, Ronald A. Walker c,4

a Department of Hydroacoustic, Marine Hydrophysical Institute, Ukrainian Academy of Sciences, 270100 Odessa, Ukraine
b Department of Mathematics, College of William and Mary, Williamsburg, VA 23187-8795, USA
c Department of Mathematics, University of Michigan, Ann Arbor, MI 48109-1109, USA

Received 2 November 1998; accepted 8 February 1999
Submitted by L. Rodman

Abstract

Let $G$ be an $n \times n$ almost periodic (AP) matrix function defined on the real line $\mathbb{R}$. By the AP factorization of $G$ we understand its representation in the form $G = G_+ AG_-$, where $G_{\pm 1} (G_{\pm 1})$ is an AP matrix function with all Fourier exponents of its entries being non-negative (respectively, non-positive) and $A(x) = \text{diag} [e^{i\lambda_1 x}, \ldots, e^{i\lambda_n x}], \lambda_1, \ldots, \lambda_n \in \mathbb{R}$. This factorization plays an important role in the consideration of systems of convolution type equations on unions of intervals. In particular, systems of $m$ equations on one interval of length $\lambda$ lead to AP factorization of matrices.
We develop a factorization techniques for matrices of the form (0.1) under various additional conditions on the off-diagonal block $f$. The cases covered include $f$ with the Fourier spectrum $\Omega(f)$ lying on a grid ($\Omega(f) \subseteq -v + h\mathbb{Z}$) and the trinomial $f$ ($\Omega(f) = \{-v, \mu, x\}$) with $-v < \mu < x$, $x + |\mu| + v \geq \lambda$. © 1999 Published by Elsevier Science Inc. All rights reserved.

AMS Classification: 42A75; 47A68; 39A10

Keywords: Almost periodic matrix functions; Factorization

1. Introduction

By definition, an almost periodic (AP) polynomial is a (finite) linear combination of the functions $e^{i\lambda x}$, $\lambda \in \mathbb{R}$:

$$f = \sum c_j e^{i\lambda_j x} \quad (c_j \in \mathbb{C}, \ \lambda_j \in \mathbb{R}).$$

The set of all AP polynomials is denoted by $APP$. It forms a (non-closed) subalgebra of the algebra $C(\mathbb{R})$ of all bounded and continuous on $\mathbb{R}$ functions. The closure of $APP$ in $C(\mathbb{R})$ is the Bohr algebra $AP$ of almost periodic functions.

We now list several properties of $AP$; they all can be found in standard monographs on almost periodic functions, such as [4, 13, 14].

For every $f \in AP$, there exists its Bohr mean value

$$M(f) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} f(x) \, dx.$$ Since $AP$ is invariant under multiplication by $e^x$, along with $M(f)$ there also exist $M_\lambda(f) = M(e^{i\lambda x})$. The set

$$\Omega(f) = \{ \lambda \in \mathbb{R} : M_\lambda(f) \neq 0 \}$$

is at most countable; it is called the Fourier spectrum of $f$. Thus, every function $f \in AP$ has a (formal) Fourier series representation

$$f \sim \sum_{\lambda \in \Omega(f)} M_\lambda(f) e^{i\lambda x}. $$

Denote by $APW$ the set of all functions $f \in AP$ with absolutely convergent Fourier series. Of course, $APW \supseteq AP$. Moreover, $APW$ is a completion of $APP$ in the Wiener norm

$$\|f\|_W = \sum_{\lambda \in \Omega(f)} |M_\lambda(f)|.$$
For any set $\Sigma \subset \mathbb{R}$, let 

$$AP_{\Sigma} = \{ f \in AP : \Omega(f) \subseteq \Sigma \},$$

$APW_{\Sigma}$ and $APP_{\Sigma}$ are defined in a similar way. We will deal mostly with the case $\Sigma = \mathbb{R}^\pm = \{ x \in \mathbb{R} : \pm x \geq 0 \}$, and will abbreviate $AP_{\mathbb{R}^\pm}$ ($APW_{\mathbb{R}^\pm}$, $APP_{\mathbb{R}^\pm}$) to $AP^\pm$ (respectively, $APW^\pm$, $APP^\pm$). Denote by $\Pi_{\Sigma}$ the natural projection of $APP$ onto $APP_{\Sigma}$:

$$\Pi_{\Sigma}f = \sum_{j \in \Sigma} M_j(f)e_j.$$

This operator extends by continuity on $APW$, but not on $AP$.

For vector and matrix functions $f$, inclusions $f \in X$ where $X = APP$, $AP$, $APW$, etc. are understood entrywise. The Fourier coefficients $M_j(f)$ are also defined entrywise, and the Fourier spectrum is given by the same formula (1.1).

An $AP$ factorization of an $n \times n$ matrix function $G$ is defined as its representation in the form

$$G(x) = G_+(x)A(x)G_-(x), \quad x \in \mathbb{R},$$

(1.2)

where

$$(G_+)^{\pm 1} \in AP^+,$$

$$(G_-)^{\pm 1} \in AP^-,$$

(1.3)

and $A = \text{diag}[e_{\lambda_1}, \ldots, e_{\lambda_n}], \lambda_j \in \mathbb{R}$. This notion was introduced in [9] and further studied in a number of subsequent publications, see [1–3,6,8,11,12,15,16,19–21]. Among other things, it was shown in [9] that the set $\{ \lambda_1, \ldots, \lambda_n \}$ is defined uniquely; the numbers $\lambda_j$ are referred to as the partial $AP$ indices of $G$. If all partial $AP$ indices equal zero (that is, $\lambda = I$), the representation (1.2) is called a canonical $AP$ factorization of $G$. We say that (1.2) is an $APP$ ($APW$) factorization of $G$ if, instead of (1.3), more restrictive conditions $(G_+)^{\pm 1} \in AP^+$, $(G_-)^{\pm 1} \in AP^-$ (respectively, $(G_+)^{\pm 1} \in APW^+$, $(G_-)^{\pm 1} \in APW^-$) hold. Of course, for an $AP$ ($APW$, $APP$) factorization of $G$ to exist it is necessary that $G$ itself is an invertible element of $AP$ (respectively, $APW$, $APP$). This necessary condition is by no means sufficient. In fact, for $n > 1$ a constructive $AP$ factorizability criterion is presently not known. It follows from the results of [7,20] (also see [10]) that a canonical $AP$ factorization of an $APW$ matrix function $G$ exists if and only if the Toeplitz operator $T_G = QG \mid \text{Im } Q$ is invertible, where $Q$ is the orthoprojection of $L^2(\mathbb{R})$ on the Hardy space in the lower half-plane, and that every such factorization, when exists, is automatically a (canonical) $APW$ factorization of $G$. Although these results show the importance of the canonical $AP$ factorization, they do not explain when, in terms of $G$, such a factorization
exists. They imply, however, that the set $\mathcal{G}_n$ of all $n \times n$ APW matrix functions, AP (and therefore APW) factorable with zero partial AP indices, is open in the uniform norm.

Consideration of Toeplitz operators with SAP (semi almost periodic, that is, continuous on $\mathbb{R}$ and having AP asymptotic behavior at $\pm \infty$) symbols suggests that the matrix

$$d(G) = M(G_+)M(G_-)$$

is a significant characteristic of the factorization (1.2), see [9]. In the case of the canonical AP factorization, the matrix $d(G)$ is defined by $G$ uniquely, and the mapping $d : G \mapsto d(G)$ is continuous.

Toeplitz operators with SAP symbols arise naturally from the consideration of convolution type equations on finite intervals [9,10]. The AP matrix functions involved have a certain algebraic structure. Namely, they are of the form

$$G = G_{\hat{\lambda},f} = \begin{bmatrix} e^{i\lambda}I_m & 0 \\ f & e^{-i\lambda}I_m \end{bmatrix},$$

(1.4)

where $\lambda > 0$ and $f$ is an $m \times m$ AP matrix function. Because of that, AP factorization of matrices (1.4) is of special interest. This constitutes the subject of our paper.

2. Preliminary results

Lemma 2.1. (i) Let $f$ be an $m \times m$ AP matrix function. Then $G_{\hat{\lambda},f}$ and $G_{\hat{\lambda},\overline{f}}$, where the bar denotes entrywise complex conjugation, are AP-factorable only simultaneously and their partial AP indices are the same. If $G_{\hat{\lambda},f}$ (and therefore $G_{\hat{\lambda},\overline{f}}$) admits a canonical AP factorization, then

$$d(G_{\hat{\lambda},f}) = J \left( d(G_{\hat{\lambda},f}) \right)^{-1} J,$$

where $J = \begin{bmatrix} I_m & 0 \\ 0 & -I_m \end{bmatrix}$.

(ii) Let now $f$ be an $m \times m$ APW matrix function, $f' = \Pi_{(-i,\lambda)}f$. Then the matrices $G_{\hat{\lambda},f'}$ and $G_{\hat{\lambda},f}$ are AP (APW) factorable only simultaneously and their partial AP indices are the same. In the case of the canonical factorization,

$$d(G_{\hat{\lambda},f}) = \begin{bmatrix} I_m \\ M_{\hat{\lambda}}(f) \end{bmatrix} d(G_{\hat{\lambda},f'}) \begin{bmatrix} I_m \\ M_{-\hat{\lambda}}(f) \end{bmatrix}.$$

Part (i) follows from the equality

$$G_{\hat{\lambda},\overline{f}} = J \left( G_{\hat{\lambda},f} \right)^{-1} J.$$
Part (ii) (as well as part (i) for \( m = 1 \)) was first established in [9] and used repeatedly after that. Due to (ii), we may (and usually will) without loss of generality suppose that \( \Omega(f) \subset (-\lambda, \lambda) \).

**Lemma 2.2.** Let \( f \) be an \( m \times m \) matrix function in \( AP_{(-\lambda, \lambda)} \). Then the matrix (1.4) admits a canonical AP factorization if and only if there exist \( m \times m \) matrix functions \( g^1_k, g^2_k \in AP_{[0, \lambda]} \) such that

\[
\Omega(g^+_k f) \cap (0, \lambda) = \emptyset \quad (k = 1, 2)
\]

and

\[
M^1_k (g^+_k f) = M(g^+_2 f) = 0, \quad M^2_k (g^+_k f) = M(g^-_1 f) = I_m.
\]

If this is the case, then

\[
d(G) = \begin{bmatrix}
-M^1_k (g^+_k f) & M(g^+_1 f) \\
-M^2_k (g^+_k f) & M(g^+_2 f)
\end{bmatrix}^{-1}.
\]

**Proof.** *Necessity.* Suppose that \( G \) admits a canonical AP factorization \( G = G_+ G_- \). Using a transformation \( G_+ \mapsto G_+ C, \; G_- \mapsto C^{-1} G_- \) with a suitable choice of a constant non-singular matrix \( C \), we may modify the factorization of \( G \) in such a way that

\[
M(G_-) = I_{2m}.
\]

Now partition the matrices \( G_+ \) and \( G_- \) into \( m \times m \) blocks

\[
G_+^{-1} = \begin{bmatrix} f^+_1 & g^+_1 \\ f^+_2 & g^+_2 \end{bmatrix}, \quad G_- = \begin{bmatrix} f^-_1 & g^-_1 \\ f^-_2 & g^-_2 \end{bmatrix}.
\]

The factorization \( G = G_+ G_- \) can be rewritten as

\[
G_+^{-1} G = G_-,
\]

which, in its turn, implies that

\[
g^+_k f = f^-_k - f^+_k e_\lambda \quad \text{and} \quad g^-_k = g^+_k e_{-\lambda} \quad (k = 1, 2).
\]

Hence, \( \Omega(g^+_k f) \subset [0, \lambda] \) and

\[
\Omega(g^+_k f) \subset \Omega(f^-_k) \cup \Omega(e_\lambda f^+_k) \subset (-\infty, 0] \cup [\lambda, +\infty).
\]

Therefore, (2.1) holds. Also, \( M^1_k (g^+_k f) = M(g^-_1 f) \) and \( M(g^+_k f) = M(f^-_k) \), so that (2.4) implies (2.2).

