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Homotopy types of homeomorphism groups of noncompact 2-manifolds

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Abstract

Suppose M is a noncompact connected PL 2-manifold and let $\mathcal{H}(M)_0$ denote the identity component of the homeomorphism group of M with the compact-open topology. In this paper we classify the homotopy type of $\mathcal{H}(M)_0$ by showing that $\mathcal{H}(M)_0$ has the homotopy type of the circle if M is the plane, an open or half open annulus, or the punctured projective plane. In all other cases we show that $\mathcal{H}(M)_0$ is homotopically trivial. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Hamstrom [4] classified the homotopy types of the identity components of homeomorphism groups of compact 2-manifolds M. In this paper we treat the case where M is noncompact. Suppose M is a PL 2-manifold and X is a compact subpolyhedron of M. We denote by $\mathcal{H}_X(M)$ the group of homeomorphisms h of M onto itself with $h|_X = id$, equipped with the compact-open topology, and by $\mathcal{H}(M)_0$ the identity component of $\mathcal{H}(M)$. Let $\mathbb{R}^2$ denote the plane, $\mathbb{S}^1$ the unit circle and $\mathbb{P}^2$ the projective plane. The following is the main result of this paper.

Theorem 1.1. *Suppose M is a noncompact connected* (*separable*) PL 2-*manifold and X is a compact subpolyhedron of M. Then*

(i) $\mathcal{H}_X(M)_0 \simeq \mathbb{S}^1$ *if* $(M, X) \cong (\mathbb{R}^2, \emptyset)$, $(\mathbb{R}^2, 1\,\mathrm{pt})$, $(\mathbb{S}^1 \times \mathbb{R}^1, \emptyset)$, $(\mathbb{S}^1 \times [0,1), \emptyset)$ *or* $(\mathbb{P}^2 \setminus 1\,\mathrm{pt}, \emptyset)$,

(ii) $\mathcal{H}_X(M)_0 \simeq *$ *in all other cases.*

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Corollary 1.1. *If M is a connected* (*separable*) 2*-manifold and X is a compact subpolyhedron of M with respect to some triangulation of M, then $\mathcal{H}_X(M)_0$ is an ℓ_2-manifold.*

In [14] we obtained a natural principal bundle connecting the homeomorphism group and the embedding space (cf. Section 2). In this paper we will seek a condition under which the fiber of this bundle is connected (Section 3). The contractibility and the ANR property of $\mathcal{H}_X(M)_0$ in the compact case will then imply the similar properties of embedding spaces and in turn the corresponding properties of $\mathcal{H}_X(M)_0$ in the noncompact case. Corollary 1.1 follows immediately from the characterization of ℓ_2-manifolds and this enables us to determine the topological type itself of $\mathcal{H}_X(M)_0$ by the homotopy invariance of infinite-dimensional manifolds.

In a succeeding paper we will investigate the subgroups of $\mathcal{H}_X(M)_0$ consisting of PL and Lipschitz homeomorphisms from the viewpoints of infinite-dimensional topological manifolds.

2. Preliminaries

Throughout the paper we follow the following conventions: Spaces are assumed to be separable and metrizable, and maps are always continuous. When A is a subset of a space X, the notations $\mathrm{Fr}_X A$, $\mathrm{cl}_X A$ and $\mathrm{Int}_X A$ denote the frontier, closure and interior of A relative to X (i.e., $\mathrm{Int}_X A = \{x \in A \mid A$ contains a neighborhood of x in $X\}$ and $\mathrm{Fr}_X A = \mathrm{cl}_X A \setminus \mathrm{Int}_X A$). On the other hand, when M is a manifold, the notations ∂M and $\mathrm{Int}\, M$ denote the boundary and interior of M as a manifold. When N is a 2-submanifold of a 2-manifold M, we always assume that N is a closed subset of M and $\mathrm{Fr}\, N = \mathrm{Fr}_M N$ is a 1-manifold transversal to ∂M. Therefore we have $\mathrm{Int}\, N = \mathrm{Int}_M N \cap \mathrm{Int}\, M$ and $\mathrm{Fr}_M N \subset \partial N$. A metrizable space X is called an ANR (absolute neighborhood retract) if any map $f : B \to X$ from a closed subset of a metrizable space Y has an extension to a neighborhood U of B. If we can always take $U = Y$, then X is called an AR (absolute retract). ANRs are locally contractible and ARs are exactly contractible ANRs (cf. [7]). Finally ℓ_2 denotes the separable Hilbert space $\{(x_n) \in \mathbb{R}^\infty \colon \sum_n x_n^2 < \infty\}$.

In [14] we investigated some extension property of embeddings of a compact 2-polyhedron into a 2-manifold, based upon the conformal mapping theorem. The result is summarized as follows: Suppose M is a PL 2-manifold and $K \subset X$ are compact subpolyhedra of M. Let $\mathcal{E}_K(X, M)$ denote the space of embeddings $f : X \hookrightarrow M$ with $f|_K = id$, equipped with the compact-open topology. We consider the subspace of proper embeddings $\mathcal{E}_K(X, M)^* = \{f \in \mathcal{E}_K(X, M)\colon\ f(X \cap \partial M) \subset \partial M,\ f(X \cap \mathrm{Int}\, M) \subset \mathrm{Int}\, M\}$. Let $\mathcal{E}_K(X, M)^*_0$ denote the connected component of the inclusion $i_X : X \subset M$ in $\mathcal{E}_K(X, M)^*$.

Theorem 2.1. *For every $f \in \mathcal{E}_K(X, M)^*$ and every neighborhood U of $f(X)$ in M, there exists a neighborhood $\mathcal{U}$ of f in $\mathcal{E}_K(X, M)^*$ and a map $\varphi : \mathcal{U} \to \mathcal{H}_{K\cup(M\setminus U)}(M)_0$ such that $\varphi(g) f = g$ for each $g \in \mathcal{U}$ and $\varphi(f) = id_M$.*

Corollary 2.1. *For any open neighborhood U of X in M, the restriction map*

$$\pi : \mathcal{H}_{K\cup(M\setminus U)}(M)_0 \to \mathcal{E}_K(X,U)_0^*, \quad \pi(f) = f|_X,$$

is a principal bundle with fiber $\mathcal{G} \equiv \mathcal{H}_{K\cup(M\setminus U)}(M)_0 \cap \mathcal{H}_X(M)$, where the group $\mathcal{G}$ acts on $\mathcal{H}_{K\cup(M\setminus U)}(M)_0$ by right composition.

Proposition 2.1. *$\mathcal{E}_K(X,M)$ and $\mathcal{E}_K(X,M)^*$ are ANRs.*

Next we recall some fundamental facts on homeomorphism groups of compact 2-manifolds.

Fact 2.1. *If N is a compact* PL 2*-manifold and Y is a compact subpolyhedron of N, then $\mathcal{H}_Y(N)$ is an ANR* ([8,9], *cf.* [14, Lemma 3.2]).

