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# Relations between matrix sets generated from linear matrix expressions and their applications

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ABSTRACT

Let $A+BXC$ and $A+BX+YC$ be two linear matrix expressions, and denote by $\{A+BXC\}$ and $\{A+BX+YC\}$ the collections of the two matrix expressions when $X$ and $Y$ run over the corresponding matrix spaces. In this paper, we study relationships between the two matrix sets $\{A_1+B_1X_1C_1\}$ and $\{A_2+B_2X_2C_2\}$, as well as the two sets $\{A_1+B_1X_1+Y_1C_1\}$ and $\{A_2+B_2X_2+Y_2C_2\}$, by using some rank formulas for matrices. In particular, we give necessary and sufficient conditions for the two matrix set inclusions $\{A_1+B_1X_1C_1\} \subseteq \{A_2+B_2X_2C_2\}$ and $\{A_1+B_1X_1+Y_1C_1\} \subseteq \{A_2+B_2X_2+Y_2C_2\}$ to hold. We also use the results obtained to characterize relations of solutions of some linear matrix equations.



## 1. Introduction

In matrix theory, there are many matrix expressions involving one or more variable matrices. These matrix expressions, often called matrix expressions or matrix functions, may vary over the choice of the variable matrices. Some of the simple matrix expressions are given by $A-BX$, $A-BXC$, $A-BX-YC$ and $A-BXC-CYD$, where $A$, $B$, $C$ and $D$ are given, and $X$ and $Y$ are variable matrices. When the variable matrices in a given matrix expression are running over certain matrix sets, the matrix expression may vary as well, such that all possible values of the matrix expression generate a matrix set. This fact prompts us to raise the following general problem: for a pair of linear matrix expressions $A_1+B_{11}X_{11}C_{11}+\cdots+B_{1k}X_{1k}C_{1k}$ and $A_2+B_{21}X_{21}C_{21}+\cdots+B_{2l}X_{2l}C_{2l}$ of the same size, give identifying conditions for

$$\{A_1+B_{11}X_{11}C_{11}+\cdots+B_{1k}X_{1k}C_{1k}\} \cap \{A_2+B_{21}X_{21}C_{21}+\cdots+B_{2l}X_{2l}C_{2l}\} \neq \emptyset,$$
$$\{A_1+B_{11}X_{11}C_{11}+\cdots+B_{1k}X_{1k}C_{1k}\} \subseteq \{A_2+B_{21}X_{21}C_{21}+\cdots+B_{2l}X_{2l}C_{2l}\}$$

to hold, respectively. Because of the multiple variable matrices and the noncommutativity of matrix multiplication, it is, however, difficult to give a complete solution to this general problem by using the conventional methods in matrix theory. In this paper, we use some rank formulas for matrices to characterize the relations between the two sets $\{A_1+B_1X_1C_1\}$ and $\{A_2+B_2X_2C_2\}$, and the two sets $\{A_1+B_1X_1+Y_1C_1\}$ and $\{A_2+B_2X_2+Y_2C_2\}$. Some applications of the results obtained on relations of solutions of matrix equations are also presented.

Throughout this paper, $\mathbb{C}^{m\times n}$ stands for the set of all $m \times n$ complex matrices. $A^*$, $r(A)$ and $\mathscr{R}(A)$ denote the conjugate transpose, rank and range (column space) of a complex matrix $A$, respectively. The Moore–Penrose inverse of $A \in \mathbb{C}^{m\times n}$, denoted by $A^\dagger$, is defined to be the unique solution $X$ satisfying the matrix equations

$$\text{(i) } AXA = A, \qquad \text{(ii) } XAX = X, \qquad \text{(iii) } (AX)^* = AX, \qquad \text{(iv) } (XA)^* = XA.$$

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The symbols $E_A$ and $F_A$ stand for the two orthogonal projectors $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$ onto the null spaces $A^*$ and $A$, respectively. The following are some known results for ranks of matrices, which will be used in the latter part of this paper.

**Lemma 1.1.** *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$. Then,*

$$r[A, B] = r(A) + r(E_A B) = r(B) + rr(E_B A), \tag{1.1}$$

$$r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) + r(CF_A) = r(C) + r(AF_C), \tag{1.2}$$

$$r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} = r(B) + r(C) + r(E_B A F_C). \tag{1.3}$$

Some formulas for extremal ranks of linear matrix expressions in [1–5] are given below.

**Lemma 1.2.** *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times k}$ and $C \in \mathbb{C}^{l\times n}$ be given. Then the maximal and minimal ranks of $A - BX$, $A - BXC$ and $A - BX - YC$ with respect to $X$ and $Y$ are given by the following closed-form formulas:*

$$\max_{X\in\mathbb{C}^{k\times n}} r(A - BX) = \min\{r[A, B],\ n\}, \tag{1.4}$$

$$\min_{X\in\mathbb{C}^{k\times n}} r(A - BX) = r[A, B] - r(B), \tag{1.5}$$

$$\max_{X\in\mathbb{C}^{k\times l}} r(A - BXC) = \min\left\{ r[A, B], r\begin{bmatrix} A \\ C \end{bmatrix} \right\}, \tag{1.6}$$

$$\min_{X\in\mathbb{C}^{k\times l}} r(A - BXC) = r[A, B] + r\begin{bmatrix} A \\ C \end{bmatrix} - r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix}, \tag{1.7}$$

$$\min_{X\in\mathbb{C}^{k\times n}, Y\in\mathbb{C}^{m\times l}} r(A - BX - YC) = r\begin{bmatrix} A & B \\ C & 0 \end{bmatrix} - r(B) - r(C). \tag{1.8}$$

The following result is well-known; see [6].