**Sufficiency.** Suppose that \( m \times m \) matrix functions \( g^+_k \in AP_{[0, \lambda]} \) satisfy (2.1), (2.2). Introduce \( f^+_k, g^-_k \) by (2.7) and \( G_+^{-1}, G_- \) according to (2.5). Then, of course,
\( G^{-1} \in AP^+, \ G_+ \in AP^- \). Direct computations show that \((2.6), (2.4)\) hold. Since \(\det G = 1\), equality \((2.6)\) implies
\[
\det G^{-1} = \det G_-. \tag{2.8}
\]
The left (right) hand side of \((2.8)\) lies in \(AP^+\) (respectively, \(AP^-\)). Therefore, both of them are constant. Due to \((2.4)\), this constant equals \(\det M(G_-) = 1\). Consequently, the matrices \(G^{-1}_+\) and \(G_-\) are invertible (before this was established, \(G^{-1}_+\) had been just a formal notation), and their inverses \(G_+, G_-^{-1}\) also lie in \(AP^+, AP^-\), respectively.

Finally,
\[
d(G) = M(G_+)M(G_-) = M(G_+^{-1})^{-1} = \begin{bmatrix} M(f^+_1) & M(g^+_1) \\ M(f^+_2) & M(g^+_2) \end{bmatrix}^{-1} = \begin{bmatrix} -M_\lambda(g^-_1f) & M(g_1^-) \\ -M_\lambda(g^-_2f) & M(g_2^-) \end{bmatrix}^{-1},
\]
so that \((2.3)\) holds.

Recall that a canonical \(AP\) factorization of a matrix \(G \in APW\) is automatically its \(APW\) factorization [20]. Hence, for \(f \in APW\) an \(AP\) solution of \((2.1), (2.2)\) automatically lies in \(APW\). A result similar to Lemma 2.2 for periodic matrix functions was established in [18].

Our next result allows one to substitute a given matrix \(G_\lambda f\) by another matrix of the same type, without changing its factorability properties.

**Lemma 2.3.** Let \(G(= G_\lambda f)\) be given by \((1.4)\) with \(f \in APP\), and let \(-v\) be the leftmost point in \(\Omega(f) \cap (-\lambda, \lambda)\). Suppose, in addition, that the Fourier coefficient \(M_{-v}(f)\) is invertible.

Then there exists an \(m \times m\) \(APP\) matrix function \(f'\) with
\[
\Omega(f') \subseteq \left\{ \sum n_j \gamma_j - \lambda : n_j \in \mathbb{Z}_+, \ \gamma_j - v \in \Omega(f) \right\} \cap (-v, v)
\]
such that \(G_\lambda f\) and \(G_{v'f'}\) are \(AP\) \((APW, APP)\) factorable only simultaneously. If these factorizations exist, then the partial \(AP\) indices of \(G_\lambda f\) and \(G_{v'f'}\) are the same.

In fact, an explicit formula for \(f'\) is available. To state it, denote by \(\Gamma = \{\gamma_1, \ldots, \gamma_d\}\) the set \((\Omega(f) \setminus \{-v\}) \cap (-\lambda, \lambda)\) shifted by \(v\): \(\Gamma = \{\Omega(f) \setminus \{-v\}) \cap (-\lambda, \lambda)\} + v\), and introduce the following function \(T_\lambda\) which acts on finite tuples with entries in \(\{1, 2, \ldots, d\}\) and assumes values in \(\mathbb{Z}_+^d\)
\[
T_\lambda((j_1, j_2, \ldots, j_n)) = (n_1, n_2, \ldots, n_d),
\]
where $n_k$ is the number of $j_i$ coinciding with $k$ (thus $n_1 + \cdots + n_d = n$). Also, represent $f$ as

$$f = a e^{-y} \left( I - \sum_{j=1}^{d} b_j e_{j_i} \right),$$

(2.10)

that is, denote $a = M_{-y}(f)$, $b_j = -a^{-1} M_{j_i-y}(f)$. Then

$$f' = \sum_{N: (N, \Gamma) \in (-y, y)} \sum_{\lambda: \lambda \in (N, \Gamma)} y_N a^{-1} e^{(N, \Gamma) - \lambda},$$

(2.11)

where $N = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$, $(N, \Gamma)$ is an abbreviated notation for $\sum_j n_j^\Gamma$, and in the latter sum $J = (j_1, \ldots, j_n) \in \{1, \ldots, d\}^n$.

Formula (2.12) simplifies significantly in the case of pairwise commuting $b_j$. Namely, it can be rewritten as

$$y_N = \frac{(n_1 + n_2 + \cdots + n_d)!}{n_1! n_2! \cdots n_d!} b_1^{n_1} b_2^{n_2} \cdots b_d^{n_d}.$$

(2.13)

The commutative case (covering, in particular, the situation $m = 1$) was considered in [3]. General formulas (2.11) and (2.12) in a slightly different form were written down in [15].

In the same paper [15], Lemma 2.3 was used to tackle the AP factorability in the following two situations.

**Theorem 2.4.** Let the off-diagonal block $f$ of the matrix function (1.4) be such that $\Omega(f) \cap (-\lambda, \lambda)$ lies to one side of the origin. Denote by $x$ the point in $\Omega(f)$ that is closest to the origin. Then:

1. $G_{x,f}$ is APP factorable provided that $M_x(f)$ is invertible,
2. $G_{x,f}$ admits a canonical AP factorization if and only if $M(f)$ is invertible (and therefore $x = 0$).

**Theorem 2.5.** Let the off-diagonal block $f$ of the matrix function (1.4) be such that there exist $x, \beta \in (-\lambda, \lambda)$, for which $|x - \beta| \geq \lambda$ and $\beta$ separates $x$ from $(\Omega(f) \setminus \{x\}) \cap (-\lambda, \lambda)$. Then:

1. $G_{x,f}$ is APP factorable provided that $M_x(f)$, $M_0(f)$ are invertible,
2. $G_{x,f}$ admits a canonical AP factorization if and only if $|x - \beta| = \lambda$ and $M_x(f)$, $M_0(f)$ are invertible.

We say that $f$ is one-sided (falls into a big gap case) if its Fourier spectrum satisfies the conditions of Theorem 2.4 (respectively, Theorem 2.5). Theorem
2.4 was, by a different method, established earlier in [12]; a big gap case for \( m = 1 \) was disposed of in [5].

3. Commensurable case

Suppose that the distances between the points of \( \Omega(f) \cap (-\lambda, \lambda) \) are commensurable, in other words,

\[
\Omega(f) \cap (-\lambda, \lambda) \subset -v + h\mathbb{Z}_+
\]

for a certain step \( h > 0 \). It was shown in [11] that for \( m = 1 \) condition (3.1) implies APP factorability of \( G_{i,f} \). An alternative approach, suggested in [15], is based on the repeated application of Lemma 2.3. Both approaches make use of a recursive reasoning, involving invertibility of certain coefficients, and therefore fail in the case \( m > 1 \). As a matter of fact, we do not know whether or not an arbitrary matrix \( G_{i,f} \) with \( f \) satisfying (3.1) is AP factorable.

The criteria of canonical AP factorability ([11, Theorems 3.2 and 3.3]), however, can be generalized to the matrix setting.

**Theorem 3.1.** Let \( \Omega(f) \subset \mathbb{M} = -v + h\mathbb{Z} \). Denote \( c_j = M_{t+nh}(f) \), where \( \tau \) is the smallest non-negative element of \( \mathbb{M} \), and let

\[
T_n = T_n(c) = (c_{j-i})_{i,j=1}^n = \begin{bmatrix}
  c_0 & c_1 & \cdots & c_{n-1} \\
  c_{-1} & c_0 & \cdots & c_{n-2} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{1-n} & c_{2-n} & \cdots & c_0 
\end{bmatrix},
\]

\[
A_n = A_n(c) = (c_{j-i-1})_{i,j=1}^n = \begin{bmatrix}
  c_{-1} & c_0 & \cdots & c_{n-2} \\
  c_{-2} & c_{-1} & \cdots & c_{n-3} \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{-n} & c_{1-n} & \cdots & c_{-1} 
\end{bmatrix}.
\]

Finally, let \( N \) be the integer part of \( \lambda/h \): \( N = \lfloor \lambda/h \rfloor \). The matrix \( G_{i,f} \) admits a canonical AP factorization if and only if one of the following three conditions holds:

1. \( v/h, \lambda/h \in \mathbb{Z} \) and \( \det T_N \neq 0 \),
2. \( v/h \in \mathbb{Z}, \lambda/h \notin \mathbb{Z} \) and \( \det T_N \det T_{N+1} \neq 0 \),
3. \( v/h \notin \mathbb{Z}, \lambda/h \in \mathbb{Z} \) and \( \det T_N \det A_N \neq 0 \).

**Proof.** Let us show first that the condition on the exponents (\( v/h \in \mathbb{Z} \) or \( \lambda/h \in \mathbb{Z} \)) is necessary for the existence of the canonical AP factorization, in other words, that zero lies on the union of the grids \( \mathbb{M} \) and \( \mathbb{M}' = -\lambda + h\mathbb{Z} \). To
this end, consider a matrix $G_{\lambda,f}$ which is $AP$ factorable with zero partial $AP$ indices. Without loss of generality we may suppose that the Fourier coefficient $M_\nu(f)$ is invertible (otherwise, consider a sufficiently small perturbation of $f$).

Applying Lemma 2.3, we construct a matrix $G_{e,f}$, admitting a canonical $AP$ factorization simultaneously with $G_{\lambda,f}$ and such that $\Omega(f') \subset \mathbb{M}' \cap (-\nu, \nu)$.

If $\mathbb{M}' \cap (-\nu, \nu) = \emptyset$, then, of course, $f' = 0$. In this case the partial $AP$ indices of $G_{e,f}$ equal $\pm \nu$ ($m$ pairs).

We conclude from here that $0 (= -\nu) \in \mathbb{M}$.

If $\mathbb{M}' \cap (-\nu, \nu) \neq \emptyset$, let us perturb the $AP$ polynomial $f'$ in such a way that its Fourier spectrum still lies in $\mathbb{M}' \cap (-\nu, \nu)$, the leftmost Fourier coefficient (say, $M_{-\mu}(f')$) is invertible, and the canonical $AP$ factorization of $G_{e,f}$ exists. Then we apply Lemma 2.3 again, obtaining the matrix $G_{\mu,f''}$ with $\Omega(f'') \subset \mathbb{M} \cap (-\mu, \mu)$ and also having a canonical $AP$ factorization.

Now, either $\mathbb{M} \cap (-\mu, \mu) = \emptyset$, which implies that $0 (= -\mu) \in \mathbb{M}'$, or this procedure can be repeated again. Observe that $\mu$ and $\lambda$ lie on the same grid $\lambda + h\mathbb{Z}$, and at the same time $(0 <) \mu < \nu < \lambda$, so that $\mu \leq \lambda - h$. Since the value of $h$ does not change under this procedure, in finitely many steps we will arrive to a matrix $G_{\lambda,f}$ with $\hat{\lambda} \in \mathbb{M}$ (or $\mathbb{M}'$), $\Omega(f') \subset \mathbb{M}' \cap (-\hat{\lambda}, \hat{\lambda})$ (respectively, $\Omega(f) \subset \mathbb{M} \cap (-\hat{\lambda}, \hat{\lambda})$), and $\hat{\lambda}$ so close to zero that $(-\hat{\lambda}, \hat{\lambda})$ does not contain any points from $\mathbb{M}'$ (respectively, $\mathbb{M}$). This means that $f = 0$, and therefore the partial $AP$ indices of $G_{\lambda,f}$ equal $\pm \hat{\lambda}$ ($m$ pairs). On the other hand, $G_{\lambda,f}$ admits a canonical $AP$ factorization simultaneously with the original matrix $G_{\lambda,f'}$.