Lemma 2.1 ([4], [12, §3]). *Suppose N is a compact connected* PL 2*-manifold and Y is a compact subpolyhedron of N.*

(i) *If* $(N,Y) \not\cong (\mathbb{D}^2,\emptyset)$, $(\mathbb{D}^2,0)$, $(\mathbb{S}^1\times[0,1],\emptyset)$, $(\mathbb{M},\emptyset)$, $(\mathbb{S}^2,\emptyset)$, $(\mathbb{S}^2,1\,\text{pt})$, $(\mathbb{S}^2,2\,\text{pts})$, $(\mathbb{T}^2,\emptyset)$, $(\mathbb{K}^2,\emptyset)$, $(\mathbb{P}^2,\emptyset)$, $(\mathbb{P}^2,1\,\text{pt})$, *then* $\mathcal{H}_Y(N)_0 \simeq *$.

(ii) *If A is a nonempty compact subset of ∂N, then $\mathcal{H}_{Y\cup A}(N)_0 \simeq *$.*

(iii) *If* $(N,Y) \cong (\mathbb{D}^2,\emptyset)$, $(\mathbb{D}^2,0)$, $(\mathbb{S}^1\times[0,1],\emptyset)$, $(\mathbb{M},\emptyset)$, $(\mathbb{S}^2,1\,\text{pt})$, $(\mathbb{S}^2,2\,\text{pts})$, $(\mathbb{P}^2,1\,\text{pt})$, $(\mathbb{K}^2,\emptyset)$, *then* $\mathcal{H}_Y(N)_0 \simeq \mathbb{S}^1$.

Proof. In [12] the PL-homeomorphism groups of compact 2-manifolds was studied in the context of semisimplicial complex. However, using Corollary 2.1 and the results in [4], we can apply the arguments and results in [12, §3] to our setting.

(i) Let L be a small regular neighborhood of the union Y_1 of the nondegenerate components of Y and let $Y_0 = Y \setminus Y_1$. Since $\mathcal{H}_Y(N)_0$ deforms into $\mathcal{H}_{L\cup Y_0}(N)_0 \cong \mathcal{H}_{\mathrm{Fr}\,L\cup Y_0}(\mathrm{cl}(N\setminus L))_0$, we may assume that $Y_1 \subset \partial N$. This case follows from [4] and [12, §3].

(ii) Let $\partial_+ N$ denote the union of the components of ∂N which meet A. Then $\mathcal{H}_{Y\cup A}(N)_0$ strongly deformation retracts onto $\mathcal{H}_{Y\cup\partial_+ N}(N)_0$, and the latter is contractible by the case (i). □

3. Relative isotopes on 2-manifolds

In Corollary 2.1 we have a principal bundle with a fiber $\mathcal{G} \equiv \mathcal{H}_X(M) \cap \mathcal{H}_K(M)_0$. In this section we will seek a sufficient condition which implies $\mathcal{G} = \mathcal{H}_X(M)_0$. Suppose M is a 2-manifold and N is a 2-submanifold of M. In [2] it is shown that (i) two homotopic essential simple closed curves in $\mathrm{Int}\,M$ and two proper arcs homotopic rel ends in M are ambient isotopic rel ∂M, (ii) every homeomorphism $h : M \to M$ homotopic to id_M is ambient isotopic to id_M. Using these results or arguments we will show that if, in addition, $h|_N = id_N$ then h is isotopic to id_M rel N under some restrictions on disks, annuli and Möbius bands components (i.e., the pieces which admit global rotations). We denote the

Möbius band, the torus and the Klein bottle by $\mathbb{M}$, $\mathbb{T}^2$ and $\mathbb{K}^2$, respectively. The symbol $\#X$ denotes the number of elements (or cardinal) of a set X.

Theorem 3.1. *Suppose M is a connected 2-manifold, N is a compact 2-submanifold of M and X is a subset of N such that*

(i) $M \neq \mathbb{T}^2, \mathbb{P}^2, \mathbb{K}^2$ *or* $X \neq \emptyset$,

(ii) (a) *if H is a disk component of N, then* $\#(H \cap X) \geqslant 2$,

(b) *if H is an annulus or Möbius band component of N, then* $H \cap X \neq \emptyset$,

(iii) (a) *if L is a disk component of* $\mathrm{cl}(M \setminus N)$, *then* $\mathrm{Fr}\, L$ *is a disjoint union of arcs or* $\#(L \cap X) \geqslant 2$,

(b) *if L is a Möbius band component of* $\mathrm{cl}(M \setminus N)$, *then* $\mathrm{Fr}\, L$ *is a disjoint union of arcs or* $L \cap X \neq \emptyset$.

If $h_t : M \to M$ is an isotopy $\mathrm{rel}\, X$ *such that* $h_0|_N = h_1|_N$, *then there exists an isotopy* $h'_t : M \to M$ $\mathrm{rel}\, N$ *such that* $h'_0 = h_0$, $h'_1 = h_1$ *and* $h'_t = h_t\,(0 \leqslant t \leqslant 1)$ *on* $M \setminus K$ *for some compact subset K of M.*

Corollary 3.1. *Under the same condition as in Theorem* 3.1, *we have* $\mathcal{H}_N(M) \cap \mathcal{H}_X(M)_0 = \mathcal{H}_N(M)_0$.

First we explain the meaning of the conditions (ii) and (iii) in Theorem 3.1. Suppose $h \in \mathcal{H}_N(M)$ and h is isotopic to $id_M\, \mathrm{rel}\, X$. In order that h is isotopic to id_M rel N, it is necessary that h does not Dehn twist along the boundary circle of any disk, Möbius band and annulus component of N. This is ensured by the condition (ii) in Theorem 3.1 (Fig. 1(a)). However, this is not sufficient because a union of some components of N and $\mathrm{cl}(M \setminus N)$ may form a disk, a Möbius band or an annulus. The condition (iii) in Theorem 3.1 is imposed to prevent Dehn twists around these pieces (Fig. 1(b)). This condition is too strong (we can replace X by $Y = \{a_1, b_1\}$), but it is simple and sufficient for our purpose.

We proceed to the verification of Theorem 3.1. We need some preliminary lemmas. Throughout this section we assume that M is a connected 2-manifold and N is a 2-submanifold of M. When G is a group and $S \subset G$, $\langle S\rangle$ denotes the subgroup of G generated by S.

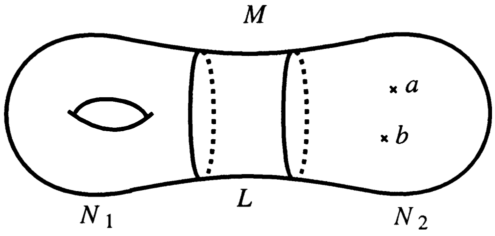
We will use the following facts from [2].

Fact 3.1.

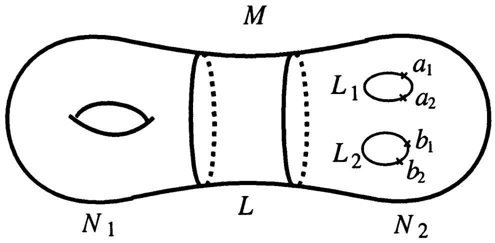
(0) ([2, Theorem 1.7]) *If a simple closed curve C in M is null-homotopic, then it bounds a disk.*

(1) ([2, Theorem 3.1]) *Suppose α and β are proper arcs in M. If they are homotopic relative to end points, then they are ambient isotopic relative to ∂M.*

(2) ([2, Theorem 4.2]) *Let C be a simple closed curve in M, which does not bound a disk or a Möbius band. Let $\alpha \in \pi_1(M, *)$ be represented by a single circuit of C and let $\alpha = \beta^k$, $k \geqslant 0$. Then $\alpha = \beta$.*



(a)



(b)

Fig. 1. h cannot Dehn twist L. (a) $N = N_1 \cup N_2$, $\mathrm{cl}(M \setminus N) = L$ and $X = \{a, b\}$, (b) $N = N_1 \cup N_2$, $\mathrm{cl}(M \setminus N) = L \cup L_1 \cup L_2$ and $X = \{a_1, a_2, b_1, b_2\}$.