**Lemma 1.3.** (a) *There exists an $X$ such that $BXC = A$ if and only if $\mathscr{R}(A) \subseteq \mathscr{R}(B)$ and $\mathscr{R}(A^*) \subseteq \mathscr{R}(C^*)$, or equivalently, $BB^\dagger A = AC^\dagger C = A$. In this case, the general solution can be written in the parametric form*

$$X = B^\dagger A C^\dagger + F_B U_1 + U_2 E_C,$$

*where $U_1$ and $U_2$ are arbitrary matrices.*

(b) *There exist $X$ and $Y$ such that $BX + YC = A$ if and only if $E_B A F_C = 0$.*

**Lemma 1.4** ([4]). *Let $p(X_1, X_2) = A - B_1X_1C_1 - B_2X_2C_2$. Then,*

$$\max_{X_1,X_2} r[p(X_1, X_2)] = \min\left\{ r[A, B_1, B_2], r\begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix}, r\begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix}, r\begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} \right\}, \tag{1.9}$$

$$\min_{X_1,X_2} r[p(X_1, X_2)] = r\begin{bmatrix} A \\ C_1 \\ C_2 \end{bmatrix} + r[A, B_1, B_2] + \max\{s_1, s_2\}, \tag{1.10}$$

*where*

$$s_1 = r\begin{bmatrix} A & B_1 \\ C_2 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_2 \\ C_2 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix},$$

$$s_2 = r\begin{bmatrix} A & B_2 \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_1 & B_2 \\ C_1 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A & B_2 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix}.$$

**Lemma 1.5.** *Let $A$, $B_i$ and $C_i$ be given such that $A - B_1X_1C_1 - \cdots - B_kX_kC_k$ is defined. Then $A - B_1X_1C_1 - \cdots - B_kX_kC_k = 0$ holds for all $X_1, \ldots, X_k$ if and only if $A = 0$ and any one of the following $2^k$ conditions:*

$$B_{i_1} = 0, \ldots, B_{i_p} = 0 \quad and \quad C_{i_{p+1}} = 0, \ldots, C_{i_k} = 0, \quad p = 1, \ldots, k \tag{1.11}$$

*holds, where $i_1, \ldots, i_k$ are a permutation of $1, \ldots, k$.*

**Proof.** Since the variable matrices $X_1, \ldots, X_k$ are all free, it is obvious that $A - B_1X_1C_1 - \cdots - B_kX_kC_k = 0$ holds for all $X_1, \ldots, X_k$ if and only if $A = 0$, and

$$B_1X_1C_1 = 0, \ldots, B_kX_kC_k = 0$$

hold for all $X_1, \ldots, X_k$, which is further equivalent to (1.11). $\square$

**Lemma 1.6.** *There exist $X, Y, Z$ such that $AXB + CY + ZD = M$ holds if and only if*

$$(E_CA)(E_CA)^{\dagger}E_CMF_D = E_CMF_D(BF_D)^{\dagger}(BF_D) = E_CMF_D, \tag{1.12}$$

*or equivalently,*

$$[I - (E_CA)(E_CA)^{\dagger}]E_CMF_D = E_CMF_D[I - (BF_D)^{\dagger}(BF_D)] = 0. \tag{1.13}$$

**Proof.** From Lemma 1.3(b), there exist $Y$ and $Z$ such that $AXB + CY + ZD = M$ holds if and only if $E_C(AXB - M)F_D = 0$. From Lemma 1.3(a), there exists an $X$ such that $E_CAXBF_D = E_CMF_D$ if and only if (1.12) holds. $\square$

## 2. Relations between some matrix sets generated from linear matrix expressions

Recall that a matrix $A$ is null if and only if $r(A) = 0$. Hence, two matrices $A$ and $B$ of the same size are equal if and only if $r(A - B) = 0$. Further, two matrix sets $\mathcal{S}_1$ and $\mathcal{S}_2$ have a common matrix, i.e., $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$, if and only if

$$\min_{A\in\mathcal{S}_1, B\in\mathcal{S}_2} r(A - B) = 0; \tag{2.1}$$

$\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if

$$\max_{A\in\mathcal{S}_1} \min_{B\in\mathcal{S}_2} r(A - B) = 0. \tag{2.2}$$

If $A - B$ can be written as a linear matrix expression with some variable matrices, then we can find the extremal ranks of this expression with respect to the variable matrices from (1.4)–(1.10).

We start with a simple result on the relations between the two linear matrix expressions $A_1 + B_1X_1$ and $A_2 + B_2X_2$ of the same size.

**Theorem 2.1.** *Let $A_1, A_2 \in \mathbb{C}^{m\times n}$, $B_1 \in \mathbb{C}^{m\times p_1}$ and $B_2 \in \mathbb{C}^{m\times p_2}$ be given, $X_1 \in \mathbb{C}^{p_1\times n}$ and $X_2 \in \mathbb{C}^{p_2\times n}$ be variable matrices, and define*

$$\mathcal{S}_1 = \{A_1 + B_1X_1 \mid X_1 \in \mathbb{C}^{p_1\times n}\} \quad \text{and} \quad \mathcal{S}_2 = \{A_2 + B_2X_2 \mid X_2 \in \mathbb{C}^{p_2\times n}\}.$$

*Then,*

(a) $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$ *if and only if* $\mathscr{R}(A_1 - A_2) \subseteq \mathscr{R}[B_1, B_2]$.
(b) $\mathcal{S}_1 \subseteq \mathcal{S}_2$ *if and only if* $\mathscr{R}[A_1 - A_2, B_1] \subseteq \mathscr{R}(B_2)$.
(c) $\mathcal{S}_1 = \mathcal{S}_2$ *if and only if* $\mathscr{R}(A_1 - A_2) \subseteq \mathscr{R}(B_1) = \mathscr{R}(B_2)$.