Hence, $0 = -\hat{\lambda} \in \mathbb{M} \cup \mathbb{M}'$.

The logic of the remaining part of the proof is as follows. If $G_{\lambda,f}$ admits a canonical $AP$ factorization, then the Riemann boundary value problem

$$
\begin{bmatrix}
\phi_1^+ \\
\phi_2^+
\end{bmatrix} =
\begin{bmatrix}
e\mathcal{I} & 0 \\
f & e_{-\lambda} \mathcal{I}
\end{bmatrix}
\begin{bmatrix}
\phi_1^- \\
\phi_2^-
\end{bmatrix},
$$

$(\phi_1^+ \in APW_{[0,\infty)}, \phi_j^- \in APW_{(-\infty,0)}, j = 1,2)$ has only a trivial solution. In other words, there is only a trivial solution of the equation

$$
e\mathcal{I} \phi_2^+ = f \phi + \phi_2^-, \quad \text{where} \quad \Omega(\phi) \subset (0, \lambda).
$$

Therefore, the homogeneous equation

$$
T_{\lambda,f} \phi = 0, \quad \phi \in APW_{[0,\lambda]},
$$

with a “Toeplitz” operator

$$
T_{\lambda,f} = \Pi_{[0,\lambda]} f \mid APW_{[0,\lambda]}
$$

has only a trivial solution. By choosing $\phi$ with a discrete Fourier spectrum lying in $-(\mathbb{M} \cup \mathbb{M}')$, we will show that this condition implies the invertibility of matrices, as stated in the theorem. To prove that the invertibility of these matrices implies the canonical $AP$ factorability of $G_{\lambda,f}$, we will simply construct its canonical factorization, making use of Lemma 2.2.
Since computations are a little different in each of the three cases, we consider them separately.

(i) Let \( \nu/h, \lambda/h \in \mathbb{Z} \). Then \( \mathbb{M} = \mathbb{M}' = h\mathbb{Z}, \lambda = Nh, \) and, according to (3.1),

\[
f = \sum_{k=-N+1}^{N-1} c_k e_{kh}.
\]

Consider

\[
\phi = \sum_{j=0}^{N-1} a_j e_{jh}.
\]

Then

\[
f \phi = \sum_{k=-N+1}^{N-1} \sum_{j=0}^{N-1} c_k a_j e_{(j+k)h}
\]

and (3.2) is equivalent to

\[
\sum_{j=0}^{N-1} c_{m-j} a_j = 0 \quad (m = 0, \ldots, N - 1).
\]

The coefficient matrix of this system differs from \( T_N \) only by a certain permutation of its rows and columns. Hence, the invertibility of \( T_N \) is necessary for the existence of the canonical \( AP \) factorization.

Suppose now that \( T_N \) is invertible. Let us find a solution of (2.1) and (2.2) in the form

\[
g_1^+ = \sum_{j=0}^{N-1} a_{1j} e_{jh}, \quad g_2^+ = \sum_{j=0}^{N-1} a_{2j} e_{jh} + I_m e_{Nh}.
\]

For such \( g_k^+ \), conditions (2.1) and (2.2) can be rewritten as

\[
[a_{10}, a_{11}, \ldots, a_{1,N-1}] T_N = [I_m, 0, \ldots, 0],
\]

\[
[a_{20}, a_{21}, \ldots, a_{2,N-1}] T_N = [-0, c_{-N+1}, \ldots, c_{-1}].
\]

These equations have a (unique) solution due to the invertibility of \( T_N \). Therefore, the desired \( g_k^+ \) exist.

(ii) Let \( \nu/h \in \mathbb{Z}, \lambda/h \notin \mathbb{Z} \). Then \( \mathbb{M} = h\mathbb{Z}, \mathbb{M}' = -\lambda + h\mathbb{Z} = -\sigma + h\mathbb{Z}, \) where \( \sigma = \lambda - Nh \) is the smallest positive element of \(-\mathbb{M}'\), and

\[
f = \sum_{k=-N}^{N} c_k e_{kh}.
\]
Consider
\[ \phi = \sum_{j=0}^{N} a_j e_{j\hbar} + \sum_{j=0}^{N-1} b_j e_{\sigma+j\hbar}. \]

For such \( \phi \),
\[ f \phi = \sum_{k=-N}^{N} \left( \sum_{j=0}^{N} c_k a_j e_{(j+k)\hbar} + \sum_{j=0}^{N-1} c_k b_j e_{\sigma+(j+k)\hbar} \right). \]

The sets \( \sigma + h\Z \) and \( h\Z \) are disjoint; therefore, \( T_{\lambda,f} \phi = 0 \) if and only if
\[ \sum_{j=0}^{N} c_{m-j}a_j = 0 \quad (m = 0, \ldots, N) \]
and
\[ \sum_{j=0}^{N-1} c_{m-j}b_j = 0 \quad (m = 0, \ldots, N-1). \]

Hence, in this case the existence of the canonical \( AP \) factorization implies the invertibility of both \( T_N \) and \( T_{N+1} \).

Conversely, let \( T_N \) and \( T_{N+1} \) be invertible. Introduce
\[ g_1^+ = \sum_{j=0}^{N} a_{1j} e_{j\hbar}, \quad g_2^+ = \sum_{j=0}^{N} a_{2j} e_{\sigma+j\hbar}, \]
where
\[
\begin{align*}
[a_{10}, a_{11}, \ldots, a_{1N}] &= [I_m, 0, \ldots, 0]T_{N+1}^{-1}, \\
[a_{20}, a_{21}, \ldots, a_{2N-1}] &= -[c_{-N}, \ldots, c_{-1}]T_N^{-1} \tag{3.4}
\end{align*}
\]
and \( a_{2,N} = I_m \).

Directly from the definition of \( g_k^+ \) it follows that \( \Omega(g_1^+) \subset [0, \lambda] \cap \mathbb{M} \), \( \Omega(g_2^+) \subset [0, \lambda] \cap (-\mathbb{M}') \) and \( \mathbb{M}_1(g_1^+) = I_m \). The products \( g_k^+ f \) have their Fourier spectra on the same grids (\( \mathbb{M} \) for \( k = 1 \), \( -\mathbb{M}' \) for \( k = 2 \)). Since \( 0 \notin \mathbb{M}' \), \( \mathbb{M}_1(g_2^+) = 0 \). For a similar reason, \( \mathbb{M}_1(g_1^+) = 0 \).

There are exactly \( N+1 \) points of the grid \( \mathbb{M} \) (and \( -\mathbb{M}' \)) in \( [0, \lambda] \). The row of respective Fourier coefficients of \( g_k^+ f \) equals \( [a_{k0}, a_{k1}, \ldots, a_{kN}]T_{N+1}^+ \). From (3.4) it follows that the latter product equals \( [I_m, 0, \ldots, 0] \) for \( k = 1 \) (so that \( \mathbb{M}(g_1^+ f) = I_m \), and
\[
[a_{20}, a_{21}, \ldots, a_{2,N-1}] [I \begin{array}{c|c}
T_N & c_N \\
\vdots & \vdots \\
c_{-N} & c_1 \\
\cdots & \cdots \\
c_{-1} & c_0
\end{array}]
\]

\[
= [a_{20}, a_{21}, \ldots, a_{2,N-1}] T_N + [c_{-N}, \ldots, c_{-1}] [a_{20}, a_{21}, \ldots, a_{2,N-1}] \begin{bmatrix}
c_N \\
\vdots \\
c_1 
\end{bmatrix} + c_0
\]

\[
= \begin{bmatrix} 0, \ldots, 0, c_0 - [c_{-N}, \ldots, c_{-1}] T_{N-1} \begin{bmatrix} c_N \\
\vdots \\
c_1 
\end{bmatrix} 
\end{bmatrix}
\]

for \( k = 2 \).

In both cases, all the Fourier coefficients corresponding to the exponents in 
\((0, \lambda)\) are equal to zero. Hence, \( g^+_k \) satisfy (2.1) and (2.2).

(iii) Finally, let \( \nu/h \not\in \mathbb{Z} \), \( \lambda/h(= N) \in \mathbb{Z} \). Then \( \mathcal{M}' = h\mathbb{Z} \), and \( \mathcal{M} = -\nu + h\mathbb{Z} \) 

can also be represented as \( \tau + h\mathbb{Z} \), where \( \tau \in (0, h) \) is the smallest positive element of \( \mathcal{M} \). In this case

\[
f = \sum_{k=-N}^{N-1} c_k e_{\tau + kh}
\]

and we consider \( \phi \) of the form

\[
\phi = \sum_{j=0}^{N-1} a_j e_{jh} + \sum_{j=1}^{N} b_j e_{-\tau + jh}.
\]

For such \( \phi \),

\[
f \phi = \sum_{k=-N}^{N-1} \left( \sum_{j=0}^{N-1} c_k a_j e_{\tau + (j+k)h} + \sum_{j=1}^{N} c_k b_j e_{(j+k)h} \right).
\]

As in case (ii), the two grids \( \mathcal{M} \) and \( \mathcal{M}' \) are disjoint, so that \( T_{\nu,f} \phi = 0 \) if and only if

\[
\sum_{j=0}^{N-1-j} c_{m-j} a_j = 0 \quad (m = 0, \ldots, N - 1)
\]

and
The coefficient matrices of these systems differ from \( T_N \) and \( \Delta_N \) only by a permutation of their rows and columns. Hence, the invertibility of \( T_N \) and \( \Delta_N \) is necessary for \( G_{\pm} \) to admit a canonical \( AP \) factorization.

Conversely, suppose that \( T_N \) and \( \Delta_N \) are invertible. Let

\[
\begin{align*}
g_1^+ &= \sum_{j=1}^{N} a_{1j} e^{-\pi jh}, \\
g_2^+ &= \sum_{j=0}^{N} a_{2j} e^{jh},
\end{align*}
\]

where \( a_{2,N} = I_m \) and other coefficients are determined by the formulas

\[
[a_{11}, a_{12}, \ldots, a_{1,N}] = [I_m, 0, \ldots, 0] \Delta_N^{-1},
\]

\[
[a_{20}, a_{21}, \ldots, a_{2,N-1}] = -[c_{-N}, \ldots, c_{-1}] T_N^{-1}. \tag{3.5}
\]

Obviously, \( \Omega(g_1^+) \subset [0, \lambda] \cap (-\mathbb{M}) \), \( \Omega(g_2^+) \subset [0, \lambda] \cap \mathbb{M}' \) and \( M_\lambda(g_1^+) = I_m \).

Due to the structure of \( f \), the Fourier spectra of \( g_1^+ f \) and \( g_2^+ f \) lie on \( \mathbb{M}' \) and \( \mathbb{M} \), respectively. Since \( 0 \in \mathbb{M} \) and \( \lambda \notin \mathbb{M} \), the coefficients \( \mathbf{M}(g_2^+ f) \) and \( \mathbf{M}(g_1^+) \) vanish.

There are exactly \( N \) points of each of the grids \( \mathbb{M} \) and \( \mathbb{M}' \) in \( [0, \lambda] \). The row of respective Fourier coefficients of \( g_1^+ f \) is \( [a_{11}, \ldots, a_{1,N}] \Delta_N = [I_m, 0, \ldots, 0] \), so that \( \mathbf{M}(g_1^+ f) = I_m \) and \( \Omega(g_1^+ f) \cap (0, \lambda) = \emptyset \).

In turn, the Fourier coefficients of \( g_2^+ f \) lying in \( \mathbb{M} \cap (0, \lambda) \) form a row

\[
[a_{20}, a_{21}, \ldots, a_{2,N-1}] | I_m \begin{bmatrix} T_N & & \\ c_{-N} & \ldots & c_{-1} \end{bmatrix} = [a_{20}, a_{21}, \ldots, a_{2,N-1}] T_N + [c_{-N}, \ldots, c_{-1}] = 0.
\]

Again, conditions (2.1) and (2.2) are satisfied. \( \square \)

The proof of Theorem 3.1 not only shows the existence of the canonical \( AP \) factorization, but also gives explicit formulas for its factors \( G_\pm \) via (2.5) and (2.7). These formulas imply, in particular, that the Fourier spectra of \( G_+ \), \( G_- \) (and, therefore, their inverses) lie in the union of the grids \( \pm \mathbb{M} \) and \( \pm \mathbb{M}' \). This union coincides with \( h \mathbb{Z} \) in case (i), with \( h \mathbb{Z} \cup (-\lambda + h \mathbb{Z}) \cup (\lambda + h \mathbb{Z}) \) in case (ii), and with \( h \mathbb{Z} \cup (-\nu + h \mathbb{Z}) \cup (\nu + h \mathbb{Z}) \) in case (iii).