(3) ([2, Lemma 4.3])
- (i) *If $M \neq \mathbb{P}$, then $\pi_1(M)$ has no torsion elements.*
- (ii) *Suppose $M \neq \mathbb{T}^2, \mathbb{K}^2$. If $\alpha, \beta \in \pi_1(M)$ and $\alpha\beta = \beta\alpha$, then $\alpha, \beta \in \langle\gamma\rangle$ for some $\gamma \in \pi_1(M)$.*

(4) ([2, p. 101, lines 5–10]) *If $M \neq \mathbb{P}^2$ and every circle component of* Fr N *is essential in M, then the inclusion induces a monomorphism $\pi_1(N, x) \to \pi_1(M, x)$ for every $x \in N$.*

(5) *Suppose M is compact, X is a closed subset of ∂M, $X \neq \emptyset$ and $h : M \to M$ is a homeomorphism with $h|_X = id_X$.*
- (i) ([2, Theorem 3.4]) *If $M = \mathbb{D}^2$ or $\mathbb{M}$ and $h|_{\partial M} : \partial M \to \partial M$ is orientation preserving, then h is isotopic to id_M* rel X.
- (ii) ([2, Proof of Theorem 6.3]) *If $M \neq \mathbb{D}^2$ and h satisfies the following condition $(*)$, then h is isotopic to id_M* rel X:
  - $(*)$ $h\ell \simeq \ell$ rel *end points for every proper arc $\ell : [0, 1] \to M$ with $\ell(0), \ell(1) \in X$ (we allow that $\ell(0) = \ell(1)$ when X is a single point).*

Comments. (4) Consider the universal covering $\pi : \widetilde{M} \to M$. By Fact 3.1(3i) $\pi^{-1}(\mathrm{Fr}_M N)$ is a union of real lines, half rays and proper arcs. If $M = \mathbb{P}^2$, then $N = \mathbb{P}^2$.

(5ii) M is a disk with k holes, ℓ handles (a handle = a torus with a hole) and m Möbius bands. The assertion is easily verified by the induction on $n = k + \ell + m$, using Fact 3.1 (1) and (5i), together with the following remarks:
- (a) When $\#X \geqslant 2$, we have $h\ell \simeq \ell$ rel. end points even if $\ell(0) = \ell(1)$.
- (b) If h is (ambient) isotopic to h_1 rel. X, then h_1 also satisfies the condition $(*)$.

(c) Since $M \neq \mathbb{D}^2$, from the condition $(*)$ it follows that for every component C of ∂M, we have $h(C) = C$ and h preserves the orientation of C.
(d) Let $C_1, \dots, C_p$ be the components of ∂M which meet X. Then h is isotopic rel. X to h_1 such that $h_1 = id$ on each C_i. Furthermore, h_1 satisfies $(*)$ for $\bigcup_i C_i$.

We also need the following remarks.

Fact 3.2. *Suppose M is a connected 2-manifold and C is a circle component of ∂M. If either* (i) $C \neq \partial M$ *or* (ii) M *is noncompact, then C is a retract of M.*

Comments. (ii) Take a half lay ℓ connecting C and ∞, and consider the regular neighborhood N of $C \cup \ell$. Since ∂N is a real line we can retract M onto N and then onto C.

Fact 3.3. *Suppose M is a compact 2-manifold, $\{M_i\}$ is a finite collection of compact connected 2-manifolds such that $M = \bigcup_i M_i$ and $\operatorname{Int} M_i \cap \operatorname{Int} M_j = \emptyset$ $(i \neq j)$.*
(i) *If M is a disk, then some M_i is a disk.*
(ii) *If M is a Möbius band, then some M_i is a disk or a Möbius band.*
(iii) *If M is an annulus, then some M_i is a disk or an essential annulus in M. (If N is a disk with r holes in M $(r \geqslant 2)$, then there exists a disk $D \subset \operatorname{Int} M$ such that $D \cap N = \partial D \subset \partial N$.)*

Lemma 3.1. *Suppose $M \neq \mathbb{K}^2$, C is a simple closed curve in M which does not bound a disk or a Möbius band in M, $x \in C$ and $\alpha \in \pi_1(M, x)$ is represented by C. If $\beta \in \pi_1(M, x)$ and $\beta^k = \alpha^\ell$ for some $k, \ell \in \mathbb{Z} \setminus \{0\}$, then $\beta \in \langle \alpha \rangle$.*

Proof. Attaching $\partial M \times [0,1)$ to $\partial M \subset M$, we may assume that $\partial M = \emptyset$. Take a covering $p : (\widetilde{M}, \tilde{x}) \to (M, x)$ such that $p_* \pi_1(\widetilde{M}, \tilde{x}) = \langle \alpha, \beta \rangle \subset \pi_1(M, x)$.

If $\widetilde{M}$ is noncompact, then by [2, Lemma 2.2] there exists a compact connected 2-submanifold N of $\widetilde{M}$ such that $\tilde{x} \in N$ and the inclusion induces an isomorphism $\pi_1(N, \tilde{x}) \to \pi_1(\widetilde{M}, \tilde{x})$. Since $\partial N \neq \emptyset$, it follows that $\pi_1(N, \tilde{x}) \cong \langle \alpha, \beta \rangle$ is a free group, so it is an infinite cyclic group $\langle \gamma \rangle$. By Fact 3.1(2) $\gamma = \alpha^{\pm 1}$, so $\beta \in \langle \alpha \rangle$.

Suppose $\widetilde{M}$ is compact. Since $\operatorname{rank} H_1(\widetilde{M}) = 0$ or 1 and $\pi_1(\widetilde{M}) \neq 1$, it follows that $\widetilde{M} \cong \mathbb{P}^2$ or $\mathbb{K}^2$ and M is closed and nonorientable. If $\widetilde{M} \cong \mathbb{K}^2$ so $\chi(\widetilde{M}) = 0$, then $\chi(M) = 0$ and $M \cong \mathbb{K}^2$, a contradiction. Therefore, $\widetilde{M} \cong \mathbb{P}^2$ and $\chi(\widetilde{M}) = 1$, so $\chi(M) = 1$ and $M \cong \mathbb{P}^2$. We have $\pi_1(M) = \langle \alpha \rangle$. □

Note that if $M = \mathbb{K}^2$ and α, β are represented by the center circles of two Möbius bands, then $\alpha^2 = \beta^2$, but $\beta \notin \langle \alpha \rangle$.

