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Orthogonal Homogeneous Polynomials

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An addition formula, Pythagorean identity, and generating function are obtained for orthogonal homogeneous polynomials of several real variables. Application is made to the study of series of such polynomials. Results include an analog of the Funk-Hecke theorem. © 1999 Academic Press

1. POLYNOMIAL IDENTITIES

Let $\mathscr{H}_n^k$ denote the vector space of real homogeneous polynomials of degree k in n real variables. Suppose $x = (x_1, x_2, \dots, x_n)$ and $\hat{x} = (x_1, x_2, \dots, x_{n-1})$. Then every polynomial $p(x) \in \mathscr{H}_n^k$ has a unique representation of the form

$$p(x) = \sum_{j=0}^{k} x_n^j p_{k-j}(\hat{x}),$$

where $p_{k-j}(\hat{x}) \in \mathscr{H}_{n-1}^{k-j}$. Thus

$$\delta_n^k = \sum_{j=0}^{k} \delta_{n-1}^j, \tag{1}$$

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where $\delta_n^k = \dim \mathscr{H}_n^k$. In [8, p. 139] it is shown that

$$\delta_n^k = \binom{n+k-1}{k} = \frac{(n+k-1)!}{(n-1)!\,k!}.$$

If $p \in \mathscr{H}_n^k$ then

$$p(x) = \sum_{|\alpha|=k} c_\alpha x^\alpha,$$

where α denotes the n-tuple of non-negative integers $(\alpha_1, \alpha_2, \dots, \alpha_n)$,

$$|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n,$$

and

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}.$$

On the vector space $\mathscr{H}$ of real polynomials in n variables, define the inner product

$$(p, q) = p\left(\frac{\partial}{\partial x}\right) q(x)\big|_{x=0}, \tag{2}$$

where

$$\frac{\partial}{\partial x} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\right),$$

and

$$p\left(\frac{\partial}{\partial x}\right) = \sum_{|\alpha|=k} c_\alpha \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}.$$

In [2] it was shown that if $p \in \mathscr{H}_n^k$, then

$$\left((x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^k, p(y)\right) = k!\,p(x). \tag{3}$$

This identity was then used in [2] to obtain the following addition formula for orthogonal homogeneous polynomials:

If $\{p_j\}_{j=1}^{\delta_n^k}$ is an orthonormal basis for $\mathscr{H}_n^k$, then

$$(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^k = k! \sum_{j=1}^{\delta_n^k} p_j(x) p_j(y). \tag{4}$$

Our first result is a generalization of this addition formula.

THEOREM 1. *Suppose* $\{p_j\}_{j=1}^{\delta_n^k} \subset \mathscr{H}_n^k$ *and* $\{q_j\}_{j=1}^{\delta_n^k} \subset \mathscr{H}_n^k$ *are normalized biorthogonal systems with respect to the inner product* (2), *and let* $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$, $y = (y_1, y_2, \dots, y_n) \in \mathbb{R}^n$. *Then*

$$(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^k = k! \sum_{j=1}^{\delta_n^k} p_j(x) q_j(y). \tag{5}$$

Proof. Since $\{p_j\}_{j=1}^{\delta_n^k}$ and $\{q_j\}_{j=1}^{\delta_n^k}$ are normalized biorthogonal systems, each are linearly independent sets [8, Section 86]. In particular, $\{p_j\}_{j=1}^{\delta_n^k}$ forms a basis for $\mathscr{H}_n^k$. Thus, there exist coefficients $c_j(y)$ such that

$$(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^k = \sum_{j=1}^{\delta_n^k} c_j(y) p_j(x).$$

Appealing to the biorthogonality, we then have

$$\left((x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^k, q_j(x)\right) = c_j(y).$$

Further, since q_j is a homogeneous polynomial of degree k, it follows from (3) that

$$\left((x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^k, q_j(x)\right) = k!\, q_j(y),$$

which completes the proof.

