Positive solutions for large elliptic systems of interacting species groups by cone index methods

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Received 22 November 2002
Submitted by C.V. Pao

Abstract

This article considers the existence of positive solutions for various systems of four nonlinear coupled elliptic partial differential equations subjected to zero Dirichlet boundary conditions. In applications, the components can be interpreted as concentrations of four interacting populations or chemicals. The species are divided into two groups which interact in prey-predator or cooperating Volterra–Lotka type of relations. Within each group, the species can interact in other possibilities. Topological cone index method is used for proving the existence of positive coexistence solutions. Sufficient conditions are found in terms of the signs of the principal eigenvalues of various simple related operators.

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1. Introduction

This article considers the existence of positive solutions for various systems of four nonlinear coupled elliptic partial differential equations subjected to zero Dirichlet boundary condition. In applications, the four components can be interpreted as concentrations of four interacting chemicals or populations. The four species can interact nonlinearly in many different ways, leading to the classifications in Sections 2 and 5. The positive solutions represent coexistence of all four species in equilibrium with each other. In [20,22], many cases are studied for three interacting species. Topological cone index method (cf. [1, 5,19]) is used for proving the existence of positive solutions. The method for calculating the cone indices of the mappings for three components in [20,22] does not apply immediately.
for the case of four components in this present article. The methods of calculation of the cone indices are extended to the case for four components in Section 4 of this paper. The extended methods are applied to study the various cases in Sections 2, 3, and 5. Sufficient conditions are found for the existence of solutions with each component positive.

In this paper, the species are divided into two groups, with a pair of species within each group. In Section 2, we assume that the two groups interact with a predator-prey relation. Each species in the first group is a predator for the prey-species in the second group. Within each group, the pair of species interact with each other in a competitive (or cooperative) manner. On the microscopic scale of immunology, for example, killer and helper T lymphocytes stimulate each others growth and proliferation through chemical mediators. They both directly or indirectly eliminate bacteria or viruses, which may compete for resources such as host cellular products and proteins (cf. [8,10]). On the macroscopic scale, we find fierce cooperating animals or people prey on less aggressive animals or people, which may compete among each other. Section 3 proves the theorems stated in Section 2, using index calculation methods explained carefully in Section 4. In Section 5, we assume that the two groups interact with a cooperating relation. Within each group, the pair of species interact in a competing or cooperating manner. For simplicity in this paper, we assume the interaction terms are of Volterra–Lotka type, which is common in many biological applications.

Positive solutions for this type of system with Dirichlet boundary condition are first found by upper-lower solution method in [15]. Studies of positive solutions for this type of system by cone index methods are made by many authors for two equations in, e.g., [4–7,19,21,23] and for three equations in [12,20,22,26]. Studies in theory and application for larger systems by various methods are made in, e.g., [9,11,13,14,16–18,24–27]. A systematic investigation of the existence of positive solutions by means of cone index method for systems with more than three equations should be of value in future research of complex biological models. The results give elegant conditions, in terms of the spectral property of simpler appropriately related operators on only one component, for the existence of positive coexistence states for the full system. Furthermore, the method of analysis here can be extended to include other boundary conditions and reactions more general than Volterra–Lotka type.

More precisely, we consider the system of elliptic equations

$$\Delta u_i + u_i \left( e_i + \sum_{j=1}^{4} a_{ij} u_j \right) = 0 \quad \text{in } \Omega; \quad u_i = 0 \quad \text{on } \partial \Omega,$$

(1.1)

where $e_i$ and $a_{ij}$ are constants satisfying

$$a_{ii} < 0 \quad \text{and} \quad e_i > 0 \quad \text{for } i = 1, 2, 3, 4.$$  

(1.2)

The constants $e_i$ and $a_{ii}, i = 1, \ldots, 4$, are the intrinsic growth rates and crowding effects of the corresponding species. The constants $a_{ij}, i \neq j$, are the interaction rates, whose signs will satisfying various assumptions according to the cases considered by the particular theorems. $\Omega$ is a bounded domain in $\mathbb{R}^N$ with smooth boundary. A solution of problem (1.1) is called positive if each component is not identically zero and nonnegative in $\overline{\Omega}$. 
2. Main theorems for predator-prey groups with competition or cooperation within the groups

