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Solutions of a Certain Class of Fractional Differintegral Equations

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Abstract—Recently, several authors demonstrated the usefulness of fractional calculus in obtaining particular solutions of a number of such familiar second-order differential equations as those associated with Gauss, Legendre, Jacobi, Chebyshev, Coulomb, Whittaker, Euler, Hermite, and Weber equations. The main object of this paper is to show how some of the latest contributions on the subject by Tu *et al.* [1], involving the associated Legendre, Euler, and Hermite equations, can be presented in a unified manner by suitably appealing to a general theorem on particular solutions of a certain class of fractional differintegral equations. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION, DEFINITIONS, AND PRELIMINARIES

The subject of fractional calculus (that is, derivatives and integrals of any real or complex order) has gained importance and popularity during the past three decades or so, due mainly to its demonstrated applications in numerous seemingly diverse fields of science and engineering (see, for details, [2]). By applying the following definition of a *fractional differintegral* (that is, *fractional derivative* and *fractional integral*) of order $\nu \in \mathbb{R}$, many authors have obtained particular solutions of a number of families of homogeneous (as well as nonhomogeneous) linear fractional differintegral equations.

DEFINITION. (See [3–5].) If the function $f(z)$ is analytic and has no branch point inside and on $\mathcal{C}$, where

$$\mathcal{C} := \{\mathcal{C}^-, \mathcal{C}^+\}. \tag{1.1}$$

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$\mathcal{C}^-$ is an integral curve along the cut joining the points z and $-\infty + i\Im(z)$, $\mathcal{C}^+$ is an integral curve along the cut joining the points z and $\infty + i\Im(z)$,

$$f_\nu(z) = cf_\nu(z) := \frac{\Gamma(\nu+1)}{2\pi i}\int_{g\mathcal{C}} \frac{f(\zeta)d\zeta}{(\zeta - z)^{\nu+1}}, \qquad (\nu \in \mathbb{R}\backslash\mathbb{Z}^-;\ \mathbb{Z} := \{-1,-2,-3,\dots\}) \tag{1.2}$$

and

$$f_{-n}(z) := \lim_{\nu \to -n} \{f_\nu(z)\}, \qquad (n \in \mathbb{N} := \{1,2,3,\dots\}), \tag{1.3}$$

where $\zeta \neq z$,

$$-\pi \leqq \arg(\zeta - z) \leqq \pi, \qquad \text{for } \mathcal{C}^-, \tag{1.4}$$

and

$$0 \leqq \arg(\zeta - z) \leqq 2\pi, \qquad \text{for } \mathcal{C}^+, \tag{1.5}$$

then $f_\nu(z)\,(\nu > 0)$ is said to be the *fractional derivative of $f(z)$ of order ν* and $f_\nu(z)\,(\nu < 0)$ is said to be the *fractional integral of $f(z)$ of order $-\nu$*, provided that

$$|f_\nu(z)| < \infty, \qquad (\nu \in \mathbb{R}). \tag{1.6}$$

REMARK 1. Throughout the present work, we shall simply write f_ν for $f_\nu(z)$ whenever the argument of the differintegrated function f is clearly understood by the surrounding context. Moreover, in case f is a many-valued function, we shall tacitly consider the *principal value* of f in our investigation. For the sake of convenience in dealing with their various (known or new) special cases, we choose also to state each of the fundamental results (Theorem 1 and 2 below) for fractional differintegral equations of a general order $\mu \in \mathbb{R}$.

We find it to be worthwhile to recall here the following useful lemmas and properties associated with the fractional differintegration which is defined above (see, e.g., [3,4]).

LEMMA 1. LINEARITY PROPERTY. *If the functions $f(z)$ and $g(z)$ are single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$(k_1 f + k_2 g)_\nu = k_1 f_\nu + k_2 g_\nu, \qquad (\nu \in \mathbb{R},\ z \in \Omega) \tag{1.7}$$

for any constants k_1 and k_2.