As a corollary, we have a generalization of the Pythagorean identity obtained in [2]:

COROLLARY. *If* $\{p_j\}_{j=1}^{\delta_n^k} \subset \mathscr{H}_n^k$ *and* $\{q_j\}_{j=1}^{\delta_n^k} \subset \mathscr{H}_n^k$ *are bi-orthonormal, then*

$$\sum_{j=1}^{\delta_n^k} p_j(s) q_j(s) = \frac{1}{k!} \tag{6}$$

for all $s = (s_1, s_2, \dots, s_n)$ *on the unit sphere* $s_1^2 + s_2^2 + \cdots + s_n^2 = 1$.

We next obtain the converse of Theorem 1.

THEOREM 2. *Suppose* $\{p_j\}_{j=1}^{\delta_n^k} \subset \mathscr{H}_n^k$ *is a linearly independent set of polynomials, and*

$$(x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^k = k! \sum_{j=1}^{\delta_n^k} p_j(x) q_j(y).$$

Then $\{q_j\}_{j=1}^{\delta_n^k} \subset \mathscr{H}_n^k$ *and* $\{p_j\}_{j=1}^{\delta_n^k}, \{q_j\}_{j=1}^{\delta_n^k}$ *are sets of normalized biorthogonal polynomials.*

Proof. For $\lambda \in \mathbb{R}$,

$$(x_1 \lambda y_1 + x_2 \lambda y_2 + \cdots + x_n \lambda y_n)^k = k! \sum_{j=1}^{\delta_n^k} p_j(x) q_j(\lambda y)$$

$$= k! \sum_{j=1}^{\delta_n^k} p_j(\lambda x) q_j(y)$$

$$= k! \sum_{j=1}^{\delta_n^k} \lambda^k p_j(x) q_j(y).$$

Thus

$$0 = \sum_{j=1}^{\delta_n^k} p_j(x) q_j(\lambda y) - \sum_{j=1}^{\delta_n^k} p_j(\lambda x) q_j(y)$$

$$= \sum_{j=1}^{\delta_n^k} p_j(x) \left[q_j(\lambda y) - \lambda^k q_j(y) \right].$$

The linear independence of the polynomials $\{p_j\}_{j=1}^{\delta_n^k}$ then implies

$$q_j(\lambda x) = \lambda^k q_j(x).$$

Thus the polynomials $\{q_j\}_{j=1}^{\delta_n^k}$ are homogeneous of degree k. Further,

$$\left((x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^k, p_l(y) \right) = k! \sum_{j=1}^{\delta_n^k} \left(q_j(y), p_l(y) \right) p_j(x).$$

But by (3),

$$\left((x_1 y_1 + x_2 y_2 + \cdots + x_n y_n)^k, p_l(y) \right) = k! p_l(x).$$

Thus

$$p_l(x) = \sum_{j=1}^{\delta_n^k} (q_j, p_l) p_j(x).$$

Appealing to the linear independence of the polynomials $\{p_j\}_{j=1}^{\delta_n^k}$, we then have

$$(q_j, p_l) = \begin{cases} 0 & \text{if } j \neq l \\ 1 & \text{if } j = l \end{cases}$$

which completes the proof.

As a corollary, we obtain the converse of the addition formula (4).

COROLLARY. *Suppose* $\{p_j\}_{j=1}^{\delta_n^k} \subset \mathscr{H}_n^k$ *is linearly independent. If*

$$(x_1y_1 + x_2y_2 + \cdots + x_ny_n)^k = k! \sum_{j=1}^{\delta_n^k} p_j(x)p_j(y),$$

then the polynomials $\{p_j\}_{j=1}^{\delta_n^k}$ *are orthonormal.*

We next generalize an identity that was central to the proofs of the major theorems of [7] and [10].