**Proof.** Applying (1.5) gives

$$\begin{aligned}\min_{M_1\in\mathcal{S}_1, M_2\in\mathcal{S}_2} r(M_1 - M_2) &= \min_{X_1,X_2} r(A_1 - A_2 + B_1X_1 - B_2X_2)\\ &= \min_{X_1,X_2} r\left(A_1 - A_2 + [B_1, -B_2]\begin{bmatrix} X_1\\ X_2\end{bmatrix}\right)\\ &= r[A_1 - A_2, B_1, B_2] - r[B_1, B_2].\end{aligned}$$

Setting both sides of the equality to zero leads to (a). From (2.2), $\mathcal{S}_1 \subseteq \mathcal{S}_2$ holds if and only if

$$\max_{X_1} \min_{X_2} r[(A_1 + B_1X_1) - (A_2 + B_2X_2)] = 0. \tag{2.3}$$

It follows from (1.5) that

$$\min_{X_2} r[(A_1 + B_1X_1) - (A_2 + B_2X_2)] = r[A_1 - A_2 + B_1X_1, B_2] - r(B_2). \tag{2.4}$$

Also by (1.6),

$$\begin{aligned}\max_{X_1} r[A_1 - A_2 + B_1X_1, B_2] &= \max_{X_1} r([A_1 - A_2, B_2] - B_1X_1[I_n, 0])\\ &= \min\left\{r[A_1 - A_2, B_1, B_2], r\begin{bmatrix} A_1 - A_2 & B_2\\ I_n & 0\end{bmatrix}\right\}\\ &= \min\{r[A_1 - A_2, B_1, B_2], r(B_2) + n\}.\end{aligned} \tag{2.5}$$

Combining (2.4) and (2.5) yields

$$\max_{X_1}\min_{X_2} r[(A_1+B_1X_1)-(A_2+B_2X_2)] = \min\{r[A_1-A_2, B_1, B_2]-r(B_2), n\}. \tag{2.6}$$

Setting the right-hand side of this equality to zero and noting that $n \neq 0$, we see that (2.3) is equivalent to $r[A_1-A_2, B_1, B_2] = r(B_2)$. Thus, we have (b). By symmetry, $\mathcal{S}_2 \subseteq \mathcal{S}_1$ if and only if $\mathscr{R}[A_1-A_2, B_2] \subseteq \mathscr{R}(B_1)$. Combining this assertion with (b) leads to (c). □

Similarly, we can show the following result.

**Theorem 2.2.** *Let $A_1, A_2 \in \mathbb{C}^{m\times n}$, $B_1 \in \mathbb{C}^{m\times p_1}$, $B_2 \in \mathbb{C}^{m\times p_2}$, $C_1 \in \mathbb{C}^{q_1\times n}$ and $C_2 \in \mathbb{C}^{q_2\times n}$ be given, and let $X_1 \in \mathbb{C}^{p_1\times q_1}$ and $X_2 \in \mathbb{C}^{p_2\times q_2}$ be two variable matrices. Also, assume that $B_i \neq 0$ and $C_i \neq 0$, $i = 1, 2$, and let*

$$\mathcal{S}_1 = \{A_1 + B_1X_1C_1 \mid X_1 \in \mathbb{C}^{p_1\times q_1}\} \quad \text{and} \quad \mathcal{S}_2 = \{A_2 + B_2X_2C_2 \mid X_2 \in \mathbb{C}^{p_2\times q_2}\}.$$

*Then,*

(a) *$\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$ if and only if*

$$\mathscr{R}(A_1-A_2) \subseteq \mathscr{R}[B_1, B_2], \qquad \mathscr{R}(A_1^*-A_2^*) \subseteq \mathscr{R}[C_1^*, C_2^*],$$
$$r\begin{bmatrix} A_1-A_2 & B_1 \\ C_2 & 0 \end{bmatrix} = r(B_1)+r(C_2), \qquad r\begin{bmatrix} A_1-A_2 & B_2 \\ C_1 & 0 \end{bmatrix} = r(B_2)+r(C_1).$$

(b) *$\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if $\mathscr{R}[A_1-A_2, B_1] \subseteq \mathscr{R}(B_2)$ and $\mathscr{R}[A_1^*-A_2^*, C_1^*] \subseteq \mathscr{R}(C_2^*)$.*

(c) *$\mathcal{S}_1 = \mathcal{S}_2$ if and only if $\mathscr{R}(A_1-A_2) \subseteq \mathscr{R}(B_1) = \mathscr{R}(B_2)$ and $\mathscr{R}(A_1^*-A_2^*) \subseteq \mathscr{R}(C_1^*) = \mathscr{R}(C_2^*)$.*

**Proof.** Observe that the difference $M_1-M_2$ for $M_1 \in \mathcal{S}_1$ and $M_2 \in \mathcal{S}_2$ can be written as $M_1-M_2 = A_1-A_2+B_1X_1C_1-B_2X_2C_2$. Applying (1.10) to $M_1-M_2$ gives

$$\min_{M_1\in\mathcal{S}_1, M_2\in\mathcal{S}_2} r(M_1-M_2) = r\begin{bmatrix} A_1-A_2 \\ C_1 \\ C_2 \end{bmatrix} + r[A_1-A_2, B_1, B_2] + \max\{s_1, s_2\}, \tag{2.7}$$

where

$$s_1 = r\begin{bmatrix} A_1-A_2 & B_1 \\ C_2 & 0 \end{bmatrix} - r\begin{bmatrix} A_1-A_2 & B_1 & B_2 \\ C_2 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A_1-A_2 & B_1 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix},$$

$$s_2 = r\begin{bmatrix} A_1-A_2 & B_2 \\ C_1 & 0 \end{bmatrix} - r\begin{bmatrix} A_1-A_2 & B_1 & B_2 \\ C_1 & 0 & 0 \end{bmatrix} - r\begin{bmatrix} A_1-A_2 & B_2 \\ C_1 & 0 \\ C_2 & 0 \end{bmatrix}.$$

Result (a) follows immediately from (2.7). From (2.2), $\mathcal{S}_1 \subseteq \mathcal{S}_2$ holds if and only if

$$\max_{X_1}\min_{X_2} r[(A_1+B_1X_1C_1)-(A_2+B_2X_2C_2)] = 0.$$

Applying (1.7) gives

$$\begin{aligned}\min_{X_2} r[(A_1+B_1X_1C_1)-(A_2+B_2X_2C_2)] &= r[A_1-A_2+B_1X_1C_1, B_2] + r\begin{bmatrix} A_1-A_2+B_1X_1C_1 \\ C_2 \end{bmatrix} \\ &\quad - r\begin{bmatrix} A_1-A_2+B_1X_1C_1 & B_2 \\ C_2 & 0 \end{bmatrix}.\end{aligned}$$