Lemma 3.2. *Suppose C is a circle component of ∂M, $x \in C$ and $\alpha \in \pi_1(M, x)$ is represented by C. If $M \neq \mathbb{D}^2$, $\mathbb{M}$ or $\mathbb{S}^1 \times [0,1] \setminus A$ (A is a compact subset of $\mathbb{S}^1 \times \{1\}$), then there exists a $\gamma \in \pi_1(M, x)$ such that $\gamma \alpha^n \neq \alpha^n \gamma$ for any $n \in \mathbb{Z} \setminus \{0\}$.*

Proof. By the claim below we have a $\gamma \in \pi_1(M,x) \setminus \langle\alpha\rangle$. If $\gamma\alpha^n = \alpha^n\gamma$ for some $n \neq 0$, then by Fact 3.1(3ii) $\alpha^n, \gamma \in \langle\beta\rangle$ for some $\beta \in \pi_1(M,x)$ and $\alpha^n = \beta^k$ for some $k \in \mathbb{Z}$. Since $\alpha \neq 1$ and $M \neq \mathbb{P}^2$, by Fact 3.1(3i) $k \neq 0$. Hence by Lemma 3.1 $\beta \in \langle\alpha\rangle$ so $\gamma \in \langle\alpha\rangle$, a contradiction. □

Claim. *Suppose M is a connected 2-manifold, C is a circle component of ∂M, $x \in C$ and $\alpha \in \pi_1(M,x)$ is represented by C. If $\pi_1(M,x) = \langle\alpha\rangle$, then $M \cong \mathbb{D}^2$ or $\mathbb{S}^1 \times [0,1] \setminus A$ for some compact subset A of $\mathbb{S}^1 \times \{1\}$.*

Proof. First we note that M does not contain any handles or Möbius bands. In fact if H is a handle or a Möbius band in M, then we can easily construct a retraction $r : M \to H$ which maps C homeomorphically onto ∂H (Fact 3.2), and we have the contradiction $\pi_1(H) = \langle r_*\alpha\rangle$. In particular, if M is compact then M is a disk or an annulus.

Suppose M is noncompact. It follows that ∂M contains no circle components other than C. In fact if C' is a circle in $\partial M \setminus C$, then we can join C and C' by a proper arc A in M and by Fact 3.2 we have a retraction $M \to C \cup A \cup C'$, a contradiction. We can write $M = \bigcup_{i=1}^{\infty} N_i$, where N_i is a compact connected 2-submanifold of M, $C \subset \mathrm{Int}_M N_1$, $N_i \subset \mathrm{Int}_M N_{i+1}$ and each component of $\mathrm{cl}(M \setminus N_i)$ is noncompact. We will show that each $N = N_i$ is an annulus. This easily implies the conclusion.

Let $C_1, \ldots, C_m$ be the components of $\partial N \setminus C$. By the above remark $C_j \not\subset \partial M$, so C_j meets a component of $\mathrm{cl}(M \setminus N)$. Let N' be a submanifold of N obtained by removing an open color of each C_j from N. It follows that $N' \cong N$, $\mathrm{Fr}\, N'$ is the union of circles C_j' associated with C_j's, each C_j' is contained in some component L_j of $\mathrm{cl}(M \setminus N')$, $\mathrm{cl}(M \setminus N') = \bigcup_j L_j$, and each L_j is noncompact. Since M contains no handles or Möbius bands (so no one point union of two circles), it follows that $L_j \cap L_{j'} = \emptyset$ $(j \neq j')$ and $L_j \cap N' = C_j'$. By Fact 3.2 N' is a retract of M, so $\pi_1(N',x) = \langle\alpha\rangle$. This implies that $N \cong N'$ is an annulus. □

The next lemma is a key point in the proof of Theorem 3.1. In [2, Lemma 6.1] the condition "the loop $h_t(x)$ is null-homotopic in M" is achieved by rotating x along C. However, this process does not keep the condition "isotopic rel N".

Lemma 3.3. *Suppose C is a circle component of $\mathrm{Fr}\, N$ which does not bound a disk or a Möbius band in M, $h : M \to M$ is a homeomorphism with $h|_N = id_N$ and $h_t : M \to M$ $(0 \leqslant t \leqslant 1)$ is a homotopy with $h_0 = h$, $h_1 = id_M$. If the following conditions are satisfied, then for any $x \in C$ the loop $m = \{h_t(x)\colon 0 \leqslant t \leqslant 1\}$ is null-homotopic in M:*

(i) $M \neq \mathbb{T}^2, \mathbb{P}^2, \mathbb{K}^2$,
(ii) *each circle component of* $\mathrm{Fr}\, N$ *is essential in* M,
(iii) *each component of* $N \not\cong \mathbb{D}^2, \mathbb{M}, \mathbb{S}^1 \times [0,1] \setminus A$ ($A \subset \mathbb{S}^1 \times \{1\}$, *compact*).

Proof. Let $\alpha = \{\ell\} \in \pi_1(M,x)$ be represented by C and let $\beta = \{m\} \in \pi_1(M,x)$. The homotopy $h_t\ell$ implies that $\alpha\beta = \beta\alpha$. Since $M \not\cong \mathbb{T}^2, \mathbb{K}^2$, by Fact 3.1(3ii) $\langle\alpha,\beta\rangle \subset \langle\delta\rangle$ for some $\delta \in \pi_1(M,x)$. Since C does not bound a disk or a Möbius band, by Fact 3.1(2)

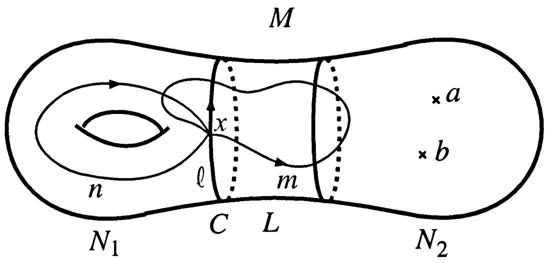


Fig. 2. The loops ℓ, m and n in Lemma 3.3.

$\delta = \alpha^{\pm 1}$ so $\beta = \alpha^k$ for some $k \in \mathbb{Z}$. Let $\alpha_1 = \{\ell\} \in \pi_1(N, x)$. By Lemma 3.2 there exists a $\gamma = \{n\} \in \pi_1(N, x)$ such that $\gamma\alpha_1^i \neq \alpha_1^i\gamma$ for any $i \in \mathbb{Z} \setminus \{0\}$ (Fig. 2). The homotopy $h_t n$ implies that $\gamma\beta = \beta\gamma$ in $\pi_1(M, x)$. Since $\pi_1(N, x) \to \pi_1(M, x)$ is monomorphic by Fact 3.1(4), $\gamma\alpha_1^k = \alpha_1^k\gamma$ in $\pi_1(N, x)$ so that $k = 0$ and $\beta = 1$ in $\pi_1(M, x)$. □

Lemma 3.4. *Suppose* $N \neq \emptyset$, $\mathrm{cl}(M \setminus N)$ *is compact, each component of* $\mathrm{Fr}\, N$ *is a circle,* $h : M \to M$ *is a homeomorphism such that* $h|_N = id_N$ *and* h *is homotopic to* id_M*. If the following conditions are satisfied, then* h *is isotopic to* id_M rel N:

(i) $M \neq \mathbb{T}^2, \mathbb{P}^2, \mathbb{K}^2$,

(ii) *each component* C *of* $\mathrm{Fr}\, N$ *does not bound a disk or a Möbius band,*

(iii) *each component of* $N \not\cong \mathbb{S}^1 \times [0, 1] \setminus A$ ($A \subset \mathbb{S}^1 \times \{1\}$, *compact*).