LEMMA 2. INDEX LAW. *If the function $f(z)$ is single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$(f_\mu)_\nu = f_{\mu+\nu} = (f_\nu)_\mu, \qquad (f_\mu \neq 0,\ f_\nu \neq 0,\ \mu,\nu \in \mathbb{R},\ z \in \Omega). \tag{1.8}$$

LEMMA 3. GENERALIZED LEIBNIZ RULE. *If the functions $f(z)$ and $g(z)$ are single-valued and analytic in some domain $\Omega \subseteq \mathbb{C}$, then*

$$(f \cdot g)_\nu = \sum_{n=0}^{\infty} \binom{\nu}{n} f_{\nu-n} \cdot g_n, \qquad (\nu \in \mathbb{R},\ z \in \Omega), \tag{1.9}$$

where g_n is the ordinary derivative of $g(z)$ of order n ($n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$), it being tacitly assumed (for simplicity) that $g(z)$ is the polynomial part (if any) of the product $f(z)g(z)$.

PROPERTY 1. For a constant λ,

$$\left(e^{\lambda z}\right)_\nu = \lambda^\nu e^{\lambda z}, \qquad (\lambda \neq 0,\ \nu \in \mathbb{R},\ z \in \mathbb{C}). \tag{1.10}$$

PROPERTY 2. For a constant λ,

$$\left(e^{-\lambda z}\right)_\nu = e^{-i\pi\nu} \lambda^\nu e^{-\lambda z}, \qquad (\lambda \neq 0,\ \nu \in \mathbb{R},\ z \in \mathbb{C}). \tag{1.11}$$

PROPERTY 3. For a constant λ,

$$\left(z^\lambda\right)_\nu = e^{-i\pi\nu} \frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)} z^{\lambda-\nu}, \qquad \left(\nu \in \mathbb{R},\ z \in \mathbb{C},\ \left|\frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)}\right| < \infty\right). \tag{1.12}$$

Some of the most recent contributions on the subject of particular solutions of fractional differintegral equations are those by Tu *et al.* [1], who considered generalizations of the associated Legendre, Euler, and Hermite equations. We recall here the *main* results of Tu *et al.* [1] as Theorems A to C below.

THEOREM A. *(See [1, p. 112, Theorem 1.1; p. 114, Theorem 1.2].) If the given function f satisfies constraint (1.6) and $f_{\beta-\alpha} \neq 0$, then the generalized nonhomogeneous associated Legendre's differential equation*

$$(1-z^n)\,\phi_m(z) - \sum_{k=1}^{n}\left[\binom{\alpha-\beta}{k}(z^n)_k + n(\beta+1)\binom{\alpha-\beta}{k-1}\left(z^{n-1}\right)_{k-1}\right]\phi_{m-k}(z) = f(z), \qquad (m\in\mathbb{Z} := \mathbb{N}_0\cup\mathbb{Z}^-,\ z^n\neq 1,\ n\in\mathbb{N}) \tag{1.13}$$

has a particular solution of the form

$$\phi(z) = \left(\left(f_{\beta-\alpha}(z)\cdot(1-z^n)^{\beta}\right)_{-1}\cdot(1-z^n)^{-\beta-1}\right)_{\alpha-\beta-m+1}, \tag{1.14}$$

provided that the second member of (1.14) exists, α and β being given constants.

Furthermore, the generalized homogeneous associated Legendre's differential equation

$$(1-z^n)\,\phi_m(z) - \sum_{k=1}^{n}\left[\binom{\alpha-\beta}{k}(z^n)_k + n(\beta+1)\binom{\alpha-\beta}{k-1}\left(z^{n-1}\right)_{k-1}\right]\phi_{m-k}(z) = 0, \qquad (m\in\mathbb{Z},\ z^n\neq 1,\ n\in\mathbb{N}) \tag{1.15}$$

has solutions of the form

$$\phi(z) = K\left((1-z^n)^{-\beta-1}\right)_{\alpha-\beta-m+1}, \tag{1.16}$$

where K is an arbitrary constant, α and β are given constants, and the second member of (1.16) is assumed to exist.

THEOREM B. *(See [1, p. 115, Theorem 1.4; p. 117, Theorem 1.5].) If the given function f satisfies constraint (1.6) and $f_{-\alpha} \neq 0$, then the generalized nonhomogeneous Euler's differential equation*

$$z^n\,\phi_m(z) + \sum_{k=1}^{n}\left[\binom{\alpha}{k}(z^n)_k + (1-2\alpha)\binom{\alpha}{k-1}\left(z^{n-1}\right)_{k-1}\right]\phi_{m-k}(z) = f(z), \qquad (m\in\mathbb{Z},\ z^n\neq 0,\ n\in\mathbb{N}) \tag{1.17}$$

has a particular solution of the form

$$\phi(z) = \left(\left(f_{-\alpha}(z)\cdot z^{1-2\alpha-n}\right)_{-1}\cdot z^{2\alpha-1}\right)_{\alpha-m+1}, \tag{1.18}$$

provided that the second member of (1.18) exists, α being a given constant.