We divide the species into two groups. Group I consists of the species \( m = 1, 2 \) and group II consists of species \( n = 3 \) and 4. In this section, we assume that groups I and II have a predator-prey relationship, with species in I as predators and species in II as prey. More precisely, we assume in this section that

\[
[C1] \quad a_{m3} \text{ and } a_{m4} \geq 0, \quad \text{for } m = 1, 2; \quad a_{n1} \text{ and } a_{n2} \leq 0, \quad \text{for } n = 3, 4.
\]

Within the two groups, we will consider 4 different cases. In the first case, the species in group I form a cooperating pair, and in group II also form a cooperating pair. More precisely, we assume

\[
[A1] \quad a_{12} \text{ and } a_{21} \geq 0; \quad a_{34} \text{ and } a_{43} \geq 0.
\]

In the second case, we assume species in group I form a cooperating pair, while in group II form a competing pair. That is

\[
[A2] \quad a_{12} \text{ and } a_{21} \geq 0; \quad a_{34} \text{ and } a_{43} \leq 0.
\]

In the third case, we assume species in group I form a competing pair, while in group II form a cooperating pair. That is

\[
[A3] \quad a_{12} \text{ and } a_{21} \leq 0; \quad a_{34} \text{ and } a_{43} \geq 0.
\]

Finally in the fourth case, we assume species in group I form a competing pair, while in group II also form a competing pair. That is

\[
[A4] \quad a_{12} \text{ and } a_{21} \leq 0; \quad a_{34} \text{ and } a_{43} \leq 0.
\]

Let \( c \) be a function defined on \( \Omega \), we will use the symbol \( \lambda_1(\Delta + c) \) to denote the first eigenvalue for the eigenvalue problem: \( \Delta u + cu = \lambda u \) in \( \Omega \), \( u = 0 \) on \( \partial \Omega \). For each \( i = 1, \ldots, 4 \), if \( \lambda_1(\Delta + \epsilon_i) > 0 \), we will use \( u_i^0 \) to denote the unique positive solution of the problem: \( \Delta u_i^0 + u_i^0[\epsilon_i + a_{ij}u_j^0] = 0 \) in \( \Omega \), \( u_i^0 = 0 \) on \( \partial \Omega \). Moreover, let \( y_i := (0, \ldots, u_i^0, \ldots, 0) \) where each of the four component is zero except the \( i \)th component as shown. For convenience, we define the following expressions:

\[
B_1^1 = [a_{11}a_{22} - a_{12}a_{21}]^{-1}[(e_1 + a_{14}e_4/|a_{44}|)a_{22}] + (e_2 + a_{24}e_4/|a_{44}|)a_{12}],
\]

\[
B_1^2 = [a_{11}a_{22} - a_{12}a_{21}]^{-1}[(e_1 + a_{14}e_4/|a_{44}|)a_{21}] + (e_2 + a_{24}e_4/|a_{44}|)a_{11}],
\]

\[
B_1^3 = [a_{11}a_{22} - a_{12}a_{21}]^{-1}[(e_1 + a_{13}e_3/|a_{33}|)a_{22}] + (e_2 + a_{23}e_3/|a_{33}|)a_{12}],
\]

\[
B_1^4 = [a_{11}a_{22} - a_{12}a_{21}]^{-1}[(e_1 + a_{13}e_3/|a_{33}|)a_{21}] + (e_2 + a_{23}e_3/|a_{33}|)a_{11}].
\]
The following theorem gives sufficient conditions for the coexistence of positive solution of problem (1.1) when the predators cooperate while the preys may either cooperate or compete. In Theorem 2.1, we will see in the proof that for each predator $i = 1, 2$, $B^i_4$ or $B^i_3$, respectively represents a bound for the $u_i$ when the prey $u_3$ or $u_4$ is absent.

