On Finite Abelian-by-Nilpotent Groups

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A class $\mathcal{H}$ of finite groups is called $n$-recognizable if a finite group all of whose $n$-generator subgroups belong to $\mathcal{H}$ also belongs to $\mathcal{H}$. For example, the classes $\mathcal{A}$ of finite abelian groups and $\mathcal{N}$ of finite nilpotent groups are 2-recognizable. In [1] it has been proved that if $\mathcal{H}$ is a $\{Q, S, D\}$-closed class of finite soluble groups which is $n$-recognizable for some $n \geq 2$ then the product $\mathcal{N} \mathcal{H}$ again is $n$-recognizable and, trivially, the class $\mathcal{A} \mathcal{H}$ is $(2n)$-recognizable. The question arises whether one can improve the latter result. However, in [5] it has been shown that the class $\mathcal{A} \mathcal{M}$ of all finite metabelian groups is 4-recognizable but not 3-recognizable.

In this note we shall present a sequence $w_1, w_2, \ldots$ of laws in three variables such that the finite group $G$ belongs to $\mathcal{A} \mathcal{N}$ if and only if $w_k(G) = 1$ for almost all $k$. In particular, $\mathcal{A} \mathcal{N}$ is 3-recognizable and we shall give an example showing that this result is best possible. In addition, it follows that every finite 3-metabelian group is abelian-by-nilpotent. This may be compared with a result of I. D. Macdonald [4] which states that the aforementioned groups are centre-by-metabelian.

The notation we shall use is fairly standard. In particular, for a word $w$, we denote by $w(G)$ the verbal subgroup of the group $G$ with respect to $w$ (see [6]). All groups considered in this paper are finite. Moreover, $\gamma_\infty(G)$ denotes the last term of the descending central series of $G$.

1. The Minimal Counterexamples

In this section we collect some information on groups which do not belong to a class of groups but all of whose proper subgroups and/or quotients do. The first result is well known.

**Lemma 1** [3, pp. 281ff.]. Let $Q$ be nonnilpotent and suppose that all proper subgroups of $Q$ are nilpotent. Then $Q$ is a semidirect product of a
normal Sylow $p$-subgroup $A$ of $Q$ with a cyclic $q$-subgroup $B = \langle b \rangle$ acting irreducibly on $A/\Phi(A)$. Moreover, $A$ is a special $p$-group.

The following reduction lemma presents some results on minimal counterexamples for the class $\mathcal{A}/\mathcal{N}$. The second part has appeared in [2], but for the convenience of the reader we include an independent proof.

**Lemma 2.** Let $G$ be a soluble group all of whose proper subgroups and quotients belong to $\mathcal{A}/\mathcal{N}$ but $G \notin \mathcal{A}/\mathcal{N}$.

(a) If $G$ is not metanilpotent then $G$ is a semidirect product of its unique minimal normal subgroup $N = F(G)$ with a complement $Q$ which is not nilpotent, but all of whose proper subgroups are. The normal Sylow subgroup of $Q$ is elementary abelian or extraspecial.

(b) [2] If $G$ is metanilpotent then $G$ is a semidirect product of $F = F(G)$, which is a $p$-group, with a cyclic $p'$-group $Q = \langle a \rangle$. Furthermore, $F'$ is the unique minimal normal subgroup of $G$ and $F' \leq Z(F)$.

**Proof.** (a) This follows immediately from [1, Satz 2.9].

(b) By minimality, the Fitting subgroup $F$ of $G$ is a $p$-group. As $G$ is metanilpotent, we have $G = FQ$, where $Q$ is a nilpotent $p'$-group. Let $N = \gamma_\infty(G)$.

Assume that $Q$ contains at least three maximal subgroups $M_1, M_2, M_3$. Let $A_i = \gamma_\infty(FM_i)$, so $A_i$ is an abelian normal subgroup of $G$. As $G = FM_iFM_j$, we have $N = A_iA_j$ and so $A_i \cap A_j \leq Z(N)$ for all $i \neq j$.

We now prove $N = (A_1 \cap A_2)A_1$, which in turn implies that $N$ is abelian contradicting the hypothesis on $G$. Now $N/A_i = C_{N/A_i}(Q) \times [N/A_i, Q] =: C_{i}/A_i \times B_i/A_i$. As $G/B_i$ is nilpotent and $N = \gamma_\infty(G)$, we have $N/A_i = [N/A_i, Q]$ and so $Q$ acts fixed point freely on $N/A_i$.