THEOREM 3. *Suppose* $(x_1, x_2, \ldots, x_n)$, $(y_1, y_2, \ldots, y_n)$ *and* $(t_1, t_2, \ldots, t_n) \in \mathbb{R}^n$. *Then*

$$\left((x_1t_1 + x_2t_2 + \cdots + x_nt_n)^k, (y_1t_1 + y_2t_2 + \cdots + y_nt_n)^k\right)$$
$$= k!(x_1y_1 + x_2y_2 + \cdots x_ny_n)^k. \tag{7}$$

Note that the inner product in the above expression is with respect to the variable $t = (t_1, t_2, \ldots, t_n)$.

Proof. Let

$$f(y,t) = (y_1t_1 + y_2t_2 + \cdots + y_nt_n)^k.$$

Then $f(y,t)$ is a homogeneous polynomial of degree k in t. Thus by (3),

$$\left((x_1t_1 + x_2t_2 + \cdots x_nt_n)^k, f(y,t)\right) = k!f(y,x)$$
$$= k!(x_1y_1 + x_2y_2 + \cdots + x_ny_n)^k.$$

Our next identity is an analog of the extended addition formula for spherical harmonics obtained in [5].

THEOREM 4. *Suppose* $\{p_j\}_{j=1}^{\delta_n^k} \subset \bigcup_{j=0}^{k} \mathscr{H}_{n-1}^k$ *is an orthonormal set. Then*

$$(1 + x_1y_1 + x_2y_2 + \cdots + x_{n-1}y_{n-1})^k = \sum_{j=1}^{\delta_n^k} \frac{k!}{(k-l_j)!} p_j(\hat{x})p_j(\hat{y}),$$

where $l_j = \deg p_j$.

Proof. Write

$$(1 + x_1 y_1 + \cdots + x_{n-1} y_{n-1})^k = \sum_{j=0}^{k} \frac{k!}{j!(k-j)!} (x_1 y_1 + \cdots + x_{n-1} y_{n-1})^j.$$

Then by the homogeneity of p_j,

$$\begin{aligned} &\left(p_j(\hat{x}), (1 + x_1 y_1 + \cdots + x_{n-1} y_{n-1})^k \right) \\ &\quad = \frac{k!}{l_j!(k-l_j)!} \left(p_j, (x_1 y_1 + \cdots + x_{n-1} y_{n-1})^{l_j} \right. \\ &\quad = \frac{k!}{l_j!(k-l_j)!} \left(l_j! p_j(\hat{y}) \right) \end{aligned} \tag{8}$$

by (3). The result now follows.

We note that if $\{q_j\}_{j=1}^{\delta_n^k} \subset \bigcup_{j=0}^{k} \mathscr{H}_{n-1}^k$ is an orthonormal set, then

$$(x_1 y_1 + \cdots + x_{n-1} y_{n-1} + x_n)^k = \sum_{j=1}^{\delta_n^k} p_j(x) q_j(\hat{y}) \tag{9}$$

where $\{p_j\}_{j=1}^{\delta_n^k}$ is an orthogonal basis for $\mathscr{H}_n^k$.

Formula (9) is analogous to the generating function for polynomial solutions of partial differential equations obtained in [3], [4], [7] and [10].

2. SERIES OF ORTHOGONAL HOMOGENEOUS POLYNOMIALS

Our first result on series of orthogonal homogeneous polynomials is an analog of the Funk–Hecke theorem. The Funk–Hecke theorem [1, p. 247] states that

$$\int_{\Sigma} f(x \cdot y) h_k(y) \, d\sigma(y) = c_k h_k(x),$$

where f is a bounded integrable function on the sphere $\Sigma \subset \mathbb{R}^n$, $h_k(x)$ is a homogeneous harmonic polynomial of degree k, and c_k is a constant depending on f and k. Since our inner product (2) is defined in terms of derivatives, we must restrict attention to functions f which are analytic. For such functions, using the inner product (2) we obtain the result of the Funk–Hecke theorem for arbitrary homogeneous polynomials.

THEOREM 5. *Suppose $f(u)$, $u \in \mathbb{R}$, is analytic in a neighborhood of* 0. *If $p_k \in \mathscr{H}_n^k$, then*

$$(p_k(y), f(x \cdot y)) = f^k(0) p_k(x),$$

where $x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$.