It is interesting to compare this observation with a general result. For an arbitrary (not necessarily block triangular) \( AP \) matrix function with its Fourier spectrum in a certain additive subgroup \( \Sigma \) of \( \mathbb{R} \), the factors \( G_\pm \) of its canonical \( AP \) factorization (if the latter exists) also have their Fourier spectra contained in \( \Sigma \), (see [17, Theorem 6.1] and [1]). In case (i) one can choose \( \Sigma = h \mathbb{Z} \), and
thus to derive the observation from the result in [1,17]. However, in cases (ii) and (iii) the obtained condition on $\Omega(G_{\pm})$ is sharper.

Using (2.3), (2.1) and formulas for $g_k^+$, one can also determine the explicit value of $d(G)$.

In case (i), (3.3) implies that $M(g_1^+)$ is the upper left $m \times m$ block of $T_N^{-1}$,

$$M(g_1^+) = a_{20} = -[0, c_{-N+1}, \ldots, c_{-1}]T_N^{-1} \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

$$M_2(g_1^+ f) = [I, 0, \ldots, 0]T_N^{-1} \begin{bmatrix} 0 \\ c_{N-1} \\ \vdots \\ c_1 \end{bmatrix}$$

and

$$M_2(g_2^+ f) = c_0 - [0, c_{-N+1}, \ldots, c_{-1}]T_N^{-1} \begin{bmatrix} 0 \\ c_{N-1} \\ \vdots \\ c_1 \end{bmatrix}.$$  

According to (2.3), this yields

$$d(G) = \begin{bmatrix} -[I, 0, \ldots, 0]T_N^{-1} \begin{bmatrix} 0 \\ c_{N-1} \\ \vdots \\ c_1 \end{bmatrix} & [I, 0, \ldots, 0]T_N^{-1} \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}^{-1} \\ \begin{bmatrix} 0 \\ c_{N-1} \\ \vdots \\ c_1 \end{bmatrix} & -c_0 & -[0, c_{-N+1}, \ldots, c_{-1}]T_N^{-1} \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix}^{-1} \end{bmatrix},$$

or, equivalently, $d(G)$ coincides with the lower right $2m \times 2m$ block of the matrix
In case (ii), $\mathbf{M}(g_1^+) = a_{10}$ is the upper left $m \times m$ block of $T^{-1}_{N+1}$. Since

$$T_{N+1} = \begin{bmatrix}
    c_0 & c_1 & \cdots & c_N \\
    c_{-1} & c_N & & \\
    \vdots & & & \\
    c_{-N} & & & T_N
\end{bmatrix}$$

and the lower right block $T_N$ in this partition is also invertible, then

$$\mathbf{M}(g_1^+) = \left( c_0 - [c_1, \ldots, c_N] T_N^{-1} \begin{bmatrix} c_{-1} \\ \vdots \\ c_{-N} \end{bmatrix} \right)^{-1}.$$ 

In turn,

$$\mathbf{M}_\lambda(g_2^+ f) = [a_{20}, a_{21}, \ldots, a_{2N-1}, I] \begin{bmatrix} c_N \\ \vdots \\ c_1 \\ c_0 \end{bmatrix} = c_0 - [c_{-N}, \ldots, c_{-1}] T_N^{-1} \begin{bmatrix} c_N \\ \vdots \\ c_1 \\ c_0 \end{bmatrix}.$$ 

Finally, $\mathbf{M}(g_2^+ ) = \mathbf{M}_\lambda(g_2^+ f) = 0$, because $\Omega(g_1^+ f) \subset \mathcal{M}$, $\Omega(g_2^+ ) \subset -\mathcal{M}'$, and $\lambda \not\in \mathcal{M}$, $0 \not\in \mathcal{M}'$. From here and (2.3),

$$\mathbf{d}(G) = \begin{bmatrix}
    0 & - \left( c_0 - [c_{-N}, \ldots, c_{-1}] T_N^{-1} \begin{bmatrix} c_N \\ \vdots \\ c_1 \end{bmatrix} \right)^{-1} \\
    c_0 - [c_1, \ldots, c_N] T_N^{-1} \begin{bmatrix} c_{-1} \\ \vdots \\ c_{-N} \end{bmatrix} & 0
\end{bmatrix}.$$
In case (iii), \( M(g^+_1) = 0 \) (since \( \Omega(g_1) \supset -\mathbb{M} \) and \( 0 \not\in \mathbb{M} \)),

\[
M(g^+_2) = a_{20} = -[c_{-N}, \ldots, c_{-1}]T_N^{-1} \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix},
\]

\( M_k(g^+_2 f) = 0 \) (since \( \Omega(g^+_2 f) \subset \mathbb{M} \), and the latter grid does not contain \( \lambda = Nh \)), and

\[
M_k(g^+_1 f) = [a_{11}, a_{12}, \ldots, a_{1N}] \begin{bmatrix} c_{N-1} \\ c_{N-2} \\ \vdots \\ c_0 \end{bmatrix} = [I, 0, \ldots, 0] \Delta_N^{-1} \begin{bmatrix} c_{N-1} \\ c_{N-2} \\ \vdots \\ c_0 \end{bmatrix}.
\]

Hence,

\[
d(G) = -\begin{bmatrix} \left( [I, 0, \ldots, 0] \Delta_N^{-1} \right)^{-1} & 0 \\ 0 & \left( [c_{-N}, \ldots, c_{-1}]T_N^{-1} \begin{bmatrix} I \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right)^{-1} \end{bmatrix}.
\]

As was mentioned at the beginning of this section, we do not know whether or not condition (3.1) guarantees the \( AP \) factorability of \( G_{k,f} \) if \( m > 1 \). However, it is true under certain additional conditions on the coefficients of \( f \). Namely, the following result holds.

**Theorem 3.2.** Let

\[
f = cR^k e_{-\gamma} \left( I + \sum_{j=1}^p z_j R^t e_{j\lambda} \right), \tag{3.6}
\]

where \( c, R \in \mathbb{C}^{m \times m}, k \in \mathbb{Z}_+, z_j \text{ are scalars, and } \det c \neq 0. \) Then the matrix function (1.4) is \( AP \) factorable.

Observe that for \( m = 1 \) every \( f \) satisfying (3.1) can be represented in the form (3.6); one can simply let \( c = M_{-\gamma}(f), R = I, z_j = c^{-1}M_{-\gamma+j\lambda}(f) \) and \( k = 0 \).
Proof. For any invertible \( m \times m \) matrix \( T \),

\[
\begin{bmatrix}
T & 0 \\
0 & Tc^{-1}
\end{bmatrix}
G_{\tilde{\lambda}, f}
\begin{bmatrix}
T^{-1} & 0 \\
0 & cT^{-1}
\end{bmatrix}
= G_{\tilde{\lambda}, \tilde{T}c^{-1}fT^{-1}}.
\]

Therefore, without loss of generality we may suppose that in (3.6) \( c = I \) and \( R \) is substituted by its similar \( R_1 = TRT^{-1} \). In its turn, if \( R_1 \) is block diagonal, then so is the matrix function

\[
f_1 = R^k_1 e_{-\tilde{\nu}} \left( I + \sum_{j=1}^{p} z_j R^j_1 e_{j\tilde{h}} \right),
\]

Since the matrices \( G_{\tilde{\lambda}, \phi_1 \oplus \phi_2} \) and \( G_{\tilde{\lambda}, \phi_1} \oplus G_{\tilde{\lambda}, \phi_2} \) differ only by a permutation of rows and columns, it suffices to consider separately the cases when \( R \) is invertible and when \( R \) is an \( I \times I \) nilpotent Jordan cell \( J_i \).

In the first case, \( f \) is of the form (2.10) with \( a = cR^k \), \( b_j = -z_j R^j \). Applying Lemma 2.3, we can substitute \( G_{\tilde{\lambda}, f} \) by the matrix \( G_{\nu, f'} \) without changing its \( AP \) factorability properties. Since the matrices \( b_j \) pairwise commute, the coefficients \( y_N \) in the formula (2.11) for \( f' \) can be computed according to (2.13), with \( d = p, \Gamma = \{ h, 2h, \ldots, ph \} \). Let

\[
x_k = \sum_{N : (N, \Gamma) = kh} \frac{(n_1 + \cdots + n_p)!}{n_1! \cdots n_p!} (-z_1)^{n_1} \cdots (-z_p)^{n_p}.
\]

Suppose that the set

\[
\left\{ k \in \mathbb{Z} \cap \left( \frac{\lambda - \nu}{h}, \frac{\lambda + \nu}{h} \right) : x_k \neq 0 \right\}
\]

is non-empty. Denote its smallest and largest members by \( \tilde{k} \) and \( \tilde{k} + \tilde{p} \), respectively. Then

\[
f' = x_k R^k e_{-\tilde{\nu}} \left( I + \sum_{j=1}^{\tilde{p}} \tilde{z}_j R^j e_{j\tilde{h}} \right),
\]

where \( \tilde{\nu} = \tilde{k}h - \lambda \) and \( \tilde{z}_j = x_{j+\tilde{k}}/x_k \).

Observe that the new off-diagonal block (3.8) has the structure similar to that of (3.6), with \( c \) changed to \( x_k I \) and the same \( R \). Since the left-most coefficient of \( f' \) invertible, Lemma 2.3 can be applied again. As was shown in [15] (where the case \( m = 1 \) was treated), after a finite number of such applications we arrive to the situation in which the corresponding set (3.7) is empty. In other words, in the resulting matrix \( G_{\tilde{\lambda}_1, f_1} \), the off-diagonal block \( f_1 \) is zero. This matrix obviously is \( APP \) factorable. Therefore, so is the original matrix \( G_{\tilde{\lambda}, f} \).

Consider now the second case, in which \( R = J_i \). Passing from (3.6) to (3.8) if necessary, we may suppose without loss of generality that \( k > 0 \). If \( k \geq l \), then
\[ f = 0 \] and the APP factorability of \( G_{\lambda,f} \) is obvious. If \( k < l \), then, after an appropriate permutation of rows and columns, \( G_{\lambda,f} \) can be represented as a direct sum of \( e_{\lambda}I_{k} \), \( e_{-\lambda}I_{k} \), and

\[
G_{\lambda,f} = \begin{bmatrix}
e_{\lambda}I_{l-k} & 0 \\
e_{-\lambda}(I_{l-k} + \sum_{j=k+1}^{p} z_{j}J_{l-k}^{j-k}e_{jh}) & e_{-\lambda}I_{l-k}
\end{bmatrix}.
\]

Hence, the matrix \( G_{\lambda,f} \) is APP factorable, provided that \( G_{\lambda,f} \) is. Observe that the latter matrix has the same structure as \( G_{\lambda,f} \) (that is, its off-diagonal block is of the form (3.6) with \( c = I \) and \( R \) being a nilpotent Jordan cell), but its size is strictly smaller. The case of \( l = 1 \) is obviously factorable, because \( J_{1} = 0 \). It remains to employ the mathematical induction principle.

4. Trinomials

Condition (3.1) is satisfied, in particular, if the number of points in \( \Omega(f) \) does not exceed two:

\[ f = ae_{x} + b e_{\beta}. \quad (4.1) \]

In fact, the case (4.1) was completely disposed of in [9], and the respective result reads as follows.

