The Effect of Delays on the Permanence for Lotka-Volterra Systems

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Abstract—In this note, two examples are given to show that delays can make two-species Lotka-Volterra cooperative systems possessing an unbounded solution. This result indicates that unlike the two-species prey-predator or competitive systems, delays in cooperative ones are not harmless in the sense of permanence. These examples also give a negative answer to a recent conjecture for general n-species Lotka-Volterra delay systems proposed in [1, p. 310].

Keywords—Lotka-Volterra systems, Discrete delay, Unbounded solutions.

Stability of Lotka-Volterra delay systems has been studied by a lot of authors. Most of the papers consider the situation at which undelayed intraspecific competitions are present. In these cases, either a Liapunov-Razumikhin functional is used or comparison theorems can be applied to obtain global attractivity of a positive equilibrium point. Essentially, the point is a global attractor if the undelayed intraspecific competition dominates over the delayed intra- (and inter-) specific competition. If the system has no undelayed intraspecific competitions, in general case, the global attractivity of a positive equilibrium point, or even the weaker property of stability-permanence of the system, is not easy to investigate. In this note, we will consider the latter case, i.e., the following two-species Lotka-Volterra delay system

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t) [r_1 - a_{11} x_1(t - \tau_{11}) + a_{12} x_2(t - \tau_{12})], \\
\dot{x}_2(t) &= x_2(t) [r_2 + a_{21} x_1(t - \tau_{21}) - a_{22} x_2(t - \tau_{22})],
\end{align*}
\]

with initial conditions

\[
x_i(t) = \phi_i(t) \geq 0, \quad t \in [-\tau_0, 0]; \quad \phi_i(0) > 0, \quad i = 1, 2,
\]

where \(x_i\) represents the density of species \(i\), and \(r_i\) the reproduction rate, \(\tau_{ij} \geq 0 (i, j = 1, 2)\) the constant time lag, and \(\tau_0 = \max \{\tau_{ij}\}\). \(a_{ij} (i, j = 1, 2)\) is nonzero constant with \(a_{ii} > 0 (i = 1, 2)\) and \(\phi_i(t) (i = 1, 2)\) continuous on \([-\tau_0, 0]\).

If all the delays \(\tau_{ij}\) are zero, then system (1) will simplify to an autonomous system of the form

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t) [r_1 - a_{11} x_1(t) + a_{12} x_2(t)], \\
\dot{x}_2(t) &= x_2(t) [r_2 + a_{21} x_1(t) - a_{22} x_2(t)].
\end{align*}
\]

In the sequel, system (3) is supposed to have a positive equilibrium point \(x^* = (x_1^*, x_2^*)\) which is also a positive one of system (1). It is well known [2] that system (3) is globally stable (we say
the system to be globally stable if \(x^*\) is globally stable in the system) if and only if the species interaction matrix \(A = (a_{ij})_{2 \times 2}\) satisfies the following condition (C):

\[
\text{Condition (C): } a_{11}a_{22} - a_{12}a_{21} > 0.
\]

**DEFINITION 1.** System (1) is permanent if there is a compact region \(K\) in the interior of \(\mathbb{R}^2_+ = \{x|x_i \geq 0; i = 1, 2\}\) such that all the solutions \(x(t) = (x_1(t), x_2(t))\) of system (1) with initial conditions (2) ultimately enter \(K\).

Recently, the permanence and global attractivity of the positive equilibrium point \(x^*\) of system (1) with \(\tau_{11} + \tau_{22} = 0\) are discussed under the Condition (C) or some stronger ones in [1,3–7].

**RESULT 1.** [6]. In the prey-predator case, i.e., \(a_{12}a_{21} < 0\), Condition (C) implies the permanence of system (1).

**REMARK 1.** In fact, [6] considered a prey-predator system with arbitrarily finite number of delays which includes system (1) as a special case.

**RESULT 2.** [5,7]. In the competitive case, i.e., \(a_{12} < 0\) and \(a_{21} < 0\), Condition (C) implies the permanence of system (1).

**REMARK 2.** The system considered in [7] also has arbitrarily finite number of delays.

**REMARK 3.** In both the above two cases, the delays are harmless for the permanence of system (1) which means that delayed system (1) remains to be permanent for any delays \(\tau_{ij}\). Note that for system (3), the permanence is equivalent to the global stability.

**RESULT 3.** [5]. In the competitive case as in Result 2, if, furthermore, \(\max\{\tau_{11}, \tau_{22}\}\) is small enough, then Condition (C) implies that positive equilibrium \(x^* = (x_1^*, x_2^*)\) is a global attractor for system (1).

If \(\tau_{11} = \tau_{22} = 0\), we know the following result.

**RESULT 4.** [8]. In the cooperative case, i.e., \(a_{12} > 0\) and \(a_{21} > 0\), Condition (C) guarantees the global attractivity of the positive equilibrium \(x^* = (x_1^*, x_2^*)\).

A natural problem from these known results is whether the Condition (C) ensures the permanence of system (1) in general. In fact, this problem is included in a recent conjecture which was proposed by Kuang in [1, p. 310] as follows.

**ASSUMPTION 1.** Assume that \(\mu_{ij}\) are nondecreasing and \(\mu_{ij}(0^+) - \mu_{ij}(-\tau^-) = 1\). Then

\[
\dot{x}_i(t) = x_i(t) \left[ r_i - \sum_{j=1}^{n} a_{ij} \int_{-\tau}^{0} x_j(t + \theta) d\mu_{ij}(\theta) \right]
\]

is permanent if and only if

\[
\dot{x}_i(t) = x_i(t) \left[ r_i - \sum_{j=1}^{n} a_{ij} x_j(t) \right]
\]

is permanent.
Unfortunately, the following two examples will show that the answer to our problem is negative. This implies that Kuang’s conjecture is not true.

**Example 1.**

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t) \left[1 - x_1(t-2) + e^{-1} x_2(t-1)\right], \\
\dot{x}_2(t) &= x_2(t) \left[1 + e^{-1} x_1(t-1) - x_2(t-2)\right].
\end{align*}
\]  

System (4) with initial condition (2) has an unbounded solution \(x(t) = (x_1(t), x_2(t)) = (e^t, e^t)\) if \(\phi_i(t) = e^t\). In this case, both diagonal delays are larger than off-diagonal ones. The following example shows that even if one of the diagonal delays is smaller than the off-diagonal delays, the system can also have an unbounded solution.

**Example 2.**

\[
\begin{align*}
\dot{x}_1(t) &= x_1(t) [1 - x_1(t-1) + e x_2(t-2)], \\
\dot{x}_2(t) &= x_2(t) \left[1 + e^{-3} x_1(t-2) - x_2(t-5)\right].
\end{align*}
\]  

Clearly, the unbounded solution given in Example 1 is also an unbounded one for system (5).

**Concluding Remark.** From the known results (Results 1, 2, 3 and 4) and the proposed examples (Examples 1 and 2), it seems that the delay effect on the permanence is more difficult to analyze for the cooperative systems than for the competitive or prey-predator ones. For two-species cooperative ones, it seems that the relationship of the magnitude of the diagonal delays and that of the off-diagonal ones will determine the permanence and global attractivity of the delay system (1).

**References**