Let $H/K$ be a $Q$-chief-factor of $N/A_i$. As $FM_i/A_i$ is nilpotent, we have $1 \neq H/K \cap Z(FM_i/K)$ and so $M_i \leq C_0(H/K)$. By the above, we have $M_i = C_0(H/K)$. Hence for $i \neq j$ the $Q$-chief-factors of $N/A_i$ and $N/A_j$ are pairwise nonisomorphic and our claim follows from the theorem of Jordan-Hölder.

So $Q$ has at most two maximal subgroups and therefore it is cyclic. The remaining statements are obvious.

The following result on minimal simple groups is well known.

**Lemma 3.** Let $G = \operatorname{PSL}(3, 3)$ or $G = Sz(q)$. Then $G$ has a soluble subgroup which is not in $\mathcal{A}/\mathcal{N}$.
2. The Main Result

In order to state the main result of this note, we first introduce the laws we shall deal with.

**Definition.** Let $x, y, z$ be variables and let $k$ be some positive integer. Then $b_k(x, y, z) = [[x, y z], [y, x z]]$ and $c_k(x, y, z) = [[x, y], [x, z]]$.

It is obvious that each group in $\mathcal{N}$ satisfies almost all laws $b_k = 1$ and $c_k = 1$. The converse of this statement is also true and this is the content of

**Theorem.** Let $G$ be a finite group. Then the following are equivalent:

(i) $G$ belongs to $\mathcal{N}$,

(ii) $G$ satisfies almost all laws $b_k = 1$ and $c_k = 1$.

The proof that (ii) implies (i) involves studying a counterexample $G$ of least possible order. By minimality, every proper subgroup and factor group of $G$ belongs to $\mathcal{N}$. In the first part of the proof we assume that $G$ is soluble, so Lemma 2 is applicable.

The following result may be of independent interest.

**Lemma 4.** Let $H$ be a group and let $a$ be a fixed point free automorphism of $H$. Then there exists a positive integer $k$ such that $[x, ka] = x$ for all $x \in H$.

**Proof.** Let $x \in H$. As $H$ is finite, there exist positive integers $r, s$ with $r < s$ and $[x, a] = [x, a]$. If for $y, z \in H$ we have $[y, a] = [z, a]$ then, by $C_n(a) = 1$, we get $y = z$. So $x = [x, a]$ for $t = s - r$. The result follows by taking $k$ to be the least common multiple of all such $t$.

**Lemma 5.** Let $Q$ be as in Lemma 1 and assume that $A$ is nonabelian special. Then there exists $a \in A \setminus Z(A)$ and some positive integer $k$ with $a = [a, kb]$.

**Proof.** Let $\bar{Q} = Q/Z(A)$ and let $a_0 \in A \setminus Z(A)$. By Lemma 4 there is some $k$ with $\bar{a}_0 = [\bar{a}_0, kb]$, so $[a_0, kb] = a_0 z$ for some $z \in Z(A)$. Let $a = a_0 z$. Then, as $[Z(A), B] = 1$, we have $[a, kb] = [a_0, kb] = a_0 z = a$ as required.

We are now in a position to deal with the first type of minimal counterexample.

**Lemma 6.** Let $G$ be as in Lemma 2(a). Then $b_k(G) \cdot c_k(G) \neq 1$ for infinitely many $k$.

**Proof.** By Lemma 4 and Lemma 5 there exists an element $1 \neq a \in A$ with
\[ a = [a, b] \] for some positive integer \( k \). We divide the proof into two parts according to whether or not \( b \) acts fixed point freely on \( N \).

First, assume that \( C_N(b) = 1 \). As clearly \( A \) acts faithfully on \( N \), there exists \( 1 \neq n \in N \) with \( [n, a] \neq 1 \). By Lemma 4 there is some \( k \) with \( [n, k^2] = n \). So for \( k = k_1 k_2 \) we have \( [a, k^2 b] = a \) and \( [n, k^2 b] = n \). Finally, we arrive at \( b_k(n, a, b) = ([n, k^2 b], [a, k] b] = [n, a] \neq 1 \) proving our assertion in this case as there obviously exist infinitely many such \( k \).

Now let \( C_N(b) \neq 1 \) and select \( 1 \neq n \in C_N(b) \). Then \( [n a, k, b] = [a, k, b] = a \).