**Theorem 4.1.** Suppose that the off-diagonal block \( f \) in (1.4) is given by (4.1). Then:

1. \( G_{\lambda,f} \) is APP factorable.
2. The partial AP indices of \( G_{\lambda,f} \) equal zero if and only if either \( \alpha\beta = 0 \), \( \det M(f) \neq 0 \), or \( \alpha\beta < 0 \), \( \lambda/(x - \beta) \in \mathbb{Z} \), and \( \det(ab) \neq 0 \).

The simplest case not covered by Theorems 4.1 and 2.4 is that of the trinomial

\[ \Pi_{(-\lambda,\lambda)}f = c_{-1}e_{-\mu} + c_{0}e_{\mu} + c_{1}e_{\lambda}, \quad (4.2) \]

with

\[ -\lambda < -\mu < \mu < \lambda, \quad \lambda, \mu > 0. \quad (4.3) \]

Suppose that the coefficient \( c_{\lambda-1} \) is invertible. Then \( f \) can be represented in the form (2.10) with \( d = 2 \), \( a = c_{\lambda-1} \), and \( b_{j} = -c_{\lambda-1}^{-1}c_{j-1} \) \((j = 1, 2)\). Formulas (2.11) and (2.12) take the form

\[ f'' = \sum_{(n_{1}, n_{2}) \in N_{1}} y_{n_{1}, n_{2}}c_{\lambda-1}^{-1}e_{n_{1}(\mu + \nu) + n_{2}(\lambda + \nu) - \lambda}, \quad (4.4) \]
where \( y_{n_1n_2} \) is the sum of all possible products of \( n_1 + n_2 \) matrices, \( n_j \) of which equal \( b_j \) \((j = 1, 2)\), and

\[
N_1 = \{ (n_1, n_2) \in \mathbb{Z}_+^2 : -v < n_1(\mu + v) + n_2(x + v) - \lambda < v \}.
\]

According to Lemma 2.3, we may transform \( G_{\lambda,f} \) into \( G_{\lambda_1,f_1} \) with \( \lambda_1 = v \), \( f_1 = f' \) given by (4.4), and this transformation does not change the AP factorability properties. Observe that for all \( (n_1, n_2) \in N_1 \), \( n_1 \geq w = [(\lambda + \mu)/(v + \mu)] \). The coefficients \( y_j \) are especially easy to compute if one of the entries \( n_1, n_2 \) is zero:

\[
y_{n_1,0} = b_1^{n_1}, \quad y_{0,n_2} = b_2^{n_2}.
\]

One case in which no other pairs appear in \( N_1 \) is covered by the following lemma.

**Lemma 4.2.** Let

\[
f_k = c_{-1,k} e_{-v_k} + c_{0,k} e_{\mu_k} + c_{1,k} e_{\lambda_k},
\]

where

\[
-\lambda_k < -v_k < 0 \leq \mu_k < \lambda_k < \lambda_k < \alpha_k < \alpha_k = v_k + \mu_k
\]

and \( c_{-1,k} \) is invertible. Define

\[
\lambda_{k+1} = \lambda_k, \quad \mu_{k+1} = \mu_k + v_k - \lambda_k, \quad w_k = \frac{\lambda_k + \mu_k}{v_k + \mu_k},
\]

\[
\alpha_{k+1} = (w_k + 1)(v_k + \mu_k) - \lambda_k, \quad v_{k+1} = \lambda_k - w_k(v_k + \mu_k),
\]

\[
c_{-1,k+1} = (-c_{-1,k}^{-1} c_{0,k} w_k c_{1,k}^{-1}), \quad c_{0,k+1} = -c_{-1,k}^{-1} c_{1,k},
\]

\[
c_{1,k+1} = (-c_{-1,k}^{-1} c_{0,k} w_{k+1} c_{1,k}^{-1})
\]

and, finally,

\[
f_{k+1} = c_{-1,k+1} e_{-v_{k+1}} + c_{0,k+1} e_{\mu_{k+1}} + c_{1,k+1} e_{\lambda_{k+1}}.
\]

Then the matrix functions \( G_{\lambda,f_k} \) and \( G_{\lambda_1,f_1,n_{-\lambda_1+1}\lambda_1} f_{k+1} \) are AP (APW, APP) factorable only simultaneously, and their partial AP indices are the same.

**Proof.** Observe that

\[
1 \cdot (\mu_k + v_k) + 1 \cdot (\alpha_k + v_k) - \lambda_k = (\alpha_k + \mu_k + v_k - \lambda_k) + v_k \geq v_k,
\]

\[
0 \cdot (\mu_k + v_k) + 2 \cdot (\alpha_k + v_k) - \lambda_k = (\alpha_k + \mu_k + v_k - \lambda_k) + (\alpha_k - \mu_k) + v_k > v_k
\]
and
\[
(w_k + 2) (\mu_k + v_k) + 0 \cdot (x_k + v_k) - \lambda_k \geq \left( \frac{\lambda_k + \mu_k}{v_k + \mu_k} + 1 \right) (\mu_k + v_k) - \lambda_k \\
= \lambda_k + 2 \mu_k + v_k - \lambda_k = 2 \mu_k + v_k \geq v_k,
\]
\[
(w_k - 1) (\mu_k + v_k) + 0 \cdot (x_k + v_k) - \lambda_k \leq \left( \frac{\lambda_k + \mu_k}{v_k + \mu_k} - 1 \right) (\mu_k + v_k) - \lambda_k \\
= -v_k.
\]
Hence, the only pairs \((n_1, n_2)\) for which \(n_1(\mu_k + v_k) + n_2(x_k + v_k) - \lambda_k \in (-v_k, v_k)\) are among \((w_k, 0), (w_k + 1, 0)\) and \((0, 1)\). It remains to use (4.5). □

Theorem 4.3 delivers sufficient conditions of \(AP\) factorability which can be derived from (possibly repeated) use of Lemma 4.2.

**Theorem 4.3.** Let \(f\) be as in (4.2) and (4.3). Then the matrix function \(G_{\lambda, f}\) is \(APP\) factorable under each of the following sets of additional requirements:

\[
\text{(1) } \det c_{-1} \neq 0 \quad \text{and} \quad \lambda - \mu \leq \min \{ (\mu + v) \left[ \frac{\lambda + \mu}{\mu + v} \right], \alpha + v \},
\]
\[
\text{(2) } \det(c_{-1}c_0) \neq 0 \quad \text{and} \quad \mu \geq 0, \quad \left[ \frac{\alpha + v}{\mu + v} \right] > \left[ \frac{\lambda + \mu}{\mu + v} \right],
\]
\[
\text{(3) } \det(c_{-1}c_0c_1) \neq 0
\]

and

\[
\text{either } \alpha + |\mu| + v > \lambda, \quad \text{or} \quad \alpha + |\mu| + v = \lambda > \alpha + v. \quad (4.9)
\]

**Proof.** Either of the requirements (1) and (2) implies that \(c_{-1}\) is invertible, \(\mu \geq 0\), and \(\alpha + \mu + v \geq \lambda\). Hence, Lemma 4.2 is applicable (with \(k = 0, c_{j,0} = c_j\) and \(f_0 = f\)).

Under the conditions (1),
\[
\left( \left[ \frac{\lambda + \mu}{\mu + v} \right] + 1 \right) (\mu + v) - \lambda \geq -\mu + \mu + v = v,
\]
so that \(x_1 \notin (-v, v)\). Therefore, \(P_{(-v,v)f_1}\) is (at most) a binomial, and the matrix \(G_{\lambda_1, f_1}\) is \(APP\) factorable due to Theorem 4.1.

In case (2), the coefficients \(c_{\pm 1,1}\) are both invertible, \(x_1 + v_1 = \mu + v \geq v = \lambda_1\), and
\[
x_1 = \left( \left[ \frac{\lambda + \mu}{\mu + v} \right] + 1 \right) (\mu + v) - \lambda \leq \left[ \frac{\alpha + v}{\mu + v} \right] (\mu + v) - \lambda \leq \alpha + v - \lambda = \mu_1.
\]
Therefore, the matrix \(G_{\lambda_1, f_1}\) satisfies the conditions of Theorem 2.5.
We turn now to the case (3). Applying Lemma 2.1 (i) if necessary, we may suppose without loss of generality that \( \lambda \geq 0 \) and, consequently, \( \alpha + \mu + \nu \geq \lambda \).

Consider first the situation when \( \alpha \geq \mu \geq \lambda \), and use (4.8) to define \( f_{k+1} \) recursively for all \( k \) such that

\[
- \lambda_k < -v_k < 0 \leq \mu_k < \alpha_k < \lambda_k \leq \alpha_k + \mu_k + \nu_k,
\]

starting with \( k = 0 \) and \( f_0 = f \). Observe that \( \alpha_{k+1} + v_{k+1} = v_k + \mu_k \), and therefore

\[
\mu_{k+2} = \alpha_{k+1} + v_{k+1} - \lambda_{k+1} = v_k + \mu_k - \lambda_{k+1} = \mu_k.
\]

Hence,

\[
\mu_k = \begin{cases} \alpha + \nu - \lambda & \text{for } k \text{ odd,} \\ \mu & \text{for } k \text{ even,} \end{cases}
\]

so that all \( \mu_k \) are automatically non-negative. Due to (4.9), at least one of the two consecutive \( \mu_k, \mu_{k+1} \) is strictly positive.

As long as (4.10) holds,

\[
\lambda_{k+1} = v_k = \lambda_{k-1} - w_{k-1}(\mu_{k-1} + v_{k-1}) \leq \lambda_{k-1} - \mu_{k-1}.
\]

Hence, the inequality \( \mu_k < \lambda_k \), and therefore (4.10), cannot hold for all \( k \in \mathbb{N} \). Denote by \( N \) the minimal value of \( k \) for which (4.10) fails. Then for all \( k = 0, 1, \ldots, N-1 \),

\[
\lambda_{k+1} - v_{k+1} = v_k - \lambda_k + v_k(\nu_k + \mu_k)
\]

\[
> v_k - \lambda_k + \left( \frac{\lambda_k + \mu_k}{v_k + \mu_k} - 1 \right)(v_k + \mu_k) = 0,
\]

\[
\alpha_{k+1} = (w_k + 1)(v_k + \mu_k) - \lambda_k > \left( \frac{\lambda_k + \mu_k}{v_k + \mu_k} \right)(v_k + \mu_k) - \lambda_k
\]

\[
= \lambda_k + \mu_k - \lambda_k \geq 0,
\]

\[
\lambda_{k+1} - \mu_{k+1} = v_k - \lambda_k - v_k + \lambda_k = \lambda_k - \alpha_k > 0,
\]

\[
\alpha_{k+1} + v_{k+1} - \lambda_{k+1} = v_k + \mu_k - v_k = \mu_k \geq 0.
\]

In particular,

\[
\nu_N < \lambda_N, \quad 0 < \alpha_N, \quad 0 \leq \mu_N < \lambda_N
\]

and

\[
\alpha_N + v_N - \lambda_N = \mu_{N-1} \geq 0.
\]
From (4.12), (4.13) it follows that the only ways in which (4.10) can fail for \( k = N \) are \( \lambda_N \geq \lambda, \nu_N \leq 0 \), and \( \lambda_N \leq \mu_N \). Condition (4.13) also shows that the case \( \nu_N \leq 0 \) is covered by \( \lambda_N \geq \lambda \). Hence, we are left with the three possibilities:

(i) \( \lambda_N \geq \lambda \),

(ii) \( -\lambda_N < -\nu_N < 0 < \mu_N = \lambda < \lambda_N \), and

(iii) \( -\lambda_N < -\nu_N < 0 < \nu_N < \min\{\mu_N, \lambda_N\} \).