**Theorem 2.1** (Cooperating predators with preys which cooperate or compete).

(i) Assume interaction relations $[C1]$ and $[A1]$. Suppose that

$$a_{11}a_{22} > a_{12}a_{21} \quad \text{and} \quad a_{33}a_{44} > a_{34}a_{43},$$

then problem (1.1) has a positive solution if the following conditions are satisfied:

- **(2.1)** $\lambda_1(\Delta + e_1) > 0, \quad \lambda_1(\Delta + e_2) > 0,$
- **(2.2)** $\lambda_1(\Delta + e_3 - \big(|a_{31}|B^1_4 + |a_{32}|B^2_3\big)) > 0,$ and
- **(2.3)** $\lambda_1(\Delta + e_4 - \big(|a_{41}|B^2_3 + |a_{42}|B^2_4\big)) > 0.$

(ii) Assume interaction relations $[C1]$ and $[A2]$. Suppose that

$$a_{11}a_{22} > a_{12}a_{21},$$

then problem (1.1) has a positive solution if the following conditions are satisfied:

- **(2.5)** $\lambda_1(\Delta + e_1) > 0, \quad \lambda_1(\Delta + e_2) > 0,$
- **(2.6)** $\lambda_1(\Delta + e_3 - \big(|a_{31}|B^1_4 + |a_{32}|B^2_3\big) - \big|a_{34}\big||e_4|/|a_{44}|) > 0,$ and
- **(2.7)** $\lambda_1(\Delta + e_4 - \big(|a_{41}|B^2_3 + |a_{42}|B^2_4\big) - |a_{43}||e_3|/|a_{33}|) > 0.$

The next theorem gives sufficient conditions for the coexistence of positive solution for problem (1.1) when the predators compete while the prey may cooperate or compete. For convenience of stating the theorem, we define the following expressions:

- $\hat{B}^1_4 = |a_{11}|^{-1}\bigg(1 + a_{14}a_{44}/|a_{44}|\bigg),$ \quad $\hat{B}^2_4 = |a_{22}|^{-1}\bigg(1 + a_{24}a_{44}/|a_{44}|\bigg),$ \quad $\hat{B}^1_3 = |a_{11}|^{-1}\bigg(1 + a_{13}a_{33}/|a_{33}|\bigg),$ \quad $\hat{B}^2_3 = |a_{22}|^{-1}\bigg(1 + a_{23}a_{33}/|a_{33}|\bigg),$ \quad $K_3 = [a_{33}a_{44} - a_{34}a_{43}]^{-1}\bigg(1 + a_{34}a_{44}/|a_{44}|\bigg),$ \quad $K_4 = [a_{33}a_{44} - a_{34}a_{43}]^{-1}\bigg(1 + a_{34}a_{44}/|a_{44}|\bigg).$

**Theorem 2.2** (Competing predators with preys which cooperate or compete).

(i) Assume interaction relations $[C1]$ and $[A3]$. Suppose that

$$a_{33}a_{44} > a_{34}a_{43},$$

then problem (1.1) has a positive solution if the following conditions are satisfied:
\[ \lambda_1 (\Delta + e_1 - |a_{12}a_{22}^{-1}|[e_2 + a_{23}K_3 + a_{24}K_4]) > 0, \]  
\[ \lambda_1 (\Delta + e_2 - |a_{21}a_{11}^{-1}|[e_1 + a_{13}K_3 + a_{14}K_4]) > 0, \]  
\[ \lambda_1 (\Delta + e_3 - (|a_{31}\tilde{B}_1^3 + |a_{32}\tilde{B}_2^2|)) > 0, \text{ and} \]  
\[ \lambda_1 (\Delta + e_4 - (|a_{41}\tilde{B}_3^1 + |a_{42}\tilde{B}_4^2|)) > 0. \]  

(ii) Assume interaction relations \([C1]\) and \([A4]\). Then problem (1.1) has a positive solution if the following conditions are satisfied:

\[ \lambda_1 (\Delta + e_1 - |a_{12}a_{22}^{-1}|[e_2 + a_{23}e_3/|a_{33}| + a_{24}e_4/|a_{44}|]) > 0, \]  
\[ \lambda_1 (\Delta + e_2 - |a_{21}a_{11}^{-1}|[e_1 + a_{13}e_3/|a_{33}| + a_{14}e_4/|a_{44}|]) > 0, \]  
\[ \lambda_1 (\Delta + e_3 - (|a_{31}\tilde{B}_1^3 + |a_{32}\tilde{B}_2^2|) - |a_{34}e_4/|a_{44}|) > 0, \]  
\[ \lambda_1 (\Delta + e_4 - (|a_{41}\tilde{B}_3^1 + |a_{42}\tilde{B}_4^2|) - |a_{43}e_3/|a_{33}|) > 0. \]

3. Proof of main theorems in Section 2

In this section, we will prove the theorems stated in the last section. In the proofs, we will use indices of various mappings from the cone of nonnegative functions into itself. In order to emphasize the main ideas of the proof of the present theorems, the details for calculating these indices are explained later in Section 4. The following lemma is needed for the proof of Theorem 2.1. It gives a priori bounds for two cooperative species under appropriate conditions.