Proof. Since $f(u)$ is analytic in a neighborhood of $0 \in \mathbb{R}$,

$$f(u) = \sum_{j=0}^{\infty} \frac{f^j(0)}{j!} u^j,$$

where the series converges absolutely and uniformly on compact subsets of an open interval $(-R, R)$. Thus all partial derivatives of the series

$$f(x \cdot y) = \sum_{j=0}^{\infty} \frac{f^j(0)}{j!} (x \cdot y)^j$$

converge absolutely and uniformly on compact subsets of the same interval, when the series is differentiated term by term. Therefore, if $p_k \in \mathscr{H}_n^k$, then

$$\begin{aligned} (p_k(y), f(x \cdot y)) &= \left(p_k(y), \sum_{j=0}^{\infty} \frac{f^j(0)}{j!} (x \cdot y)^j \right) \\ &= \sum_{j=0}^{\infty} \frac{f_j(0)}{j!} \left(\left(p_k(y), (x \cdot y)^j \right) \right) \\ &= f^{(k)}(0) p_k(x), \end{aligned}$$

since

$$\left(p_k(y), (x \cdot y)^j \right) = \begin{cases} 0 & \text{if } j \neq k, \\ k! p_k(x) & \text{if } j = k \end{cases}$$

where the later result is a consequence of (3).

Using the Schwarz inequality and the Pythagorean identity of [2] or its generalization (6), we easily establish the convergence of infinite series of orthogonal homogeneous polynomials.

THEOREM 6. *Suppose $\{p_k^j\}_{j=1}^{\delta_n^k} \subset \mathscr{H}_n^k$, is an orthonormal set. Then the series*

$$\sum_{k=0}^{\infty} \sum_{j=1}^{\delta_n^k} a_{kj} p_k^j(x), \qquad a_{kj} \in \mathbb{R}, \tag{10}$$

converges absolutely and uniformly on compact subsets of the open ball

$$|x| = \left(x_1^2 + x_2^2 + \cdots + x_n^2\right)^{1/2} < R, \text{ where}$$

$$R^{-1} = \overline{\lim_{k\to\infty}} \left(\frac{\|a_k\|}{\sqrt{k!}}\right)^{1/k} \tag{11}$$

and

$$\|a_k\| = \left(\sum_{j=1}^{\delta_n^k} a_{kj}^2\right)^{1/2}.$$

Proof. Since the polynomials $p_k^j(x)$ are homogeneous of degree k, $p_k^j(x) = r^k p_k^j(x/r)$, where $r = \|x\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$. Thus,

$$\begin{aligned}
\left|\sum_{k=0}^{\infty}\sum_{j=1}^{\delta_n^k} \alpha_{kj} p_k^j(x)\right| &= \left|\sum_{k=0}^{\infty} r^k \sum_{j=1}^{\delta_n^k} a_{kj} p_k^j(x/r)\right| \\
&\le \sum_{k=0}^{\infty} r^k \left|\sum_{j=1}^{\delta_n^k} a_{kj} p_k^j(x/r)\right| \\
&\le \sum_{k=0}^{\infty} r^k \left(\sum_{j=1}^{\delta_n^k} a_{kj}\right)^{1/2} \left(\sum_{j=1}^{\delta_n^k} \left[p_k^j(x/r)\right]^2\right)^{1/2} \\
&= \sum_{k=0}^{\infty} \frac{\|a_k\|}{\sqrt{k!}} r^k,
\end{aligned}$$

where the last equality follows from the Pythagorean identity of [2] or its generalization (6). Thus the series (10) converges absolutely and uniformly on compact subsets of the open ball $\|x\| < R$, where R is given by (11).

The radius of convergence of the series (10), in the special case where all of the polynomials $p_k^j(x)$ are harmonic, is given in [6]. Note that for $n = 1$, the result (11) reduces to the long established radius of convergence of a power series in a single variable.

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