In cases (i) and (ii), \( \Pi(\lambda_N, \lambda_N)\) is (at most) a binomial, and the matrix \( G_{\lambda_N, \lambda_N} \) is \( APP \) factorable due to Theorem 4.1. In case (iii), all three points \(-\nu_N, \lambda_N, \mu_N\) of \( \Omega(f_N) \) are distinct, and formulas (4.7), applied consequently with \( k = 0, \ldots, N - 1 \), show that the coefficients \( c_{\lambda, \lambda} \) of \( f_N \) are invertible. Due to (4.13), the \( AP \) polynomial \( f_N \) falls into the big gap case. According to Theorem 2.5, the matrix \( G_{\lambda_N, \lambda_N} \) is \( APP \) factorable. Due to Lemma 4.2 applied \( N \) times, this implies the \( APP \) factorability of the original matrix \( G_{\lambda, \lambda} \).

Finally, if \( \alpha + \nu < \lambda \), then one application of Lemma 4.2 leads to the matrix \( G_{\lambda, f} \) with \( \Omega(f) \subseteq \{-\nu, \mu, \alpha\} \). But then

\[ \Omega(f) \subseteq \{-\alpha, -\mu, v\} \]

and \( v - (-\alpha) = v + \alpha = v + \mu = v = \lambda \), \( -\mu = \lambda - (\alpha + v) > 0 \). Hence, either \( \Pi(\alpha, \alpha) \) is (at most) a binomial, or \( G_{\lambda, f} \) satisfies the conditions of the already established part of case (iii) (with \( \lambda \mapsto \lambda, \nu \mapsto \nu, \mu \mapsto \min\{-\mu, \nu\}, \alpha \mapsto \max\{-\mu, \nu\} \)). In both cases, \( G_{\lambda, f} \) is then \( APP \) factorable due to Lemma 2.1. \( \square \).

Remarks. 1. Parts (1) and (2) of Theorem 4.3, combined with part (1) of Lemma 2.1, lead to sufficient \( APP \) factorability conditions for matrix functions \( G_{\lambda, f} \) with a singular coefficient \( c_{-1} \). Part (3), however, is invariant under the transformation \( G_{\lambda, f} \mapsto G_{\lambda, f} \) of Lemma 2.1.

2. Conditions (4.3) in part (3) are essential: even for \( m = 1 \), there exist matrix functions \( G_{\lambda, f} \) with \( f \) given by (4.3) with \( \mu = 0, \alpha + \nu = \lambda \) which are not \( AP \) factorable [12]. In fact, [12,3] contain an \( AP \) factorability criterion for such matrix functions (with \( m = 1 \) or, more generally, with commuting coefficients \( c_j \)). The case of non-commuting coefficients, on the other hand, remains open.

3. If \( \mu_{N-1} \geq 0 \), then an \( AP \) factorization of \( G_{\lambda_N, f_N} \) (and therefore of \( G_{\lambda, f} \)) in cases (ii) and (iii) of part (3) is not canonical, because the gap between the exponents \(-\nu_N\) and \( \lambda_N \) is strictly greater than \( \lambda_N \): \( \lambda_N + \nu_N = \lambda_N + \mu_{N-1} > \lambda_N \).

Let us consider now a canonical \( AP \) factorization of matrix functions \( G_{\lambda, f} \) with trinomial off-diagonal block \( f \). We start with a necessary condition.

Lemma 4.4. Let the off-diagonal block \( f \) of the matrix function \( G_{\lambda, f} \) equal \( c_{-1} e_{-\nu} + c_0 e_{\mu} + c_1 e_{\nu} \) with \( 2\mu > 0 \). Suppose that \( G_{\lambda, f} \) admits a canonical \( AP \) factorization. Then \( \nu \mu > 0 \), \( |\nu| < \lambda \), and the coefficient \( c_{-1} \) is invertible.
For \( \mu > 0 \) this result follows directly from Lemma 6.6 in [15], where a more general case of an arbitrary APP block \( f \) with \( \Omega(f) \cap (-\lambda, 0] \) consisting of at most one point was considered. The result for \( \mu < 0 \) follows from that for \( \mu > 0 \) and 2.1.

If \((x + \nu/\mu + \nu)\) is rational, then the block (4.2) satisfies (3.1), and the criterion of the canonical AP factorability is delivered by Theorem 3.1. Therefore, we may without loss of generality suppose that \((x + \nu)/(\mu + \nu) \not\in \mathbb{Q}\). This condition is imposed in the next two theorems.

**Theorem 4.5.** Let the off-diagonal block \( f \) of the matrix function \( G_{\lambda, f} \) be given by (4.2), (4.3) with \( \mu = 0, x + \nu > \lambda, \) and \( x/\nu \) irrational. Then \( G_{\lambda, f} \) admits a canonical AP factorization if and only if \( \det c_0 \neq 0 \).

**Proof.** Necessity. Suppose that \( G_{\lambda, f} \) admits a canonical AP factorization. Then so do all its sufficiently small perturbations, in particular, a matrix function \( G_{\lambda, f_1} \) with

\[
\Pi(-\lambda, \lambda)f_1 = c_{-1,1} e_{-\nu} + c_{0,1} + c_{1,1} e_x,
\]

\( c_{0,1} = c_0, c_{1,1} = c_1, \) and \( c_{-1,1} \) close enough to \( c_{-1,1} \). Choose \( c_{-1,1} \) to be invertible. Then Lemma 4.2 is applicable (with \( f_k = f_1, \lambda_k = \lambda \)). Hence, a matrix function \( G_{\lambda_2, f_2} \) with

\[
\lambda_2 = v = x_2 + \nu, \quad \mu_2 = x + \nu - \lambda \in (0, \nu), \quad x_2 > 0,
\]

\( \Omega(f_2) \subseteq \{-\nu_2, \mu_2, x_2\} \) and

\[
M_{-\nu_2}(f_2) = (-c_{-1,1}^{-1} c_0)^{\lfloor \lambda/\nu \rfloor} c_{-1,1}^{-1}
\]

also admits a canonical AP factorization.

Due to Theorem 2.4, the intersection \( \Omega(f_2) \cap (-\nu, 0] \) must be non-empty. Hence, \( \Omega(f_2) \cap (-\nu, 0] = \{-\nu_2\} \). From Lemma 4.4 we conclude that \( M_{-\nu_2}(f_2) \) is invertible. Since \( \lambda > \nu \), this implies the invertibility of \( c_0 \).

Sufficiency. Without loss of generality, we may suppose that \( c_0 = -I_m (= -I) \), that is,

\[
G_{\lambda, f} = G_0 = \begin{bmatrix} e_{\lambda} & 0 \\ c_{-1} e_{-\nu} - I + c_1 e_x & e_{-\lambda} \end{bmatrix}.
\]

Represent \( \nu/\alpha \) as a continued fraction

\[
\frac{\nu}{\alpha} = n_1 + \frac{1}{n_2 + \frac{1}{n_3 + \ldots}}
\]

and introduce the sequences \( \beta_k, \lambda_k \in \mathbb{R}, \ c_{\pm 1,k} \in \mathbb{C}^{m \times m} \) according to the following recursive formulas:
Let also $a_1 = I$, $b_1 = c_1$,

\[ a_j = \begin{cases} 
c_{-1,j}^* e_{\beta_j} + \sum_{s=0}^{n_j-1} c_{1,j-1}^s c_{-1,j-1} e_{s\beta_j+\beta_j-\beta_j} & \text{if } j > 1 \text{ is odd}, \\
c_{-1,j}^* e_{\beta_j} - I + c_{1,j} e_{\beta_j-1} & \text{if } j \text{ is even}.
\end{cases} \]

\[ b_j = \begin{cases} 
c_{1,j-2}^* e_{\beta_j-3} \cdots c_{1,j-1}^s c_{1,0}^1 c_{1,0} & \text{if } j > 1 \text{ is odd}, \\
c_{1,j-2}^* c_{1,j-3}^s c_{1,j-4}^s \cdots c_{1,1}^s c_{1,0}^1 c_{1,0} & \text{if } j \text{ is even}.
\end{cases} \]

Finally, let

\[ G_j = \begin{bmatrix} a_j \left(c_{-1,j}^* e_{\beta_j} + \sum_{s=0}^{n_j-1} c_{1,j-1}^s c_{-1,j-1} e_{s\beta_j+\beta_j-\beta_j}\right) & I + a_j \sum_{s=0}^{n_j-2} c_{1,j-1}^s b_j e_{(s+1)\beta_j-1} & c_{1,j-1}^* b_j e_{-\beta_j} \\
-c_{-1,j} e_{\beta_j} - I + c_{1,j} e_{\beta_j-1} & c_{1,j}^* b_j e_{-\beta_j} & \end{bmatrix}, \]

if $j \in \mathbb{N}$ is odd, and

\[ G_j = \begin{bmatrix} a_j c_{1,j-1}^* e_{\beta_j} & I + a_j \sum_{s=0}^{n_j-2} c_{1,j-1}^s b_j e_{(s+1)\beta_j-1} \\
-c_{-1,j} e_{-\beta_j-1} - I + c_{1,j} e_{\beta_j} & c_{1,j}^* b_j e_{-\beta_j} \end{bmatrix}, \]

if $j \in \mathbb{N}$ is even.

Direct computations show that

\[ Y_j^+ G_j = G_{j-1} \text{ for } j \text{ even, and } G_j Y_j^- = G_{j-1} \text{ for } j \text{ odd}, \tag{4.14} \]

where

\[ Y_1^- = \begin{bmatrix} I - \phi_1^+ c_{1,0} e_{-\beta_{-1}} - \phi_1^+ e_{-\beta_0} \phi_1^+ e_{-\beta_{-1}} & \phi_1^+ e_{-\beta_0} \\
-c_{1,0} \phi_1 e_{-\beta_{-1}} e_{\beta_{0}} + c_{1,0} e_{-\beta_{-1}} I & I + c_{1,0} \phi_1 e_{-\beta_{-1}} \end{bmatrix}, \]

and for $j > 1$,

\[ Y_j^\pm = \begin{bmatrix} y_{11} & y_{12} \\
y_{21} & y_{22} \end{bmatrix} \]

with
where $G_j$ and $G_j^{-}$ are given by the formulas

$$
Y_{11} = \begin{cases}
I - \sum_{k=0}^{n_j-1} c_{1,j-1} e_k b_j, \\
I + a_{j-1} \sum_{k=0}^{n_j-1} c_{1,j-2} e_k b_j - \sum_{k=0}^{n_j-1} (c_{1,j-2} e_k b_j) b_{j-1},
\end{cases}
$$

$$
Y_{12} = \begin{cases}
\left(-e_{-j-2} I + \sum_{k=0}^{n_j-2} c_{1,j-2} e_{k+1} b_j, \\
-a_{j-1} \left(e_{-j-2} I + \sum_{k=0}^{n_j-1} c_{1,j-2} e_k b_j - \sum_{k=0}^{n_j-1} (c_{1,j-2} e_k b_j) b_{j-1}\right),
\end{cases}
$$

$$
Y_{21} = \begin{cases}
a_{j+1} \sum_{k=0}^{n_j-1} c_{1,j-1} e_k b_j, \\
c_{1,j-1} \sum_{k=0}^{n_j-1} c_{1,j-1} e_k b_j - \sum_{k=0}^{n_j-1} (c_{1,j-2} e_k b_j) b_{j+1},
\end{cases}
$$

$$
Y_{22} = \begin{cases}
I + a_{j-1} e_{-j-2} \sum_{k=0}^{n_j-1} (c_{1,j-2} e_k b_j), \\
I - c_{1,j-1} \sum_{k=0}^{n_j-1} c_{1,j-2} e_k b_j, \\
for \ j \ \ {\text{odd, and}}
\end{cases}
$$

if $j$ is odd, and

$$
\begin{align*}
&\begin{cases}
a_{j} c_{1,j-1} e_k b_j, \\
I + a_{j} c_{1,j-1} \sum_{k=0}^{n_j-1} c_{1,j} e_{k+1} b_j,
\end{cases} \\
&\begin{cases}
-a_{j} c_{1,j} e_k b_j, \\
I + a_{j} c_{1,j} \sum_{k=0}^{n_j-1} c_{1,j-1} e_k b_j,
\end{cases}
\end{align*}
$$

if $j$ is even, respectively. In these formulas, $N_j$ is an abbreviation for $[j/\beta_j]$. From (4.14) and (4.15) it follows that, for any $l \in \mathbb{N}$,

$$
G_0 = \left( Y_2^+ \cdots Y_{2[l/2]}^+ G_l^+ \right) \left( G_1^+ Y_{2[l/2]+1}^- \cdots Y_1^- \right). \tag{4.16}
$$