If we assume that h is isotopic to id_M, then the condition (iii) is weakened to the condition:

(iii)′ each component of $N \not\cong \mathbb{S}^1 \times [0, 1], \mathbb{S}^1 \times [0, 1)$.

Proof. Let $h_t : h \simeq id_M$ be any homotopy and let $L_1, \ldots, L_m$ be the components of $\mathrm{cl}(M \setminus N)$. By Lemma 3.3 the loop $h_t(x) \simeq *$ in M for any $x \in \mathrm{Fr}\, N = \bigcup_j \mathrm{Fr}\, L_j$. We must find an isotopy $h|_{L_j} \simeq id_{L_j}$ rel $\mathrm{Fr}\, L_j$.

Let $f : [0, 1] \to L_j$ be any path with $f(0), f(1) \in \mathrm{Fr}\, L_j$. The homotopy $h_t f$ yields a contraction of the loop $hf \cdot h_t(f(1)) \cdot f^{-1} \cdot (h_t(f(0)))^{-1}$ in M. Since $h_t(f(0))$, $h_t(f(1)) \simeq *$, it follows that $hf \cdot f^{-1} \simeq *$ in M. Since $\pi_1(L_j) \to \pi_1(M)$ is monomorphic by Fact 3.1(4), the loop $hf \cdot f^{-1} \simeq *$ in L_j, and the desired isotopy is obtained by Fact 3.1(5ii). □

Fig. 3 illustrates an original idea to prove Lemma 3.4 and Theorem 3.1: Consider the loop $m = n_1 f n_2 f^{-1}$ (f^{-1} is the inverse path of f). Any isotopy h_t: $id_M \simeq h\,\mathrm{rel}\,\{a, b\}$ induces a homotopy $h_t m$: $m \simeq n_1(hf)n_2(hf)^{-1}$ in $M \setminus \{a, b\}$. Modify the homotopy $h_t m$ to simplify the intersection of the image of $h_t m$ and $\mathrm{Fr}\, N$, and obtain a homotopy $F : \mathbb{S}^1 \times [0, 1] \to M \setminus \{a, b\}$ shown in Fig. 3. The homotopies $F|_{A_1}$ in N_1 and $F|_{A_2}$ in $N_2 \setminus \{a, b\}$ imply that $k_i = 0$ ($i = 1, 2$), and the homotopy $F|_{B_1}$ in L implies that $f \simeq hf$ rel end points in L as required.

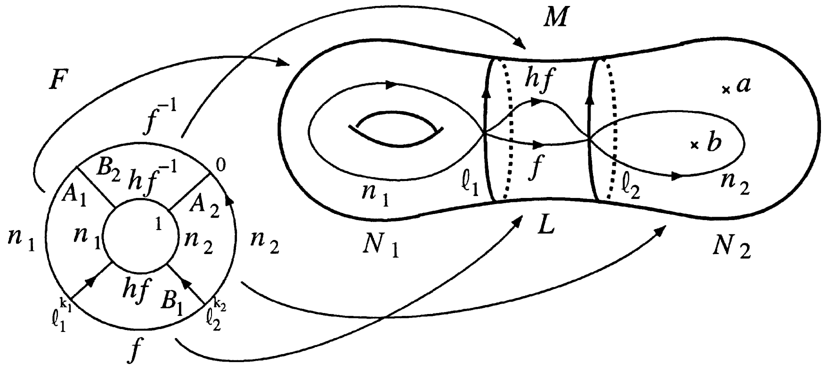


Fig. 3.

Proof of Theorem 3.1. We can assume that X is a finite set, since there exists a finite subset Y of X such that (M, N, Y) satisfies the conditions (i)–(iii) in Theorem 3.1. Replacing h_t by $h_1^{-1}h_t$, we may assume that $h_1 = id_M$.

(I) The case where M is compact: Let $N_1, \ldots, N_n$ be the components of N and $L = \mathrm{cl}(M \setminus N)$. Let $K_1, \ldots, K_p$ be the components of L which are disks or Möbius bands and let $L_1, \ldots, L_q$ be the remaining components. For each j we can write

$$\partial L_j = \left(\bigcup_{i=1}^{k(j)} A_i^j\right) \cup \left(\bigcup_{i=1}^{\ell(j)} B_i^j\right) \cup \left(\bigcup_{i=1}^{m(j)} C_i^j\right),$$

where A_i^j's are the circle components of $\mathrm{Fr}\, L_j$, B_i^j's are the components of ∂L_j which contain some arc components of $\mathrm{Fr}\, L_j$ and C_i^j's are the remaining components of ∂L_j. We choose disjoint collars E_i^j of A_i^j and F_i^j of B_i^j in L_j and set $\hat{A}_i^j = \partial E_i^j \setminus A_i^j$, $\widehat{B}_i^j = \partial F_i^j \setminus B_i^j$ and

$$L_j' = \mathrm{cl}\left(L_j \setminus \left(\bigcup_{i=1}^{k(j)} E_i^j\right) \cup \left(\bigcup_{i=1}^{\ell(j)} F_i^j\right)\right),$$

$$N' = N \cup \left(\bigcup_{k=1}^{p} K_k\right) \cup \left(\bigcup_{j=1}^{q} \left[\left(\bigcup_{i=1}^{k(j)} E_i^j\right) \cup \left(\bigcup_{i=1}^{\ell(j)} F_i^j\right)\right]\right).$$

Note that

$$\mathrm{Fr}\, N' = \bigcup_{j=1}^{q} \left[\left(\bigcup_{i=1}^{k(j)} \hat{A}_i^j\right) \cup \left(\bigcup_{i=1}^{\ell(j)} \widehat{B}_i^j\right)\right] \subset \mathrm{Int}\, M.$$

Since $\mathcal{H}_\partial(\mathbb{D}) \simeq \mathcal{H}_\partial(\mathbb{M}) \simeq *$ by Fact 3.1(5i), we can isotope h_0 rel N to an $h' \in \mathcal{H}_{N'}(M)$.

By the construction (M, N', X, h') satisfies the following conditions:

(1) N' is a 2-submanifold of M, every component of $\mathrm{Fr}\, N'$ is a circle and $X \subset \mathrm{Int}_M N'$.

(2) $h'|_{N'} = id_{N'}$ and h' is isotopic to id_M rel X.

(3) Suppose C is a component of $\operatorname{Fr} N'$. If C bounds a disk D then $\#(D \cap X) \geqslant 2$, and if C bounds a Möbius band E then $E \cap X \neq \emptyset$.

(4) If H is an annulus component of N' then $H \cap X \neq \emptyset$.

To see (3) first note that M is the union of compact 2-manifolds N_i's, E_i^j's, F_i^j's, K_k's and L'_j's, which have disjoint interiors. Suppose G is a compact connected 2-manifold in M with $\partial G \subset \operatorname{Fr} N'$. Since $G \subset \operatorname{Int} M$ and each F_i^j meets ∂M, it follows that G is the union of N_i's, E_i^j's, L'_j's and K_k's contained in G. Since E_i^j is an annulus and $L'_j \cong L_j$ is not a disk or a Möbius band, from Fact 3.3 it follows that (i) if G is a disk, then G contains a disk which is some N_i or K_k with $K_k \subset G \subset \operatorname{Int} M$, so $\#(G \cap X) \geqslant 2$, (ii) if G is a Möbius band, then G contains a disk or a Möbius band which is some N_i or K_k with $K_k \subset G \subset \operatorname{Int} M$, so $G \cap X \neq \emptyset$.