It is difficult to find the maximal rank of the expression with respect to $X_1$. From Lemma 1.3(a), $\mathcal{S}_1 \subseteq \mathcal{S}_2$ holds if and only if

$$r[A_1-A_2+B_1X_1C_1, B_2] = r(B_2) \quad \text{and} \quad r[A_1^*-A_2^*+C_1^*X_1^*C_1^*, B_2^*] = r(C_2) \tag{2.8}$$

hold for all $X_1$. Applying (1.6) gives

$$\max_{X_1} r[A_1-A_2+B_1X_1C_1, B_2] = \min\left\{ r[A_1-A_2, B_1, B_2], \quad r\begin{bmatrix} A_1-A_2 & B_2 \\ C_1 & 0 \end{bmatrix}\right\},$$

$$\max_{X_1} r[A_1^*-A_2^*+C_1^*X_1^*B_1^*, C_2^*] = \min\left\{ r\begin{bmatrix} A_1-A_2 \\ C_1 \\ C_2 \end{bmatrix}, \quad r\begin{bmatrix} A_1-A_2 & B_1 \\ C_2 & 0 \end{bmatrix}\right\}.$$

Thus, the first equality in (2.8) is equivalent to

$$r[A_1-A_2, B_1, B_2] = r(B_2) \quad \text{or} \quad r\begin{bmatrix} A_1-A_2 & B_2 \\ C_1 & 0 \end{bmatrix} = r(B_2), \tag{2.9}$$

and the second equality in (2.8) is equivalent to

$$r\begin{bmatrix} A_1 - A_2 \\ C_1 \\ C_2 \end{bmatrix} = r(C_2) \quad \text{or} \quad r\begin{bmatrix} A_1 - A_2 & B_1 \\ C_2 & 0 \end{bmatrix} = r(C_2). \tag{2.10}$$

Note that $B_1, B_2, C_1$ and $C_2$ are nonzero. Hence, (2.9) and (2.10) are equivalent to (b). □

If at least one of $B_1, B_2, C_1$ and $C_2$ is zero, the relations between the two sets $\mathcal{S}_1$ and $\mathcal{S}_2$ in Theorem 2.2 become trivial.

**Theorem 2.3.** *Let $A_1, A_2 \in \mathbb{C}^{m\times n}$, $B_1 \in \mathbb{C}^{m\times p_1}$, $B_2 \in \mathbb{C}^{m\times p_2}$, $C_1 \in \mathbb{C}^{q_1\times n}$ and $C_2 \in \mathbb{C}^{q_2\times n}$ be given, $X_1 \in \mathbb{C}^{p_1\times n}$, $Y_1 \in \mathbb{C}^{m\times q_1}$, $X_2 \in \mathbb{C}^{p_2\times n}$ and $Y_2 \in \mathbb{C}^{m\times q_2}$ be four variable matrices. Also, assume that $B_i \neq 0$ and $C_i \neq 0$ for $i = 1, 2$, and let*

$$\mathcal{S}_1 = \{A_1 + B_1X_1 + Y_1C_1 \mid X_1 \in \mathbb{C}^{p_1\times n},\ Y_1 \in \mathbb{C}^{m\times q_1}\}, \tag{2.11}$$

$$\mathcal{S}_2 = \{A_2 + B_2X_2 + Y_2C_2 \mid X_2 \in \mathbb{C}^{p_2\times n},\ Y_2 \in \mathbb{C}^{m\times q_2}\}. \tag{2.12}$$

*Also, define*

$$N = \begin{bmatrix} A_1 - A_2 & B_1 & B_2 \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \end{bmatrix}.$$

*Then,*

(a) *$\mathcal{S}_1 \cap \mathcal{S}_2 \neq \emptyset$ if and only if $r(N) = r\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + r[B_1, B_2]$.*
(b) *$\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if any one of the three conditions* (i) $r(B_2) = m$, (ii) $r(C_2) = n$ *and* (iii) $r(N) = r(B_2) + r(C_2)$ *holds.*
(c) *$\mathcal{S}_1 = \mathcal{S}_2$ if and only if any one of the three conditions* (i) $r(B_1) = m$, (ii) $r(C_1) = n$ *and* (iii) $r(N) = r(B_1) + r(C_1)$ *and any one of the three conditions* (iv) $r(B_2) = m$, (v) $r(C_2) = n$ *and* (vi) $r(N) = r(B_2) + r(C_2)$ *hold.*

**Proof.** Note that the difference $M_1 - M_2$ for $M_1 \in \mathcal{S}_1$ and $M_2 \in \mathcal{S}_2$ can be written as

$$M_1 - M_2 = A_1 - A_2 + B_1X_1 + Y_1C_1 - B_2X_2 - Y_2C_2 = A_1 - A_2 + [B_1, -B_2]\begin{bmatrix} X_1 \\ X_2 \end{bmatrix} + [Y_1, Y_2]\begin{bmatrix} C_1 \\ -C_2 \end{bmatrix}.$$

Applying (1.8) to this expression gives

$$\min_{M_1\in\mathcal{S}_1, M_2\in\mathcal{S}_2} r(M_1 - M_2) = r(N) - r\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} - r[B_1, B_2]. \tag{2.13}$$

Setting the right-hand side of (2.13) to zero yields (a). It can also be derived from (1.9) and (1.10) that

$$\max_{M_1\in\mathcal{S}_1} \min_{M_2\in\mathcal{S}_2} r(M_1 - M_2) = \min\{m - r(B_2), n - r(C_2), r(N) - r(B_2) - r(C_2)\}.$$

Setting the right-hand side to zero yields (b). Result (c) follows from (b). □

Two more general results are given below.