Let us choose $l$ in such a way that

$$
a + v + \sum_{s=1}^{l} n_s \beta_{s-1} \geq \lambda > \beta_{l-1} + \sum_{s=1}^{l-1} n_s \beta_{s-1} \tag{4.17}
$$
the existence of such an $l$ was shown in [12, p. 229]). Then directly from the definitions of $Y^+_j$, $G^+_j$ it follows that

$$Y^+_j, G^+_j \in APW^+ \quad Y^-_j, G^-_j \in APW^- \quad (j = 1, \ldots, l).$$

Since

$$\det Y^+_j = \det Y^-_j = \det G^+_j = \det G^-_j = 1,$$

the inverse matrices fall into the same classes:

$$(Y^+_j)^{-1}, (G^+_j)^{-1} \in APW^+ \quad (Y^-_j)^{-1}, (G^-_j)^{-1} \in APW^- \quad (j = 1, \ldots, l).$$

Hence, the representation (4.16) delivers a canonical $APW$ factorization of the matrix function $G_0$. □

**Remark.** In the setting of the sufficiency part of Theorem 4.5, but under the additional condition $\det(c_1c_1^*) \neq 0$, a canonical factorization of the matrix function $G_{z,f}$ was constructed (also recursively) in [12, Theorem 6.1]. Formulas for the factors $G_{z}$, obtained in [12], contain $c_1^*$ and therefore cannot be used directly in the case of singular $c_{\pm 1}$. It is possible to rewrite these formulas in such a way that $c_1^*$ eventually cancel out. However, the amount of computations involved is no less than in the independent proof given above.

**Theorem 4.6.** Let the off-diagonal block $f$ of the matrix function $G_{z,f}$ be given by (4.2) and (4.3) with $\mu \neq 0$, $\alpha + v = \lambda$, and $\beta = (\alpha + v)/(\mu + v)$ irrational. Then $G_{z,f}$ admits a canonical $AP$ factorization if and only if $\det(c_1c_1^*) \neq 0$.

**Proof.** Using Lemma 2.1 if appropriate, we may suppose without loss of generality that $\mu > 0$. Due to Lemma 4.4, the invertibility of $c_{-1}$ is then necessary for $G_{z,f}$ to admit a canonical $AP$ factorization. Suppose therefore that $c_{-1}$ is invertible. Then Lemma 4.2 can be applied, according to which $G_{z,f}$ admits a canonical $AP$ factorization only simultaneously with $G_{z_1,f_1}$. Here $f_1$ is given by (4.8), in which $f_0 = f$, $k = 0$, $\mu_1 = \alpha + v - \lambda = 0$, $z_1 + v_1 = \mu + v > v = \lambda_1$, and

$$\frac{z_1}{v_1} = \frac{(w_0 + 1)(\mu + v) - \lambda}{\lambda - w_0(\mu + v)} = \frac{w_0 + 1 - \beta}{\beta - w_0}$$

is irrational along with $\beta$. In particular, $z_1v_1 \neq 0$, so that the constant term of $f_1$ coincides with $-c_{-1}^*c_1$.

Observe that $\Pi_{(-\lambda_1, \lambda_1]}f_1$ either is a trinomial satisfying conditions of Theorem 4.5 (if both $-v_1$ and $z_1$ lie in the interval $(-\lambda_1, \lambda_1)$), or it is one-sided. In both cases, $G_{z_1,f_1}$ admits a canonical $AP$ factorization if and only if $M(f_1)$ is invertible. The latter condition is equivalent to the invertibility of $c_1$. □

We turn now to the case $\mu \neq 0$, $\alpha + v \neq \lambda$. 
Theorem 4.7. Let the off-diagonal block \( f \) of the matrix function \( G_{\lambda,f} \) be given by (4.2) and (4.3) with \( \alpha + \nu \neq \lambda \leq \alpha + |\mu| + \nu \), and either \( \mu > 0 \), \( \lambda/(\mu + \nu) \in \mathbb{Z} \) or \( \mu < 0 \), \( \lambda/(\alpha - \mu) \in \mathbb{Z} \). Then \( G_{\lambda,f} \) admits a canonical AP factorization if and only if \( \det c_{-1} c_0 \neq 0 \) (for \( \mu > 0 \)) or \( \det c_0 c_1 \neq 0 \) (for \( \mu < 0 \)).

Proof. Due to Lemma 2.1(i), it suffices to consider the case \( \mu > 0 \). Necessity of the condition \( \det c_{-1} \neq 0 \) then follows from Lemma 4.4. If this condition is satisfied, Lemma 4.2 implies that \( G_{\lambda,f} \) admits a canonical AP factorization only simultaneously with \( G_1 = G_{\hat{\lambda}, f_1} \), where \( f_1 \) is given by (4.8) with \( k = 0 \) and \( f_0 = f \). Since

\[
\frac{\lambda}{\mu + \nu} < \frac{\hat{\lambda} + \mu}{\mu + \nu} < \frac{\hat{\lambda} + \mu + \nu}{\mu + \nu},
\]

condition \( \lambda/(\mu + \nu) \in \mathbb{Z} \) implies that \( w = [(\hat{\lambda} + \mu)/(\mu + \nu)] = \lambda/(\mu + \nu) \).

Therefore,

\[ v_1 = \hat{\lambda} - w(\mu + \nu) = 0, \quad \alpha_1 = v + \mu > v = \lambda_1, \quad \text{and} \quad \mu_1 = \alpha + v - \hat{\lambda} \neq 0. \]

Hence, \( \Pi_{(\alpha, \lambda_1)} f_1 \) is one-sided, with the constant term \( \mathbf{M}(f_1) \) equal \( (-c_{-1}^1 c_0)^w c_{-1}^1 \). According to Theorem 2.4, \( G_1 \) (and therefore \( G_{\lambda,f} \)) admits a canonical AP factorization if and only if the matrix \( (-c_{-1}^1 c_0)^w c_{-1}^1 \) (or, equivalently, \( c_0 \)) is invertible. \( \square \)

Our subsequent consideration depends on the sign of \( \alpha + \nu - \hat{\lambda} \).

Theorem 4.8. Let the off-diagonal block \( f \) of the matrix function \( G_{\lambda,f} \) be given by (4.2) and (4.3) with \( \lambda < \alpha + \nu \) and either \( \mu > 0 \), \( \hat{\lambda}/(\mu + \nu) \notin \mathbb{Z} \), or \( \mu < 0 \), \( \hat{\lambda}/(\alpha - \mu) \notin \mathbb{Z} \). Define the sequences \( \lambda_k, \alpha_k, v_k, \mu_k \) by the recursive relations (4.6) with the initial conditions \( \lambda_0 = \lambda, \mu_0 = |\mu| \),

\[
\alpha_0 = \begin{cases} \alpha, & v_0 = \begin{cases} v, & \mu > 0, \\ \alpha, & \mu < 0. \end{cases} \end{cases}
\]

Let \( N \) be the first value of \( k \) for which at least one of the inequalities

\[ 0 < v_k < \lambda_k, \quad \mu_k < \alpha_k < \hat{\lambda}_k, \]

fails. \(^5\) Then \( G_{\lambda,f} \) admits a canonical AP factorization if and only if all the coefficients \( c_{-1}, c_0, c_1 \) are invertible, and \( v_N = 0 \) or \( v_N > 0 \), \( \lambda_N/(\mu_N + v_N) \in \mathbb{Z} \).

---

\(^5\) Observe that \( \mu_0 > 0 \) and \( \alpha_0 + v_0 > \hat{\lambda}_0 \). Therefore, the existence of such \( N(\in \mathbb{N}) \) follows from the proof of Theorem 4.3, case (3).
Proof. Due to Lemma 2.1(i), it suffices to consider the case $\mu > 0$. Suppose first that $\det(c_{-1}c_0c_1) \neq 0$. Theorem 4.3 then implies that $G_{\lambda,f}$ is AP factorable, and its partial AP indices are the same as those of the matrix $G_{\lambda_{N},f_{N}}$. Due to (4.11), $\mu_k > 0$ for all $k = 1, \ldots, N$. Then, according to Remark 3 after Theorem 4.3, an AP factorization of $G_{\lambda,f}$ is canonical if and only if $\alpha_N \geq \lambda_N$ and the partial AP indices of the matrix $G_{\lambda_{N},f_{N}}$ with

$$f_N = c_{-1,N} e_{v_N} + c_{0,N} e_{\mu_N}, \quad (4.19)$$

are both positive, another application of Lemma 4.4 shows that the coefficient $c_{-1,1} = (c_{-1}c_0c_1)^{([\lambda + \mu]/(v + \mu))}c_{-1}^{-1}$ is invertible. This implies the invertibility of $c_0$.

If $N > 1$, then the reasoning of the previous paragraph can be applied to $f_1$ in place of $f$. Hence, the coefficient $c_{0,1}$ also is invertible.

If $N = 1$, then the canonical AP factorability of $G_{\lambda_{1},f_{1}}$ implies that $\alpha_1 \geq \lambda_1$ and the matrix $G_{\lambda_{1},f_{1}}$, with $f_1$ given by (4.19), also admits a canonical AP factorization. Since $v_1 = \lambda - [(\lambda + \mu)/(v + \mu)](v + \mu)$ differs from zero due to the condition $\lambda/(\mu + v) \notin \mathbb{Z}$, this is only possible if both coefficients of $f_1$ are invertible (Theorem 4.1). This again implies the invertibility of $c_{0,1}$.

Taking into consideration the formula $c_{0,1} = -c_{-1}^{-1}c_1$, we conclude that (both for $N = 1$ and $N > 1$) the coefficient $c_1$ is invertible as well. □

Theorem 4.9. Let the off-diagonal block $f$ of the matrix function $G_{\lambda,f}$ be given by (4.2) and (4.3) with $\alpha + v < \lambda \leq \alpha + |\mu| + v$ and $\lambda/(\mu + v) \notin \mathbb{Z}$ ($\lambda/(\lambda - \mu) \notin \mathbb{Z}$) if $\mu > 0$ (respectively, $\mu < 0$). Define the sequences $\lambda_k, \alpha_k, v_k, \mu_k$ by the recursive relations (4.6) with the initial conditions
\( \mu_1 = \lambda - (\alpha + v) \),

\[ \hat{\lambda}_1 = \begin{cases} v, & \alpha_1 = \frac{\lambda - w(\mu + v)}{\mu}, \\ \alpha, & \frac{\lambda - w(\alpha - \mu)}{\mu} \end{cases} \quad (4.20) \]

\[ v_1 = \begin{cases} (w + 1)(\mu + v) - \lambda, & w = \left\lfloor \frac{\lambda + \mu}{\mu + v} \right\rfloor, \\ (w + 1)(\alpha - \mu) - \lambda, & \text{if } \begin{cases} \mu > 0, \\ \mu < 0. \end{cases} \end{cases} \quad (4.21) \]

Let \( N \) be the first value of \( k \) for which at least one of inequalities (4.18) fails. Then \( G_{\hat{\alpha}, f} \) admits a canonical AP factorization if and only if all the coefficients \( c_{-1}, c_0, c_1 \) are invertible, and one of the following three conditions holds:

(i) \( \alpha_1 < 0, \frac{\lambda}{\alpha_1 - \lambda} \in \mathbb{Z} \),

(ii) \( v_N = 0 \),

(iii) \( v_N > 0, \frac{\lambda}{\mu + v} \in \mathbb{Z} \).