As for (4), H is the union of N_i's E_i^j's, F_i^j's and K_k's contained in H, and H contains at least one N_i, which is a disk with r holes. If $r \leqslant 1$ then by the assumption $N_i \cap X \neq \emptyset$. If $r \geqslant 2$ then we can find a disk D in $\operatorname{Int} H$ such that $D \cap N_i = \partial D \subset \partial N_i$ (Fact 3.3(iii)). Since $D \subset \operatorname{Int} N' \subset \operatorname{Int} M$, D is a union of N_i's and K_k's and we can conclude that it coincides with some $K_k (\subset \operatorname{Int} M)$, which meets X. These imply (4).

It remains to show that h' is isotopic to id_M rel N' under the conditions (1)–(4).

(i) When $X \subset \operatorname{Int} N'$, we can apply Lemma 3.4 to the triple $(M \setminus X, N' \setminus X, h'|_{M \setminus X})$. To verify the condition (iii) in Lemma 3.4, note that (a) each component of $N' \setminus X$ takes of the form $H \setminus X$ for some component of H of N', and, in particular, (b) if $H \setminus X \cong \mathbb{S}^1 \times [0,1)$, then H is a disk and $\#H \cap X \geqslant 2$, a contradiction. Therefore $h'|_{M \setminus X}$ is isotopic to $id_{M \setminus X}$ rel $N' \setminus X$. Extending this isotopy over M by id_X we have the required isotopy $h' \simeq id_M$ rel N'.

(ii) In the case where $X \not\subset \operatorname{Int} N'$, let $C = \partial N' \cap \partial M$ and consider $(\widetilde{M} = M \cup_C C \times [0,1],\ \widetilde{N} = N' \cup_C C \times [0,1], X, \tilde{h})$, where $\tilde{h}$ is the extension of h' by $id_{C \times [0,1]}$. Then (a) $X \subset \operatorname{Int} \widetilde{N}$ and $(\widetilde{M}, \widetilde{N}, X, \tilde{h})$ satisfies (1)–(4), and (b) an isotopy of $\tilde{h}$ to $id_{\widetilde{M}}$ rel $\widetilde{N}$ restricts to an isotopy of h' to id_M rel N'. (Alternatively, we can modify the isotopy of h' to id_M rel X to an isotopy rel $X \cup V$, where V is a neighborhood of $X \cap \partial M$ in M. We can replace X so that $X \subset \operatorname{Int} N'$.) This completes the proof of the case (I).

(II) The case where M is noncompact: Choose a compact connected 2-submanifolds L_0 and L of M such that $h_t(N) \subset \operatorname{Int}_M L_0$ $(0 \leqslant t \leqslant 1)$ and $L_0 \subset \operatorname{Int}_M L$. Let $N_1 = N \cup \operatorname{cl}(L \setminus L_0)$. Since N_1 is a subpolyhedron of L with respect to some triangulation of L (cf. [2]), by Corollary 2.1 we have the principal bundle: $\mathcal{H}(L)_0 \to \mathcal{E}(N_1, L)_0^*$. Let $f_t \in \mathcal{H}(L)_0$, $f_1 = id_L$, be any lift (= extension) of the path $e_t \in \mathcal{E}(N_1, L)_0^*$ defined by $e_t|_N = h_t|_N$ and $e_t = id$ on $\operatorname{cl}(L \setminus L_0)$.

We can apply the case (I) to (L, N_1, X_1, f_t), $X_1 = X \cup \operatorname{cl}(L \setminus L_0)$. For the condition (iii) in Theorem 3.1, when E is a component of $\operatorname{cl}(L \setminus N_1) = \operatorname{cl}(L_0 \setminus N)$, (a) if $E \cap \operatorname{Fr} L_0 = \emptyset$, then E is a component of $\operatorname{cl}(M \setminus N)$ and (b) if $E \cap \operatorname{Fr} L_0 \neq \emptyset$, then E contains a component of $\operatorname{Fr} L_0$ and $\operatorname{Fr} L_0 \subset \operatorname{cl}(L \setminus L_0) \subset X_1$ (it also follows that $\operatorname{Fr}_L E$ is not connected since $E \cap \operatorname{Fr} N \neq \emptyset$, so if E is a disk or a Möbius band, then $\operatorname{Fr}_L E$ is a disjoint union of arcs).

Therefore we have an isotopy $k_t : L \to L$ rel N_1 such that $k_0 = f_0$, $k_1 = id_L$. We can extend f_t and k_t to M by id. The required isotopy h'_t is defined by $h'_t = k_t f_t^{-1} h_t$. □

Proof of Corollary 3.1. Let $\mathcal{G}_1$ denote the unit path-component of a topological group $\mathcal{G}$. Theorem 3.1 implies $\mathcal{H}_N(M) \cap \mathcal{H}_X(M)_1 = \mathcal{H}_N(M)_1$. When M is compact, from Fact 2.1 it follows that $\mathcal{H}_K(M)_0 = \mathcal{H}_K(M)_1$ for any compact subpolyhedron K of M. Since X can be replaced by a finite subset Y of X as in the above proof, we have $\mathcal{H}_N(M) \cap \mathcal{H}_X(M)_0 \subset \mathcal{H}_N(M) \cap \mathcal{H}_Y(M)_0 = \mathcal{H}_N(M)_0$. The noncompact case follows from the same argument when we will show that $\mathcal{H}_K(M)_0$ is an ANR (Propositions 4.1, 4.2) in the next section.

4. The homotopy types of the identity components of homeomorphism groups of noncompact 2-manifolds

In this final section we will prove Theorem 1.1 and Corollary 1.1. Below we assume that M is a *noncompact* connected PL 2-manifold and X is a compact subpolyhedron of M. We set $M_0 = X$ and write as $M = \bigcup_{i=0}^{\infty} M_i$, where for each $i \geqslant 1$ (a) M_i is a nonempty compact connected PL 2-submanifold of M and $M_{i-1} \subset \mathrm{Int}_M M_i$, (b) for each component L of $\mathrm{cl}(M \setminus M_i)$, L is noncompact and $L \cap M_{i+1}$ is connected and (c) $M_1 \cap \partial M \neq \emptyset$ if $\partial M \neq \emptyset$. Taking a subsequence, we have the following cases:

(i) each M_i is a disk,
(ii) each M_i is an annulus,
(iii) each M_i is a Möbius band, and
(iv) each M_i is not a disk, an annulus or a Möbius band.

In (ii) the inclusion $M_i \subset M_{i+1}$ is essential, otherwise a boundary circle of M_i bounds a disk component in $\mathrm{cl}(M \setminus M_i)$, and it contradicts the condition (b).