**Lemma 3.1.** Consider the following Dirichlet problem:

\[ \Delta v_i + v_i \left( b_i + g_i(x) + \sum_{j=1}^{2} c_{ij}v_j \right) = 0 \quad \text{in } \Omega; \quad v_i = 0 \quad \text{on } \partial \Omega, \quad \text{for } i = 1, 2, \]  

where \( g_i(x) \) are nonpositive continuous functions on \( \Omega \) and \( b_i, c_{ij} \) are constants satisfying \( b_i > 0, c_{ii} < 0 \) for \( i = 1, 2, c_{12} \geq 0, c_{21} \geq 0 \). Suppose that \( c_{11}c_{22} > c_{12}c_{21} \). Then any positive solution \((v_1, v_2)\), with \( v_i \in C^1(\overline{\Omega}), i = 1, 2 \), must satisfy:

\[ v_1 \leq \left[ b_1c_{21} + b_2c_{12} \right] \left/ \left| c_{11}c_{22} - c_{12}c_{21} \right| \right., \quad \text{and} \]  
\[ v_2 \leq \left[ b_1c_{21} + b_2c_{12} \right] \left/ \left| c_{11}c_{22} - c_{12}c_{21} \right| \right. \quad \text{in } \Omega. \]

**Proof.** On the \( x-y \) plane, the two lines \( b_i + c_{1i}x + c_{2i}y = 0, i = 1, 2 \), intersect at \((x_0, y_0)\) where

\[ x_0 := (-b_1c_{22} + b_2c_{12})/(c_{11}c_{22} - c_{12}c_{21}), \]
\[ y_0 := (b_1c_{21} - b_2c_{11})/(c_{11}c_{22} - c_{12}c_{21}). \]

The assumptions on \( b_i \) and \( c_{ij} \) of this lemma implies that \( x_0 \) and \( y_0 \) are positive. Let \( \lambda \) be a positive number satisfying \( c_{21}/|c_{22}| < \lambda < |c_{11}|/c_{12} \). (Here, \(|c_{11}|/c_{12}\) is replaced with
+∞ if c_{12} = 0.) For each δ > 0, the pair of constant functions v^δ_1 := x_0 + δ, v^δ_2 := y_0 + kδ on $\overline{\Omega}$ satisfy
\[
\Delta v^\delta_i + v^\delta_i \left[ b_i + g_i(x) + c_{1i}v^\delta_i + c_{12}v^\delta_2 \right] < 0 \quad \text{in } \Omega,
\]
\[v^\delta_i > 0 \quad \text{on } \partial \Omega, \text{ for } i = 1, 2.
\]
That is, they form a family of coupled upper solutions for problem (3.1). For $M > 0$ sufficiently large, the positive solution $(v_1, v_2)$ of problem (3.1) satisfies $v_i(x) < v^M_i$, $i = 1, 2$. Let $J := \{ \delta \in (0, M] : \text{for both } i = 1, 2, v_i(x) < v^i_1 \text{ for all } x \in \overline{\Omega} \}$. Suppose the set $J$ has a positive glb $\delta > 0$; and let there be a point $x \in \Omega$ where $v_i(x) = v^i_1$ for some $i$. We may assume, without loss of generality, that $i = 1$. For $x \in \Omega$, $u > 0$, define $f_I(x, u) = u(b_1 + g_1(x) + c_{11}u)$; and let $P$ be a large positive constant such that $|\partial f_1/\partial u| < P$ for all $x \in \overline{\Omega}$, $0 \leq u \leq v^M_1$. Consider the expression
\[
\Delta (v_1(x) - v^1_1) - P(v_1(x) - v^1_1)
\]
\[= \Delta v_1(x) + v_1(x) \left[ b_1 + g_1(x) + c_{11}v_1(x) + c_{12}v_2(x) \right] - \left[ \Delta v^1_1 + v^1_1 \left[ b_1 + g_1(x) + c_{11}v^1_1 + c_{12}v^2_2 \right] \right] + f_1(x, v^1_1) + v^1_1 c_{12}v^2_2
\]
\[= f_1(x, v_1(x)) - v_1(x) c_{12}v_2(x) - P(v_1(x) - v^1_1) > 0 \quad \text{for all } x \in \Omega.
\]
The last inequality is true due to the facts that $(v^1_1, v^2_2)$ is a strict upper solution, $c_{12} > 0$, and the choice of large $P$. The maximum principle asserts that $v_1(x) = v^1_1$ on $\overline{\Omega}$. This contradiction implies that the positive glb $\delta$ can be reduced, and cannot be positive. Thus, we must have $v_1(x) \leq x_0$ and $v_2(x) \leq y_0$ in $\overline{\Omega}$. □