**Proof.** As in Theorems 4.7 and 4.8, it suffices to consider the case \( \mu > 0 \). Then, due to Lemma 4.4, the invertibility of \( c_{-1} \) is necessary for a canonical AP factorization of \( G_{\hat{\alpha}, f} \) to exist. Provided that \( c_{-1} \) is invertible, \( G_{\hat{\alpha}, f} \) admits a canonical AP factorization only simultaneously with \( G_{\hat{\alpha}, \tilde{f}} \), where

\[ \tilde{f} = (-c_{-1}^{-1} c_0)^w c_{-1}^{-1} e_{-\alpha_1} + (-c_{-1}^{-1} c_1) e_{-\mu_1} + (-c_{-1}^{-1} c_0)^{w+1} c_{-1}^{-1} e_{v_1} \]

and \( \alpha_1, \mu_1, v_1 \) are given by (4.20). Observe that \( \alpha_1 \neq 0 \) because \( \frac{\lambda}{\alpha_1 - \lambda} \notin \mathbb{Z} \).

Moreover,

\[ v_1 > \frac{\lambda + \mu}{\mu + v} (\mu + v) - \lambda = \mu > 0, \quad \alpha_1 < \lambda - \left( \frac{\lambda + \mu}{\mu + v} - 1 \right) (\mu + v) \]

(4.22)

and

\[ \alpha_1 + v_1 = \mu + v > v = \lambda_1. \]

(4.23)

We now consider the situations \( N = 1 \) and \( N > 1 \) separately.

(a) \( \mu_1 > 0 \). According to (4.21), then either \( v_1 > \lambda_1 \) or \( \alpha_1 \leq \mu_1 \). For \( v_1 > \lambda_1 \), Lemma 2.1 (ii) allows us to substitute \( \tilde{f} \) by the binomial

\[ (-c_{-1}^{-1} c_0)^w c_{-1}^{-1} e_{-\alpha_1} + (-c_{-1}^{-1} c_1) e_{-\mu_1}. \]

(4.24)

Since \( \mu_1 > 0 \), Theorem 4.1 implies that \( G_{\hat{\alpha}, f} \) admits a canonical AP factorization if and only if (i) holds and both coefficients in (4.23) are invertible, that is, \( \det c_{-1}^{-1} c_0 c_1^{-1} \neq 0 \).
If \( \lambda_1 \leq \mu_1 \) but \( v_1 < \lambda_1 \), then \( \lambda_1 > 0 \) due to (4.22), \( v_1 > 0 \) due to (4.21), and \( 0 < \lambda_1/(\mu_1 + v_1) < \lambda_1/(\lambda_1 + v_1) < 1 \). Hence, neither of the conditions (i)–(iii) holds. On the other hand, \( G_{\lambda_1,f} \) does not admit a canonical \( AP \) factorization because \( \hat{f} \) falls into a big gap case with a gap \( \lambda_1 + v_1 > \lambda_1 \) (Theorem 2.5).

(b) \( N \geq 2 \). Then (4.18) holds for \( k = 1 \). Since \( \mu_1 > 0 \), we can write that

\[
-\lambda_1 < -\mu_1 < 0 < v_1 < \lambda_1.
\]

If \( \lambda_1/(\mu_1 + v_1) \in \mathbb{Z} \), then Theorem 4.7 (with \( \alpha, \mu, \nu, \lambda \) changed to \( v_1, -\mu_1, \lambda_1 \) and \( \lambda_1 \), respectively) shows that \( G_{\lambda_1,f} \) admits a canonical \( AP \) factorization if and only if the coefficients \( -c_{-1}^{-1}c_1 \) and \( -c_{-1}^{-1}c_0 ^{-1}c_1^{-1} \) of \( \hat{f} \) are invertible, that is, if and only if \( \det c_{-1}c_0 c_1 \neq 0 \). On the other hand, it follows from (4.6) that \( v_2 = 0 \).

Thus, (ii) holds with \( N = 2 \). Conversely, \( N = 2 \) and (ii) imply that \( \lambda_1/(\mu_1 + v_1) \in \mathbb{Z} \).

If \( \lambda_1/(\mu_1 + v_1) \notin \mathbb{Z} \), then, due to (4.22) and (4.24), the matrix \( G_{\lambda_1,f} \) satisfies the conditions of Theorem 4.8 (again, with \( \alpha, \mu, \nu, \lambda \) changed to \( v_1, -\mu_1, \lambda_1, \lambda_1 \)). Since the middle exponent \( -\mu_1 \) of \( \hat{f} \) is negative, the initial values for the corresponding sequences (4.6) are given by (4.20). According to Theorem 4.8, a canonical \( AP \) factorization of \( G_{\lambda_1,f} \) exists if and only if all the coefficients of \( \hat{f} \) are invertible (equivalently, \( \det c_{-1}c_0 c_1 \neq 0 \)) and (ii) or (iii) holds. It remains to observe that situation (i) for \( N \geq 2 \) is excluded because of (4.24).

Theorems 4.5–4.9, cover all the cases of a trinomial off-diagonal block (4.2) with \( \alpha + |\mu| + v \geq \lambda \), except for \( \alpha + v = \lambda \), \( \mu = 0 \). Partial results available for the latter case were discussed in Remark 2 after Theorem 4.3. The case \( \alpha + |\mu| + v < \lambda \) remains open.

Finally, we consider the case of a trinomial (4.2) and (4.3), for which the intersection

\[
\left[ \frac{\lambda - \mu}{\mu + v}, \frac{\lambda + \mu}{\alpha + v} \right] \cap \mathbb{Z}
\]

is not empty. We use the standard notation \( [x] \) for the smallest integer number greater than or equal to \( x \); of course, \( [x] = [x] + 1 \) for \( x \in \mathbb{R} \setminus \mathbb{Z} \) and \( [x] = [x] = x \) for \( x \in \mathbb{Z} \).

**Theorem 4.10.** Let the off-diagonal block \( f \) of the matrix function \( G_{\lambda,f} \) be given by (4.2), (4.3) and, in addition,

\[
\left[ \frac{\lambda - \mu}{\mu + v}, \frac{\lambda + \mu}{\alpha + v} \right] \cap \mathbb{Z} \neq \emptyset.
\]
Denote
\[ K = \begin{bmatrix} \lambda - \mu \\ \mu + v \end{bmatrix}, \quad p = \begin{bmatrix} (\lambda - K(\mu + v)) \\ \lambda - \mu \end{bmatrix}, \quad N = \begin{bmatrix} v \\ \lambda - \mu \end{bmatrix}, \]

Then \( G_{\lambda,f} \) admits a canonical AP factorization if and only if \( c_{-1} \) is invertible and one of the following three conditions holds:

(i) \( \frac{\lambda - K(\mu + v)}{\lambda - \mu}, \frac{v}{\lambda - \mu} \in \mathbb{Z} \) and \( \det T_N(z) \neq 0 \),

(ii) \( \frac{\lambda - K(\mu + v)}{\lambda - \mu} \in \mathbb{Z}, \frac{v}{\lambda - \mu} \notin \mathbb{Z} \) and \( \det T_N(z) \det T_{N+1}(z) \neq 0 \),

(iii) \( \frac{\lambda - K(\mu + v)}{\lambda - \mu} \notin \mathbb{Z}, \frac{v}{\lambda - \mu} \in \mathbb{Z} \) and \( \det T_N(z) \det A_N(z) \neq 0 \).

Here \( z_j = y_{k-p+j,p+j} \); for \( n_1, n_2 \in \mathbb{Z}_+ \), \( y_{n_1,n_2} \) is the sum of all possible products of \( n_1 + n_2 \) matrices, \( n_j \) of which equal \(-c_{-1}^{-1}c_{-j-1} \) \((j = 1,2)\) and otherwise \( y_{n_1,n_2} = 0 \). The matrices \( T_n(z) \), \( A_n(z) \) are defined by the sequence \( z_j \) as in Theorem 3.1.

**Proof.** Condition
\[ \left\{ \frac{\lambda - \mu \lambda + \mu}{\mu + v \lambda + v} \right\} \cap \mathbb{Z} \neq \emptyset \]
implies, in particular, that \((\lambda - \mu)/(\mu + v) \leq (\lambda + \mu)/(\lambda + v)\), and therefore \( \mu > 0 \). As in the proof of Theorems 4.6–4.8, we can now use Lemma 4.4 to conclude that the invertibility of \( c_{-1} \) is necessary for a canonical AP factorability of \( G_{\lambda,f} \). If this condition is satisfied, we still can use Lemma 2.3 to replace \( G_{\lambda,f} \) by the matrix function \( G_{\lambda,f} \) with the same AP factorability properties. However, Lemma 4.2 is no more applicable, because the condition \( \lambda \leq \lambda + \mu + v \) does not necessarily hold. Instead, we need to retreat to the general formula (4.4).

Observe that
\[ \frac{\lambda - \mu}{\mu + v} \leq K \leq \frac{\lambda + \mu}{\lambda + v} \]
and
\[ (n_1 + n_2)(\mu + v) \leq n_1(\mu + v) + n_2(\lambda + v) \leq (n_1 + n_2)(\lambda + v). \]
Therefore,
\[ n_1(\mu + v) + n_2(\lambda + v) - \lambda \leq (K - 1)(\lambda + v) - \lambda \leq \left( \frac{\lambda + \mu}{\lambda + v} - 1 \right)(\lambda + v) - \lambda \]
\[ = \lambda + \mu - v - \lambda = -v \]
if \( n_1 + n_2 \leq K - 1 \), and
\[ n_1(\mu + v) + n_2(\lambda + v) - \lambda \geq (K + 1)(\mu + v) - \lambda \geq \frac{\lambda + v}{\mu + v}(\mu + v) - \lambda \]
\[ = \lambda + v - \lambda = v \]
if \( n_1 + n_2 \geq K + 1 \). Hence, the only possible pairs \( \{n_1, n_2\} \) appearing in (4.4) are such that \( n_1 + n_2 = K \). In other words,
\[ f' = \Pi_{(\mu, \lambda)} \sum_j y_{K-j} e_{-1}^{j} e_j(\mu + v - \lambda). \]  
(4.26)

According to (4.25), the distances between the exponents of \( f' \) are commensurable, with \( h = \lambda - \mu \). It remains to introduce \( \tau = p(\mu - \mu) + K(\mu + v) - \lambda \), rewrite \( f' \) as
\[ f' = \Pi_{(\mu, \lambda)} \sum_j y_{K-j} e_{-1}^{j} e_j(\mu + v + \tau) \]
and use Theorem 3.1.  

With no further restrictions on the coefficients \( c_j \), we do not know whether a (not necessarily canonical) \( AP \) factorization of \( G_{\lambda, f} \) always exists in the setting of Theorem 4.10. However, if the coefficients \( c_{-1}, c_0, c_1 \) are invertible and \( c_0 c_{-1} c_1 = c_1 c_{-1} c_0 \), then the (also invertible) matrices \( b_j = -e_{-1}^{j_1} e_{j-1}^{j_1} \) \((j = 1, 2)\) commute. Hence, (4.25) can be rewritten as
\[ f' = \Pi_{(\mu, \lambda)} \sum_j \frac{K!}{j!(K-j)!} b_1^{K-j} b_2^{j} e_{-1}^{j_1} e_j(\mu + v + K(\mu + v) - \lambda). \]

This matrix function is actually of the form (3.6), with \( R = c_{-1} b_1^{-1} b_2 c_{-1} \). Due to Theorem 3.2, the matrix function \( G_{\mu, f'} \) (and therefore \( G_{\lambda, f} \)) is \( AP \) factorable. In other words, the following result holds.

**Theorem 4.11.** Let the off-diagonal block \( f \) of the matrix function \( G_{\lambda, f} \) be given by (4.2) and (4.3) with
\[
\left[ \frac{\lambda - \mu}{\mu + v}, \frac{\lambda + \mu}{\mu + v} \right] \cap \mathbb{Z} \neq \emptyset,
\]
all the coefficients $c_{-1}$, $c_0$, $c_1$ be invertible, and $c_0 c_{-1}^{-1} c_1 = c_1 c_{-1}^{-1} c_0$. Then $G_{k,f}$ is APP factorable.

References