**Theorem 2.4.** *Let $A_1, A_2 \in \mathbb{C}^{m\times n}$, $B_1 \in \mathbb{C}^{m\times p_1}$, $B_2 \in \mathbb{C}^{m\times p_2}$, $B_3 \in \mathbb{C}^{m\times p_3}$, $C_1 \in \mathbb{C}^{q_1\times n}$, $C_2 \in \mathbb{C}^{q_2\times n}$ and $C_3 \in \mathbb{C}^{q_3\times n}$ be given, and $X_1 \in \mathbb{C}^{p_1\times n}$, $X_2 \in \mathbb{C}^{p_2\times n}$, $Y_1 \in \mathbb{C}^{m\times p_1}$, $Y_2 \in \mathbb{C}^{m\times p_2}$ and $Z \in \mathbb{C}^{p_3\times q_3}$ be five variable matrices. Also, assume that $B_i \neq 0$ and $C_i \neq 0$, $i = 1, 2, 3$, and define*

$$\mathcal{S}_1 = \{A_1 + B_1X_1 + Y_1C_1 \mid X_1 \in \mathbb{C}^{p_1\times n},\ Y_1 \in \mathbb{C}^{m\times q_1}\}, \tag{2.14}$$

$$\mathcal{S}_2 = \{A_2 + B_2X_2 + Y_2C_2 + B_3ZC_3 \mid X_2 \in \mathbb{C}^{p_2\times n},\ Y_2 \in \mathbb{C}^{m\times q_2},\ Z \in \mathbb{C}^{p_3\times q_3}\}. \tag{2.15}$$

*Then, $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if one of the following three conditions:*

$$\text{(i) } r(C_2) = n, \qquad \text{(ii) } r[B_2, B_3] = m, \qquad \text{(iii) } r\begin{bmatrix} A_1 - A_2 & B_1 & B_2 & B_3 \\ C_1 & 0 & 0 & 0 \\ C_2 & 0 & 0 & 0 \end{bmatrix} = r[B_2, B_3] + r(C_2), \tag{2.16}$$

*and one of the following three conditions:*

$$\text{(iv) } r(B_2) = m, \qquad \text{(v) } r\begin{bmatrix} C_2 \\ C_3 \end{bmatrix} = n, \qquad \text{(vi) } r\begin{bmatrix} A_1 - A_2 & B_1 & B_2 \\ C_1 & 0 & 0 \\ C_2 & 0 & 0 \\ C_3 & 0 & 0 \end{bmatrix} = r\begin{bmatrix} C_2 \\ C_3 \end{bmatrix} + r(B_2) \tag{2.17}$$

*holds.*

**Proof.** Note that $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if the matrix equation

$$B_2X_2 + Y_2C_2 + B_3ZC_3 = A_1 - A_2 + B_1X_1 + Y_1C_1$$

is solvable for $X_2$, $Y_2$ and $Z$, which, from Lemma 1.6, is equivalent to

$$[I - (E_{B_2}B_3)(E_{B_2}B_3)^{\dagger}](E_{B_2}A_1F_{C_2} - E_{B_2}A_2F_{C_2} + E_{B_2}B_1X_1F_{C_2} + E_{B_2}Y_1C_1F_{C_2}) = 0, \tag{2.18}$$

$$(E_{B_2}A_1F_{C_2} - E_{B_2}A_2F_{C_2} + E_{B_2}B_1X_1F_{C_2} + E_{B_2}Y_1C_1F_{C_2})[I - (C_3F_{C_2})^{\dagger}(C_3F_{C_2})] = 0 \tag{2.19}$$

for all $X_1$ and $Y_1$. From Lemma 1.5, (2.18) holds if and only if one of the following three conditions holds:

$$[I - (E_{B_2}B_3)(E_{B_2}B_3)^{\dagger}]E_{B_2} = 0, \tag{2.20}$$

$$\begin{bmatrix} [I - (E_{B_2}B_3)(E_{B_2}B_3)^{\dagger}](E_{B_2}A_1F_{C_2} - E_{B_2}A_2F_{C_2}) & [I - (E_{B_2}B_3)(E_{B_2}B_3)^{\dagger}]E_{B_2}B_1 \\ C_1F_{C_2} & 0 \end{bmatrix} = 0, \tag{2.21}$$

$$F_{C_2} = 0. \tag{2.22}$$

From (1.4)–(1.6),

$$\begin{aligned} r([I - (E_{B_2}B_3)(E_{B_2}B_3)^{\dagger}]E_{B_2}) &= r[E_{B_2}B_3, E_{B_2}] - r(E_{B_2}B_3) \\ &= r(E_{B_2}) - r[B_2, B_3] + r(B_2) = m - r[B_2, B_3], \end{aligned} \tag{2.23}$$

$$\begin{aligned} & r\begin{bmatrix} [I - (E_{B_2}B_3)(E_{B_2}B_3)^{\dagger}](E_{B_2}A_1F_{C_2} - E_{B_2}A_2F_{C_2}) & [I - (E_{B_2}B_3)(E_{B_2}B_3)^{\dagger}]E_{B_2}B_1 \\ C_1F_{C_2} & 0 \end{bmatrix} \\ &= r\begin{bmatrix} E_{B_2}A_1F_{C_2} - E_{B_2}A_2F_{C_2} & E_{B_2}B_1 & E_{B_2}B_3 \\ C_1F_{C_2} & 0 & 0 \end{bmatrix} - r(E_{B_2}B_3) \\ &= r\begin{bmatrix} A_1 - A_2 & B_1 & B_2 & B_3 \\ C_1 & 0 & 0 & 0 \\ C_2 & 0 & 0 & 0 \end{bmatrix} - r[B_2, B_3] - r(C_2). \end{aligned} \tag{2.24}$$

Hence, (2.20)–(2.22) are equivalent to (2.16). Similarly, we can show that (2.19) is equivalent to (2.17). □