Lemma 4.1. *In the cases* (i)–(iii) *it follows that*

(i) (a) $\partial M = \emptyset \Rightarrow M \cong \mathbb{R}^2$,
(b) $\partial M \neq \emptyset \Rightarrow M \cong \mathbb{D} \setminus A$, *where A is a nonempty* 0*-dimensional compact subset of* $\partial \mathbb{D}$;

(ii) (a) $\partial M = \emptyset \Rightarrow M \cong \mathbb{S}^1 \times \mathbb{R}^1$,
(b) $\partial M \neq \emptyset \Rightarrow$
(b)$_1$ $M \cong \mathbb{S}^1 \times [0, 1)$,
(b)$_2$ $M \cong \mathbb{S}^1 \times [0, 1) \setminus A$, *where A is a nonempty* 0*-dimensional compact subset of* $\mathbb{S}^1 \times \{0\}$,
(b)$_3$ $M \cong \mathbb{S}^1 \times [0, 1] \setminus A$, *where A is a nonempty* 0*-dimensional compact subset of* $\mathbb{S}^1 \times \{0, 1\}$;

(iii) (a) $\partial M = \emptyset \Rightarrow M \cong \mathbb{P}^2 \setminus 1\,\mathrm{pt}$,
(b) $\partial M \neq \emptyset \Rightarrow M \cong \mathbb{M} \setminus A$, *where A is a nonempty* 0*-dimensional compact subset of* $\partial \mathbb{M}$.

In the case (ii)(b)$_3$ we may further assume that M_1 meets both $\mathbb{S}^1 \times \{0\}$ and $\mathbb{S}^1 \times \{1\}$.

We choose a metric d on M with $d \leqslant 1$ and metrize $\mathcal{H}_X(M)$ by the metric ρ defined by

$$\rho(f, g) = \sum_{i=1}^{\infty} \frac{1}{2^i} \sup_{x \in M_i} d\big(f(x), g(x)\big).$$

We separate the following two cases:

(I) $(M,X)\cong(\mathbb{R}^2,\emptyset)$, $(\mathbb{R}^2,1\,\mathrm{pt})$, $(\mathbb{S}^1\times\mathbb{R}^1,\emptyset)$, $(\mathbb{S}^1\times[0,1),\emptyset)$, $(\mathbb{P}^2\setminus 1\,\mathrm{pt},\emptyset)$.

(II) (M,X) is not case (I).

Case (II): First we treat case (II) and prove the following statements:

Proposition 4.1. *In case* (II), *we have* (1) $\mathcal{H}_X(M)_0\simeq *$ *and* (2) $\mathcal{H}_X(M)_0$ *is an ANR.*

We use the following notation: For each $j\geqslant 1$ let $U_j=\mathrm{Int}_M M_j$ and $L_j=\mathrm{Fr}_M M_j$, and for each $j>i\geqslant k\geqslant 0$ let $\mathcal{H}_{k,j}=\mathcal{H}_{M_k\cup(M\setminus U_j)}(M)_0$, $\mathcal{U}^i_{k,j}=\mathcal{E}_{M_k}(M_i,U_j)^*_0$ and let $\pi^i_{k,j}:\mathcal{H}_{k,j}\to\mathcal{U}^i_{k,j}$ denote the restriction map, $\pi^i_{k,j}(h)=h|_{M_i}$.

Lemma 4.2.

(1) $\mathcal{H}_{k,j}\cong\mathcal{H}_{M_k\cup L_j}(M_j)_0$ *is an AR.*

(2) *The map* $\pi^i_{k,j}:\mathcal{H}_{k,j}\to\mathcal{U}^i_{k,j}$ *is a principal bundle with the structure group* $\mathcal{H}_{k,j}\cap\mathcal{H}_{M_i}(M)\cong\mathcal{G}^i_{k,j}\equiv\mathcal{H}_{M_k\cup L_j}(M_j)_0\cap\mathcal{H}_{M_i\cup L_j}(M_j)$ (*under the restriction map*).

(3) $\mathcal{U}^i_{k,j}$ *is an open subset of* $\mathcal{E}_{M_k}(M_i,M)^*_0$, $\mathrm{cl}\,\mathcal{U}^i_{k,j}\subset\mathcal{U}^i_{k,j+1}$ *and* $\mathcal{E}_{M_k}(M_i,M)^*_0=\bigcup_{j>i}\mathcal{U}^i_{k,j}$.

Proof. The statement (1) follows from Fact 2.1 and Lemma 2.1(ii), and (2) follows from Corollary 2.1. For (3), note that $\mathcal{E}_{M_k}(M_i,M)^*_0$ is path connected (Proposition 2.1) and each $f\in\mathcal{E}_{M_k}(M_i,M)^*_0$ is isotopic to the inclusion $M_i\subset M$ in a compact subset of M. $\square$

Lemma 4.3. *In case* (II), *for each* $j>i>k\geqslant 0$, (a) $\mathcal{G}^i_{k,j}$ *is an AR,* (b) *the restriction map* $\pi^i_{k,j}:\mathcal{H}_{k,j}\to\mathcal{U}^i_{k,j}$ *is a trivial bundle and* (c) $\mathcal{U}^i_{k,j}$ *is also an AR.*

Proof. Once we show that $(*)$ $\mathcal{G}^i_{k,j}=\mathcal{H}_{M_i\cup L_j}(M_j)_0$, then (a) the fiber $\mathcal{G}^i_{k,j}$ is an AR by Fact 2.1 and Lemma 2.1(ii), so (b) the principal bundle has a global section and it is trivial and (c) follows from Lemma 4.2(1). It remains to prove $(*)$.

(1) The cases (i)(a), (ii)(a), (iii)(a), (ii)(b)$_1$ and (iv) (under the condition (II)): We can apply Theorem 3.1 to $(\widetilde{M}_j=M_j\cup_{L_j}L_j\times[0,1]$, $\widetilde{M}_i=M_i\cup_{L_j}L_j\times[0,1]$, $\widetilde{M}_k=M_k\cup_{L_j}L_j\times[0,1])$. We can verify the conditions (ii) and (iii) in Theorem 3.1 as follows: (ii) By the assumption $(M_i,X)\not\cong(\mathbb{D},\emptyset)$, $(\mathbb{D},1\,\mathrm{pt})$, $(\mathbb{S}^1\times[0,1],\emptyset)$, $(\mathbb{M},\emptyset)$ for each $i\geqslant 1$. (iii) If H is a component of $\mathrm{cl}_{\widetilde{M}_j}(\widetilde{M}_j\setminus\widetilde{M}_i)=\mathrm{cl}_M(M_j\setminus M_i)$, then H contains a component of L_j. (Also, H meets both M_i and L_j (if $H\cap L_j=\emptyset$ then H is a compact component of $\mathrm{cl}(M\setminus M_i)$, a contradiction.), so $\mathrm{Fr}_{\widetilde{M}_j}H$ is not connected. Hence if H is a disk or a Möbius band, then $\mathrm{Fr}_{\widetilde{M}_j}H$ is a disjoint union of arcs.) By Corollary 3.1 (Compact case) it follows that $\mathcal{H}_{\widetilde{M}_k}(\widetilde{M}_j)_0\cap\mathcal{H}_{\widetilde{M}_i}(\widetilde{M}_j)=\mathcal{H}_{\widetilde{M}_i}(\widetilde{M}_j)_0$ and this implies $(*)$.