This gives

\[
\begin{align*}
c_{k_1}(n a, a, b) &= [[n a, k_1 a], [n a, k_1 b]] \\
&= [[n, k_1 a]^a, a] \\
&= [n, k_1, a]^a.
\end{align*}
\]

Now \( c_{k_1}(G) = 1 \) would imply \( [n, k_1, a] = 1 \) and, as the orders of \( N \) and \( A \) are coprime, this gives \( [n, a] = 1 \), contradicting the choice of \( n \).

The second type of a minimal counterexample needs some more preparatory remarks.

**Lemma 7.** Let \( G \) be as in Lemma 2(b) and let \( N = \gamma_r(G) \). Then \( a \) acts fixed point freely on \( N/N' \).

**Proof.** This follows from an argument similar to that used in the proof of Lemma 2(b).

Now we rule out the second type of minimal counterexample.

**Lemma 8.** Let \( G \) be as in Lemma 2(b). Then \( b_k(G) \neq 1 \) for infinitely many \( k \).

**Proof.** As \( N = \gamma_r(G) \) is nonabelian, we can select \( n_1, n_2 \in N \) with \( [n_1, n_2] \neq 1 \). By Lemma 7 and Lemma 4 there exists a positive integer \( u \) with \( [n_i, a] \equiv n_i \pmod{N'} \), so \( [n_i, a] = n_i w_i \) for some suitable \( w_i \in N' \) \((i = 1, 2)\). Then

\[
\begin{align*}
b_u(n_1, n_2, a) &= [[n_1, a], [n_2, a]] \\
&= [n_1 w_1, n_2 w_2] \\
&= [n_1, n_2] \neq 1, \quad \text{as} \quad N' \leq Z(N).
\end{align*}
\]

In proving the Theorem it suffices by Lemma 6 and Lemma 8 to show that a finite group satisfying almost all laws \( b_k = 1 \) and \( c_k = 1 \) is soluble. A minimal counterexample to this assertion clearly is a minimal simple group (see [7]).
LEMMA 9. Let \( q = p^k \geq 4 \) be a prime power. Then \( b_k(\text{PSL}(2, q)) \neq 1 \) for all \( k \).

Proof. By assumption there exists \( r \in \text{GF}(q) \) with \( r^2 \neq 1 \). The following can be easily verified:

\[
\begin{bmatrix}
(1, 1), (r^{-1}, 0)
\end{bmatrix}
= \begin{bmatrix}
1, \quad (1 - r^2)^k
\end{bmatrix}
\]
\[
\begin{bmatrix}
(1, 0), (r^{-1}, 0)
\end{bmatrix}
= \begin{bmatrix}
1, \quad 0
\end{bmatrix}
\]
\[
\begin{bmatrix}
(1, 1), (0, r)
\end{bmatrix}
= \begin{bmatrix}
(r^{-2} - 1)^k, \quad 1
\end{bmatrix}
\]

This implies

\[
b_k \left( \begin{bmatrix}
(1, 1), (1, 0)
\end{bmatrix}, \begin{bmatrix}
1, \quad 0
\end{bmatrix}, \begin{bmatrix}
(r^{-1}, 0)
\end{bmatrix} \right) \neq \pm 1.
\]

The proof of the Theorem now is complete as by Lemma 3 the remaining minimal simple groups are no minimal counterexamples.

COROLLARY. Let \( G \) be a finite group all of whose 3-generator subgroups belong to \( \mathcal{N} \). Then \( G \in \mathcal{N} \).

Proof. This follows from the Theorem as in the sequences of words there are only three variables.

Another Corollary provides some information on 3-metabelian groups.

COROLLARY. Let \( G \) be a (finite) 3-metabelian group. Then \( \gamma_\infty(G) \) is abelian.

3. AN EXAMPLE

In view of the Corollaries stated above the question arises whether \( \mathcal{N} \) is 2-recognizable. This, however, is not the case as the following shows.

EXAMPLE. Let \( p \) be an odd prime and let \( N = \langle x, y \rangle \) be the nonabelian group of order \( p^3 \) and exponent \( p \). One easily verifies that there exists an automorphism \( z \) of \( N \) inverting the two generators \( x, y \). Let \( G \) be the split extension of \( N \) with \( \langle z \rangle \).

Then \( \gamma_\infty(G) = N \) is not abelian, so \( G \notin \mathcal{N} \). But every proper subgroup or quotient has order dividing \( 2p^2 \) or \( p^3 \) and so lies in \( \mathcal{N} \). Also the group \( G \) cannot be generated by two elements as can be seen by considering \( G/\Phi(N) \). This shows that \( \mathcal{N} \) is not 2-recognizable and so our result is best possible.