Furthermore, the generalized homogeneous Euler's differential equation

$$z^n\,\phi_m(z) + \sum_{k=1}^{n}\left[\binom{\alpha}{k}(z^n)_k + (1-2\alpha)\binom{\alpha}{k-1}\left(z^{n-1}\right)_{k-1}\right]\phi_{m-k}(z) = 0, \qquad (m\in\mathbb{Z},\ z^n\neq 0,\ n\in\mathbb{N}) \tag{1.19}$$

has solutions of the form

$$\phi(z) = K\left(z^{2\alpha-1}\right)_{\alpha-m+1}, \tag{1.20}$$

where K is an arbitrary constant, α is a given constant, and the second member of (1.20) is assumed to exist.

THEOREM C. *(See [1, p. 118, Theorem 1.7; p. 120, Theorem 1.8].) If the given function f satisfies constraint (1.6) and $f_\alpha \neq 0$, then the generalized nonhomogeneous Hermite's differential equation*

$$z^n\,\phi_m(z) + \sum_{k=1}^{n} \binom{-\alpha}{k} (z^n)_k\,\phi_{m-k}(z) - \sum_{k=0}^{n+1} \binom{-\alpha}{k} \left(z^{n+1}\right)_k\,\phi_{m-k-1}(z) = f(z), \qquad (m \in \mathbb{Z},\ z^n \neq 0,\ n \in \mathbb{N}_0)\,, \tag{1.21}$$

has a particular solution of the form

$$\phi(z) = \left(\left(f_\alpha(z)\cdot z^{-n}\cdot \exp\left(-\frac{1}{2}z^2\right)\right)_{-1}\cdot \exp\left(\frac{1}{2}z^2\right)\right)_{1-m-\alpha}, \tag{1.22}$$

provided that the second member of (1.22) exists, α being a given constant.

Furthermore, the generalized homogeneous Hermite's differential equation

$$z^n\,\phi_m(z) + \sum_{k=1}^{n} \binom{-\alpha}{k} (z^n)_k\,\phi_{m-k}(z) - \sum_{k=0}^{n+1} \binom{-\alpha}{k} \left(z^{n+1}\right)_k\,\phi_{m-k-1}(z) = 0, \qquad (m \in \mathbb{Z},\ z^n \neq 0,\ n \in \mathbb{N}_0) \tag{1.23}$$

has solutions of the form

$$\phi(z) = K\left(\exp\left(\frac{1}{2}z^2\right)\right)_{1-m-\alpha}, \tag{1.24}$$

where K is an arbitrary constant, α is a given constant, and the second member of (1.24) is assumed to exist.

In this paper, we aim at presenting a unification (and generalization) of each of the above results (Theorems A to C) by appropriately applying a general theorem on particular solutions of a certain class of fractional differintegral equations.

2. A GENERAL THEOREM AND ITS CONSEQUENCES

The following general theorem (due to Tu *et al.* [6]) unifies as well as extends a considerably large number of widely scattered results on the solutions of various families of homogeneous and nonhomogeneous fractional differintegral equations.

THEOREM 1. *(See [6, Theorems 1 and 2].) Let $P(z;p)$ and $Q(z;q)$ be polynomials in z of degrees p and q, respectively, defined by*

$$P(z;p) := \sum_{k=0}^{p} a_k\,z^{p-k} = a_0 \prod_{j=1}^{p} (z - z_j)\,, \qquad (a_0 \neq 0,\ p \in \mathbb{N}) \tag{2.1}$$

and

$$Q(z;q) := \sum_{k=0}^{q} b_k\,z^{q-k}, \qquad (b_0 \neq 0,\ q \in \mathbb{N})\,. \tag{2.2}$$

Suppose also that $f_{-\nu}\ (\neq 0)$ exists for a given function f.