**Theorem 2.5.** *Let $A_1, A_2 \in \mathbb{C}^{m\times n}$, $B_1 \in \mathbb{C}^{m\times p_1}$, $B_2 \in \mathbb{C}^{m\times p_2}$, $C_1 \in \mathbb{C}^{q_1\times n}$ and $C_2 \in \mathbb{C}^{q_2\times n}$ be given, and $X_1 \in \mathbb{C}^{p_1\times q_1}$, $X_2 \in \mathbb{C}^{p_2\times q_2}$, $Y \in \mathbb{C}^{p_3\times n}$ and $Z \in \mathbb{C}^{m\times q_3}$ be four variable matrices. Also, assume that $B_i \neq 0$ and $C_i \neq 0$, $i = 1, 2, 3$, and define*

$$\mathcal{S}_1 = \{A_1 + B_1X_1C_1 \mid X_1 \in \mathbb{C}^{p_1\times q_1}\}, \tag{2.25}$$

$$\mathcal{S}_2 = \{A_2 + B_2X_2C_2 + B_3Y + ZC_3 \mid X_2 \in \mathbb{C}^{p_2\times q_2},\ Y \in \mathbb{C}^{p_3\times n},\ Z \in \mathbb{C}^{m\times q_3}\}. \tag{2.26}$$

*Then, $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if*

$$r\begin{bmatrix} A_1 - A_2 & B_1 & B_2 & B_3 \\ C_3 & 0 & 0 & 0 \end{bmatrix} = r[B_2, B_3] + r(C_3), \qquad r\begin{bmatrix} A_1 - A_2 & B_3 \\ C_1 & 0 \\ C_2 & 0 \\ C_3 & 0 \end{bmatrix} = r\begin{bmatrix} C_2 \\ C_3 \end{bmatrix} + r(B_3). \tag{2.27}$$

**Proof.** It is obvious that $\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if the matrix equation

$$B_2X_2C_2 + B_3Y + ZC_3 = A_1 - A_2 + B_1X_1C_1$$

is solvable for $X_2$, $Y$ and $Z$, which by Lemma 1.6 is equivalent to

$$[I - (E_{B_3}B_2)(E_{B_3}B_2)^{\dagger}](E_{B_3}A_1F_{C_3} - E_{B_3}A_2F_{C_3} + E_{B_3}B_1X_1C_1F_{C_3}) = 0, \tag{2.28}$$

$$(E_{B_3}A_1F_{C_3} - E_{B_3}A_2F_{C_3} + E_{B_3}B_1X_1C_1F_{C_3})[I - (C_2F_{C_3})^{\dagger}(C_2F_{C_3})] = 0 \tag{2.29}$$

for all $X_1$. Applying Lemma 1.5 to (2.27) and (2.28) gives

$$[I - (E_{B_3}B_2)(E_{B_3}B_2)^{\dagger}][E_{B_3}A_1F_{C_3} - E_{B_3}A_2F_{C_3}, E_{B_3}B_1] = 0, \tag{2.30}$$

$$\begin{bmatrix} E_{B_3}A_1F_{C_3} - E_{B_3}A_2F_{C_3} \\ C_1F_{C_3} \end{bmatrix}[I - (C_2F_{C_3})^{\dagger}(C_2F_{C_3})] = 0, \tag{2.31}$$

where by Lemma 1.1,

$$\begin{aligned} & r([I-(E_{B_3}B_2)(E_{B_3}B_2)^{\dagger}][E_{B_3}A_1F_{C_3}-E_{B_3}A_2F_{C_3}, E_{B_3}B_1]) \\ & \quad = r[E_{B_3}B_1, E_{B_3}B_2, E_{B_3}A_1F_{C_3}-E_{B_3}A_2F_{C_3}] - r(E_{B_3}B_2) \\ & \quad = \begin{bmatrix} A_1-A_2 & B_1 & B_2 & B_3 \\ C_3 & 0 & 0 & 0 \end{bmatrix} - r[B_2, B_3] - r(C_3), \end{aligned} \tag{2.32}$$

and

$$\begin{aligned} & r\left(\begin{bmatrix} E_{B_3}A_1F_{C_3}-E_{B_3}A_2F_{C_3} \\ C_1F_{C_3} \end{bmatrix}[I-(C_2F_{C_3})^{\dagger}(C_2F_{C_3})]\right) \\ & \quad = r\begin{bmatrix} E_{B_3}A_1F_{C_3}-E_{B_3}A_2F_{C_3} \\ C_1F_{C_3} \\ C_2F_{C_3} \end{bmatrix} - r(C_2F_{C_3}) = r\begin{bmatrix} A_1-A_2 & B_3 \\ C_1 & 0 \\ C_2 & 0 \\ C_3 & 0 \end{bmatrix} - r\begin{bmatrix} C_2 \\ C_3 \end{bmatrix} - r(B_3). \end{aligned} \tag{2.33}$$

Setting both sides of (2.32) and (2.33) to zero leads to (2.27). □

## 3. Some applications

The results in Section 2 can be used for investigating various relations between solutions of linear matrix equations. For example, it is of interest to see whether the least-squares solution of the perturbed equation $(A+\delta A)X = B+\delta B$ is also the least-squares solution of the original equation $AX = B$. These problems can easily be solved by investigating the relations between the corresponding solution sets.