(2) The cases (i)(b), (ii)(b)$_3$ and (iii)(b): Since $M_1\cap\partial M\neq\emptyset$ and M_1 meets both $\mathbb{S}^1\times\{0\}$ and $\mathbb{S}^1\times\{1\}$ in the case (ii)(b)$_3$, it follows that $\mathrm{cl}(M_j\setminus M_i)$ is a disjoint union of disks, thus $\mathcal{H}_{M_i\cup L_j}(M_j)=\mathcal{H}_{M_i\cup L_j}(M_j)_0$ by Fact 3.1(5-i). This implies $(*)$.

(3) The remaining case (ii)(b)$_2$: It follows that (a) $\mathrm{cl}(M_j\setminus M_i)$ is a disjoint union of disks D_k and an annulus H and (b) $D_k\cap M_i$ is an arc, $D_k\cap L_j$ is a disjoint union of arcs

($\neq \emptyset$) and $H \cap M_i$, $H \cap L_j$ are the boundary circles of H. Since $N = M_i \cup (\bigcup_k D_k)$ is an annulus and $N \cap L_j \neq \emptyset$, from Theorem 3.1 it follows that $\mathcal{H}_{M_k \cup L_j}(M_j)_0 \cap \mathcal{H}_{N \cup L_j}(M_j) = \mathcal{H}_{N \cup L_j}(M_j)_0$. Each $f \in \mathcal{G}^i_{k,j}$ is isotopic rel $M_i \cup L_j$ to $f' \in \mathcal{H}_{N \cup L_j}(M_j)$. Since f' is isotopic to id rel $M_k \cup L_j$, it follows that $f' \in \mathcal{H}_{N \cup L_j}(M_j)_0$ and so $f \in \mathcal{H}_{M_i \cup L_j}(M_j)_0$. This completes the proof. □

Lemma 4.4. *In case* (II)*, for each* $i \geqslant k \geqslant 0$*,*

(a) $\mathcal{E}_{M_k}(M_i, M)^*_0$ *is an AR,*
(b) *the restriction map* $\pi : \mathcal{H}_{M_k}(M)_0 \to \mathcal{E}_{M_k}(M_i, M)^*_0$ *is a trivial principal bundle with fiber* $\mathcal{H}_{M_i}(M)_0$*,*
(c) $\mathcal{H}_{M_k}(M)_0$ *strongly deformation retracts onto* $\mathcal{H}_{M_i}(M)_0$.

Proof. By Lemma 4.3(c) each $\mathcal{U}^i_{k,j}$ $(j > i)$ is an AR. Thus by Fact 4.2(3) $\mathcal{E}_{M_k}(M_i, M)^*_0$ is also an AR and it strongly deformation retracts onto the single point set $\{M_i \subset M\}$. Hence the principal bundle

$$\mathcal{G}^i_k \equiv \mathcal{H}_{M_k}(M)_0 \cap \mathcal{H}_{M_i}(M) \subset \mathcal{H}_{M_k}(M)_0 \to \mathcal{E}_{M_k}(M_i, M)^*_0$$

is trivial and $\mathcal{H}_{M_k}(M)_0$ strongly deformation retracts onto the fiber $\mathcal{G}^i_k$. In particular, $\mathcal{G}^i_k$ is connected and $\mathcal{G}^i_k = \mathcal{H}_{M_i}(M)_0$. □

Proof of Proposition 4.1(1). By Lemma 4.4(c), for each $i \geqslant 0$ there exists a strong deformation retraction h^i_t $(0 \leqslant t \leqslant 1)$ of $\mathcal{H}_{M_i}(M)_0$ onto $\mathcal{H}_{M_{i+1}}(M)_0$. A strong deformation retraction h_t $(0 \leqslant t \leqslant \infty)$ of $\mathcal{H}_X(M)_0$ onto $\{id_M\}$ is defined as follows:

$$h_t(f) = h^i_{t-i} h^{i-1}_1 \cdots h^0_1(f) \quad (f \in \mathcal{H}_X(M)_0,\ i \geqslant 0,\ i \leqslant t \leqslant i+1)$$
$$h_\infty(f) = id_M.$$

Since diam $\mathcal{H}_{M_i}(M)_0 \leqslant 1/2^i \to 0$, the map $h : \mathcal{H}_X(M)_0 \times [0, \infty] \to \mathcal{H}_X(M)_0$ is continuous.

(In the cases (i), (ii) and (iii), the same conclusion follows from Lemma 2.1(i), (ii) by taking the end compactification of M.) □

For the proof of Proposition 4.1(2), we will apply Hanner's criterion of ANRs:

Fact 4.1 [6]. *A metric space* X *is an ANR iff for any* $\varepsilon > 0$ *there is an ANR* Y *and maps* $f : X \to Y$ *and* $g : Y \to X$ *such that* gf *is* ε*-homotopic to* id_X.

Proof of Proposition 4.1(2). By Lemma 4.4(b) and Proposition 4.1(1) for each $i \geqslant 1$ we have the trivial principal bundle

$$\mathcal{H}_{M_i}(M)_0 \subset \mathcal{H}_X(M)_0 \xrightarrow{\pi} \mathcal{E}_X(M_i, M)^*_0 \quad \text{with } \mathcal{H}_{M_i}(M)_0 \simeq *.$$

It follows that π admits a section s, and the map $s\pi$ is fiber preserving homotopic to $id_{\mathcal{H}_X(M)_0}$ over $\mathcal{E}_X(M_i, M)^*_0$. Since each fiber of π has diam $\leqslant 1/2^i$, this homotopy is a $1/2^i$-homotopy. Since $\mathcal{E}_X(M_i, M)^*_0$ is an ANR (Proposition 2.1), by Fact 4.1 $\mathcal{H}_X(M)_0$ is also an ANR. □

Case (I): The next statements follow from Lemma 2.1(iii) and Fact 2.1 by taking the end compactification of M.

Proposition 4.2. *In case* (I), *we have* (1) $\mathcal{H}_X(M)_0 \simeq \mathbb{S}^1$ *and* (2) $\mathcal{H}_X(M)_0$ *is an ANR.*

Theorem 1.1 follows from Propositions 4.1, 4.2, and Corollary 1.1 now follows from the following characterization of ℓ_2-manifold topological groups.

Fact 4.2 [1]. *A topological group is an ℓ_2-manifold iff it is a separable, non-locally compact, completely metrizable ANR.*

Proof of Corollary 1.1. Since M is locally compact and locally connected, $\mathcal{H}(M)$ is a topological group and $\mathcal{H}_X(M)$ is a closed subgroup of $\mathcal{H}(M)$. Since M is locally compact and second countable, $\mathcal{H}(M)$ is also second countable. A complete metric ρ on $\mathcal{H}(M)$ is defined by

$$\rho(f, g) = d_\infty(f, g) + d_\infty\bigl(f^{-1}, g^{-1}\bigr),$$
$$d_\infty(f, g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \sup_{x \in M_n} d\bigl(f(x), g(x)\bigr)$$

for $f, g \in \mathcal{H}(M)$, where d is a complete metric on M with $d \leqslant 1$. Since $\mathcal{H}_X(M)_0 \cong \mathcal{H}_X(M)_0 \times s$ [3], $\mathcal{H}_X(M)_0$ is not locally compact. Finally, by Propositions 4.1, 4.2 $\mathcal{H}_X(M)_0$ is an ANR. This completes the proof. □

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