Then the nonhomogeneous linear ordinary fractional differintegral equation

$$P(z;p)\phi_\mu(z) + \left[\sum_{k=1}^{p} \binom{\nu}{k} P_k(z;p) + \sum_{k=1}^{q} \binom{\nu}{k-1} Q_{k-1}(z;q)\right] \times\phi_{\mu-k}(z) + \binom{\nu}{q} q!\,b_0\,\phi_{\mu-q-1}(z) = f(z), \qquad (\mu,\nu \in \mathbb{R};\ p,q \in \mathbb{N}) \tag{2.3}$$

has a particular solution of the form

$$\phi(z) = \left(\left(\frac{f_{-\nu}(z)}{P(z;p)} e^{H(z;p,q)} \right)_{-1} e^{-H(z;p,q)} \right)_{\nu-\mu+1}, \qquad (z \in \mathbb{C} \setminus \{z_1, \dots, z_p\}), \tag{2.4}$$

where, for convenience,

$$H(z;p,q) := \int^z \frac{Q(\zeta;q)}{P(\zeta;p)} \, d\zeta, \qquad (z \in \mathbb{C} \setminus \{z_1, \dots, z_p\}), \tag{2.5}$$

provided that the second member of (2.4) exists.

Furthermore, the homogeneous linear ordinary fractional differintegral equation

$$\begin{aligned} P(z;p)\phi_\mu(z) + \left[\sum_{k=1}^{p} \binom{\nu}{k} P_k(z;p) + \sum_{k=1}^{q} \binom{\nu}{k-1} Q_{k-1}(z;q) \right] \phi_{\mu-k}(z) \\ + \binom{\nu}{q} q!\, b_0\, \phi_{\mu-q-1}(z) = 0, \qquad (\mu, \nu \in \mathbb{R};\ p, q \in \mathbb{N}) \end{aligned} \tag{2.6}$$

has solutions of the form

$$\phi(z) = K \left(e^{-H(z;p,q)} \right)_{\nu-\mu+1}, \tag{2.7}$$

where K is an arbitrary constant and $H(z;p,q)$ is given by (2.5), it being provided that the second member of (2.7) exists.

REMARK 2. It should be remarked in passing that Tu *et al.* [6, Section 3] also gave the solutions of several general families of *partial* fractional differintegral equations analogous to (2.3) and (2.6).

With a view to applying Theorem 1, we set

$$a_0 = \xi, \quad (\xi \neq 0), \qquad a_1 = \cdots = a_{p-1} = 0, \qquad \text{and} \qquad a_p = \eta \tag{2.8}$$

and

$$b_0 = \lambda, \quad (\lambda \neq 0) \qquad \text{and} \qquad b_1 = \cdots = b_q = 0, \tag{2.9}$$

so that definitions (2.1) and (2.2) immediately yield

$$P(z;p) = \xi z^p + \eta, \qquad (\xi \neq 0,\ p \in \mathbb{N}) \tag{2.10}$$

and

$$Q(z;q) = \lambda z^q, \qquad (\lambda \neq 0,\ q \in \mathbb{N}). \tag{2.11}$$

We thus find from (2.5), (2.10), and (2.11) that

$$H(z;p,q) = \int^z \frac{\lambda t^q}{\xi t^p + \eta} \, dt, \qquad \left(z^p \neq -\frac{\eta}{\zeta};\ p, q \in \mathbb{N} \right). \tag{2.12}$$

In particular, if $q = p - 1$ or $q = p + 1$, (2.12) leads us to

$$H(z;p,q) = \begin{cases} \dfrac{\lambda}{\xi p} \log\left(\xi z^p + \eta\right), & (q = p - 1), \\ \dfrac{\lambda z^2}{2\xi} - \dfrac{\lambda \eta}{\xi} \displaystyle\int^z \frac{t}{\xi t^p + \eta} \, dt, & (q = p + 1). \end{cases} \tag{2.13}$$

A *special* case of Theorem 1 can now be stated as the following theorem.