**Theorem 3.1.** *Let $A_1 \in \mathbb{C}^{m_1\times n}$, $A_2 \in \mathbb{C}^{m_2\times n}$, $B_1 \in \mathbb{C}^{m_1\times p}$, and $B_2 \in \mathbb{C}^{m_2\times p}$ be given, and assume that $A_1X_1 = B_1$ and $A_2X_2 = B_2$ are consistent, respectively. Also, let*

$$\mathcal{S}_1 = \{X_1 \in \mathbb{C}^{n\times p} \mid A_1X_1 = B_1\} \quad \text{and} \quad \mathcal{S}_2 = \{X_2 \in \mathbb{C}^{n\times p} \mid A_2X_2 = B_2\}.$$

*Then,*

$$\min_{X_1\in\mathcal{S}_1, X_2\in\mathcal{S}_2} r(X_1-X_2) = r\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} - r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \tag{3.1}$$

$$\max_{X_1\in\mathcal{S}_1} r(A_2X_1-B_2) = r\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} - r(A_1). \tag{3.2}$$

*Hence,*

(a) *$A_1X_1 = B_1$ and $A_2X_2 = B_2$ have a common solution if and only if $r\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} = r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$, i.e., $\mathscr{R}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$.*

(b) *$\mathcal{S}_1 \subseteq \mathcal{S}_2$ if and only if $r\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} = r(A_1)$, i.e., $\mathscr{R}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ and $\mathscr{R}(A_2^*) \subseteq \mathscr{R}(A_1^*)$.*

(c) *$\mathcal{S}_1 = \mathcal{S}_2$ if and only if $r\begin{bmatrix} A_1 & B_1 \\ A_2 & B_2 \end{bmatrix} = r(A_1) = r(A_2)$, i.e., $\mathscr{R}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}$ and $\mathscr{R}(A_2^*) = \mathscr{R}(A_1^*)$.*

**Proof.** Note from Lemma 1.3(a) that

$$\mathcal{S}_1 = \{A_1^{\dagger}B_1 + (I_n - A_1^{\dagger}A_1)V_1 \mid V_1 \in \mathbb{C}^{n\times p}\} \quad \text{and} \quad \mathcal{S}_2 = \{A_2^{\dagger}B_2 + (I_n - A_2^{\dagger}A_2)V_2 \mid V_2 \in \mathbb{C}^{n\times p}\}.$$

Applying Theorem 2.1 to these two sets and simplifying by using (1.1)–(1.3) leads to (3.1) and (3.2). □

**Corollary 3.2.** *Let $A, \delta A \in \mathbb{C}^{m\times n}$, $B, \delta B \in \mathbb{C}^{m\times p}$, be given, and assume that $AX = B$ and $(A+\delta A)Y = B+\delta B$ are both consistent. Then, every solution of $(A+\delta A)Y = B+\delta B$ is a solution of $AX = B$ if and only if $\mathscr{R}\begin{bmatrix} B \\ B+\delta B \end{bmatrix} \subseteq \mathscr{R}\begin{bmatrix} A \\ A+\delta A \end{bmatrix}$ and $\mathscr{R}(A^*) \subseteq \mathscr{R}[(A+\delta A)^*]$.*

**Corollary 3.3.** *Let $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{m\times p}$ and $M \in \mathbb{C}^{p\times m}$ be given, and assume that $AX = B$ is consistent. Also, let $\mathcal{S}_1 = \{X \in \mathbb{C}^{n\times p} \mid AX = B\}$ and $\mathcal{S}_2 = \{X \in \mathbb{C}^{n\times p} \mid MAX = MB\}$. Then, $\mathcal{S}_1 = \mathcal{S}_2$ if and only if $r(MA) = r(A)$.*

For any two matrix equations $A_1X_1 = B_1$ and $A_2X_2 = B_2$, where $X_1$ and $X_2$ have the same order, the corresponding two normal equations are $A_1^*A_1X_1 = A_1^*B_1$ and $A_2^*A_2X_2 = A_2^*B_2$. From Lemma 1.3(a), the general solutions of these two norm equations (least-squares solutions of $A_1X_1 = B_1$ and $A_2X_2 = B_2$) are given by

$$X_1 = (A_1^*A_1)^{\dagger}A_1^*B_1 + (I_n - A_1^{\dagger}A_1)V_1, \qquad X_2 = (A_2^*A_2)^{\dagger}A_2^*B_2 + (I_n - A_2^{\dagger}A_2)V_2,$$

where $V_1$ and $V_2$ are arbitrary. Relations between these two matrix expressions can be investigated by a similar approach.

Suppose that the two matrix equations $A_1X = B_1$ and $A_2X = B_2$ of the same size have a common solution. Then, the equation $(A_1 + A_2)X = B_1 + B_2$ is consistent as well. Also define

$$\mathcal{S}_3 = \{X \in \mathbb{C}^{n\times p} \mid A_1X = B_1 \text{ and } A_2X = B_2\}, \qquad \mathcal{S}_4 = \{X \in \mathbb{C}^{n\times p} \mid (A_1 + A_2)X = B_1 + B_2\}.$$

Then, it is easy to see that $\mathcal{S}_3 \subseteq \mathcal{S}_4$. In [7], the set equality $\mathcal{S}_3 = \mathcal{S}_4$ and its statistical applications were discussed. Notice that the elements in $\mathcal{S}_3$ and $\mathcal{S}_4$ can be written as two linear matrix expressions. We are able to give a new solution to this problem by the matrix rank method.

**Theorem 3.4.** *Let $A_1, A_2 \in \mathbb{C}^{m\times n}$, $B_1, B_2 \in \mathbb{C}^{m\times p}$ be given, and assume that $A_1X = B_1$ and $A_2X = B_2$ have a common solution. Then,*

$$\max_{X\in\mathcal{S}_4} r\left(\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} X - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\right) = \min\left\{ r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} - r(A_1 + A_2),\, p \right\}. \tag{3.3}$$

*Hence,*

(a) $\mathcal{S}_3 = \mathcal{S}_4$ *if and only if* $r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = r(A_1 + A_2)$, *i.e.*, $\mathscr{R}(A_i^*) \subseteq \mathscr{R}(A_1^* + A_2^*)$, $i = 1, 2$.

(b) *In particular, if* $\mathscr{R}(A_1^*) \cap \mathscr{R}(A_2^*) = \{0\}$, *then* $\mathcal{S}_3 = \mathcal{S}_4$.