THEOREM 2. *Suppose that $f_{-\nu}$ ($\neq 0$) exists for a given function f. Then the nonhomogeneous linear ordinary fractional differintegral equation*

$$\begin{aligned}(\xi z^p+\eta)\,\phi_\mu(z)+\left[\sum_{k=1}^{p}\binom{\nu}{k}(\xi z^p+\eta)_k+\sum_{k=1}^{q}\binom{\nu}{k-1}(\lambda z^q)_{k-1}\right]\phi_{\mu-k}(z)\\ +\lambda\binom{\nu}{q}q!\,\phi_{\mu-q-1}(z)=f(z),\qquad(\mu,\nu\in\mathbb{R};\ \xi\neq 0;\ \lambda\neq 0;\ p,q\in\mathbb{N})\end{aligned}\tag{2.14}$$

has a particular solution of the form

$$\begin{aligned}\phi(z)=\left(\left(\frac{f_{-\nu}(z)}{\xi z^p+\eta}e^{H(z;p,q)}\right)_{-1}e^{-H(z;p,q)}\right)_{\nu-\mu+1}\\ \left(z\in\mathbb{C}\setminus\left\{z:z^p=-\frac{\eta}{\xi}\,(p\in\mathbb{N})\right\}\right),\end{aligned}\tag{2.15}$$

where $H(z;p,q)$ is given by (2.12), it being provided that the second member of (2.15) exists.

Furthermore, the homogeneous linear ordinary fractional differintegral equation

$$\begin{aligned}(\xi z^p+\eta)\,\phi_\mu(z)+\left[\sum_{k=1}^{p}\binom{\nu}{k}(\xi z^p+\eta)_k+\sum_{k=1}^{q}\binom{\nu}{k-1}(\lambda z^q)_{k-1}\right]\phi_{\mu-k}(z)\\ +\lambda\binom{\nu}{q}q!\,\phi_{\mu-q-1}(z)=0,\qquad(\mu,\nu\in\mathbb{R};\ \xi\neq 0;\ \lambda\neq 0;\ p,q\in\mathbb{N})\end{aligned}\tag{2.16}$$

has solutions of the form

$$\phi(z)=K\left(e^{-H(z;p,q)}\right)_{\nu-\mu+1},\tag{2.17}$$

where K is an arbitrary constant and $H(z;p,q)$ is given by (2.12), it being provided that the second member of (2.17) exists.

As already observed in conclusion by Tu *et al.* [6], either or both of the polynomials $P(z;p)$ and $Q(z;q)$, involved in Theorem 1 (and hence, also in Theorem 2), can be of degree 0 *as well.* Thus, in definitions (2.1) and (2.2) (as also in Theorems 1 and 2), $\mathbb{N}$ may easily be replaced (if and where needed) by $\mathbb{N}_0$.

REMARK 3. The function $H(z;p,q)$ given by (2.13) would *further* simplify considerably if (for example) $\eta=0$, and we thus find from (2.13) that

$$H(z;p,q)|_{\eta=0}=\begin{cases}\dfrac{\lambda}{\xi p}\log(\xi z^p), & (q=p-1),\\ \dfrac{\lambda z^2}{2\xi}, & (q=p+1).\end{cases}\tag{2.18}$$

3. FURTHER DEDUCTIONS FROM THEOREM 2

In the preceding section, we have already shown how simply Theorem 2 would follow as a *special* case of Theorem 1 of [6]. Theorem 2, in turn, provides a unification (and generalization) of Theorems A to C (see Section 1). First of all, in view of Lemma 1 and the case $q=p-1$ in (2.13), Theorem 2 yields Theorem A as one of its special cases when

$$\begin{aligned}\mu=m\quad(m\in\mathbb{Z}),\qquad\nu=\alpha-\beta,\qquad p=q+1=n,\quad(n\in\mathbb{N}),\qquad\xi=-1,\\ \eta=1,\qquad\text{and}\qquad\lambda=-n\,(\beta+1).\end{aligned}\tag{3.1}$$

Next, by appealing to Lemma 1 and the case $q = p - 1$ in (2.18), we can derive Theorem B as a special case of Theorem 2 when

$$\mu = m, \quad (m \in \mathbb{Z}), \qquad \nu = \alpha, \qquad p = q + 1 = n \quad (n \in \mathbb{N}), \qquad \xi = 1, \\ \eta = 0, \qquad \text{and} \qquad \lambda = 1 - 2\alpha. \tag{3.2}$$

Finally, if we apply Lemma 1 and the case $q = p+1$ in (2.18), we shall readily obtain Theorem C as yet another special case of Theorem 2 when

$$\mu = m, \quad (m \in \mathbb{Z}), \qquad \nu = -\alpha \qquad p = q - 1 = n, \quad (n \in \mathbb{N}_0), \qquad \xi = 1, \\ \eta = 0, \qquad \text{and} \qquad \lambda = -1. \tag{3.3}$$

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