**Proof.** Since $A_1X = B_1$ and $A_2X = B_2$ have a common solution, the two equations

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} X = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \text{and} \quad (A_1 + A_2)X = B_1 + B_2 \tag{3.4}$$

are consistent. Then by Lemma 1.3(a),

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}\begin{bmatrix} A_1 \\ A_2 \end{bmatrix}^{\dagger}\begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \quad \text{and} \quad (A_1 + A_2)(A_1 + A_2)^{\dagger}(B_1 + B_2) = B_1 + B_2 \tag{3.5}$$

hold. In these cases, the general solution of $(A_1 + A_2)X = B_1 + B_2$ can be written as

$$X = (A_1 + A_2)^{\dagger}(B_1 + B_2) + [I_n - (A_1 + A_2)^{\dagger}(A_1 + A_2)]U,$$

where $U$ is an arbitrary matrix. Let $A = A_1 + A_2$. Substituting this $X$ into the first equation in (3.4) and applying (1.4), we obtain

$$\begin{aligned}
\max_X r\left(\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} X - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}\right) &= \max_U r\left(\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} A^{\dagger}(B_1 + B_2) - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} + \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}(I_n - A^{\dagger}A)U\right) \\
&= \min\left\{ r\left[\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} A^{\dagger}(B_1 + B_2) - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}(I_n - A^{\dagger}A)\right], p \right\} \\
&= \min\left\{ r\begin{bmatrix} \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} A^{\dagger}(B_1 + B_2) - \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} & \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} \\ 0 & A \end{bmatrix} - r(A),\, p \right\} \quad \text{(by (1.2))}.
\end{aligned}$$

Simplifying the block matrix on the right-hand side by using (3.5) and block elementary matrix operations gives (3.3). □

Applying Theorem 3.4 to the sets of the least-squares solutions of the two equations $A_1X = B_1$ and $A_2X = B_2$ yields the following result.

**Corollary 3.5.** *Let $A_1, A_2 \in \mathbb{C}^{m\times n}$, $B_1, B_2 \in \mathbb{C}^{m\times p}$ be given, and assume that $A_1X = B_1$ and $A_2X = B_2$ have a common least-squares solution. Also, let*

$$\mathcal{S}_5 = \{X \mid A_1^*A_1X = A_1^*B_1,\, A_2^*A_2X = A_2^*B_2\}, \qquad \mathcal{S}_6 = \{X \mid (A_1^*A_1 + A_2^*A_2)X = A_1^*B_1 + A_2^*B_2\}.$$

*Then $\mathcal{S}_5 = \mathcal{S}_6$.*

More results on relations of solution sets of linear matrix equations can be derived. For instance, let

$$\mathcal{S}_7 = \{X \in \mathbb{C}^{n\times p} \mid AXB = C\}, \qquad \mathcal{S}_8 = \{X_1 + X_2 \in \mathbb{C}^{n\times p} \mid A_1X_1B_1 = C_1,\, A_2X_2B_2 = C_2\}. \tag{3.6}$$

Then, both sets can be written as in (2.11) and (2.12) by using Lemma 1.3(a). Applying Theorem 2.3 to (3.6) yields the following known result.

**Corollary 3.6** ([8], *Theorem 2.1*). *Assume that the three matrix equations in (3.6) are consistent. Also, define*

$$D = \begin{bmatrix} C & 0 & 0 \\ 0 & -C_1 & 0 \\ 0 & 0 & -C_2 \end{bmatrix}, \qquad P = \begin{bmatrix} A & A \\ A_1 & 0 \\ 0 & A_2 \end{bmatrix}, \qquad Q = \begin{bmatrix} B & B_1 & 0 \\ B & 0 & B_2 \end{bmatrix}.$$

*Then,*

(a) $\mathcal{S}_7 \cap \mathcal{S}_8 \neq \emptyset$ *if and only if* $r\begin{bmatrix} D & P \\ Q & 0 \end{bmatrix} = r(P) + r(Q)$, *or equivalently* , $E_P D F_Q = 0$.

(b) $\mathcal{S}_7 \supseteq \mathcal{S}_8$ *if and only if* $A = 0$, *or* $B = 0$, *or* $r\begin{bmatrix} D & P \\ Q & 0 \end{bmatrix} = r(A_1) + r(A_2) + r(B_1) + r(B_2)$, *or equivalently*, $\mathscr{R}(A^*) \subseteq \mathscr{R}(A_i^*)$, $\mathscr{R}(B) \subseteq \mathscr{R}(B_i)$ *and* $C = AA_1^{\dagger}C_1B_1^{\dagger}B + AA_2^{\dagger}C_2B_2^{\dagger}B$, $i = 1, 2$.

(c) $\mathcal{S}_7 \subseteq \mathcal{S}_8$ *if and only if* $\mathscr{R}(A_1^*) \cap \mathscr{R}(A_2^*) = \{0\}$, *or* $\mathscr{R}(B_1) \cap \mathscr{R}(B_2) = \{0\}$, *or*

$$r\begin{bmatrix} D & P \\ Q & 0 \end{bmatrix} = r\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} + r[B_1, B_2] + r(A) + r(B).$$

Consequences of Corollary 3.6 and their applications were also given in [8]. In addition to equalities for matrices in matrix sets generated from linear matrix expressions, we may also consider other kinds of relations for matrices in the matrix sets. For example, for two matrix sets $\mathcal{S}_1$ and $\mathcal{S}_2$ generated from linear matrix expressions, it would be of interest to give necessary and sufficient conditions for

(a) the existence of $X_1 \in \mathcal{S}_1$ and $X_2 \in \mathcal{S}_2$ such that $X_1^*X_2 = 0$, namely, $X_1$ and $X_2$ are orthogonal;
(b) the existence of $X_1 \in \mathcal{S}_1$ and $X_2 \in \mathcal{S}_2$ and some number $\lambda$ such that $X_1 = \lambda X_2$, namely, $X_1$ and $X_2$ are parallel.

These two problems can be solved by deriving certain closed-form formulas for the minimal ranks of $X_1^*X_2$ and $X_1 - \lambda X_2$ first, and then by setting the rank formulas equal to zero.

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