**Remark 3.1.** Note that the proof of the above lemma uses an extension of a sweeping principle to quasimonotone nondecreasing elliptic systems. The arguments are similar to [13, Theorem 1.4-3] and [17, Lemma 2.1].

We will next prove Theorem 2.1(i) by the following procedure. Under the hypotheses of this part, we use Lemma 3.1 to obtain a bound for all nonnegative solutions of (3.2) below. We thus define a bounded set $D$ in $[C^+_{\Omega}(\overline{\Omega})]^d$ containing all solutions of (3.2). Then we define various subsets of $D$ containing solutions with certain components identically zero. The solutions will be fixed points of appropriate positive compact mappings on $D$. We will show that the index of the mapping on $D$ is equal to one, by homotopy invariance and normalization property (cf. [1]). By appropriate deformations and homotopic invariance principle again we will show that the indices are zero on the various subsets of $D$ described above. By the additive property of the indices of the maps on disjoint open subsets (cf. [1]), we will conclude by index formula (3.5) below that there must exist a solution of (1.1) with each component positive.

**Proof of Theorem 2.1(i).** Assume [C1], [A1] and (2.1) to (2.4). Consider any nonnegative solutions of the problem
\[-\Delta u_i = \theta_i u_i \left[ e_i + \sum_{j=1}^{4} a_{ij} u_j \right] \quad \text{in } \Omega,\]

\[u_i = 0 \quad \text{on } \partial \Omega, \quad \text{for } i = 1, \ldots, 4,\]

(3.2)

for \( \theta_i \in (0, 1) \). By Lemma 3.1, the second part of \([C1]\), and second part of (2.1), we have

\[
\begin{align*}
\{ & u_3 \leq \left[ e_3 [a_{44}^3 + e_4 a_{34}^3] \right]/[a_{33} a_{44} - a_{34} a_{43}] := \bar{u}_3, \\
& u_4 \leq \left[ e_4 a_{43} + e_4 [a_{34}] \right]/[a_{33} a_{44} - a_{34} a_{43}] := \bar{u}_4. 
\end{align*}
\]

(3.3)

For \( i = 1, 2, \) let

\[
\begin{align*}
b_i &= \theta_i [a_i^3 u_3 + a_i^4 u_4], \\
g_i(x) &= \theta_i [a_i^3 u_3 + a_i^4 u_4 + a_i^3 \bar{u}_3 + a_i^4 \bar{u}_4] \leq 0 \\
\text{and} \quad c_{ij} &= a_{ij},
\end{align*}
\]

we apply Lemma 3.1 again to obtain an uniform bound for \( u_1 \) and \( u_2 \), for \( \theta_i \in (0, 1) \).

Note that we have made use of the second condition of \([C1]\) and first condition of (2.1). In case \( \theta_1 \) (or \( \theta_4 \)) is equal to zero, \( u_3 \) (or \( u_4 \)) will be the trivial function, and \( u_4 \) (or \( u_3 \)) will be bounded above by \( e_4/[a_{44}] \) (or \( e_4/[a_{33}] \)). The subsequent bound on \( u_1 \) and \( u_2 \) will be again established by Lemma 3.1 as before if both \( \theta_1 \) and \( \theta_2 \) are positive. If one of the \( \theta_1 \) or \( \theta_2 \) is zero, the corresponding \( u_1 \) or \( u_2 \) will be the trivial function, and the bound on the other component can be established by the corresponding scalar equation uniformly for \( \theta_i \in [0, 1] \). In any case, there is a constant \( M > 0 \) such that all components of all nonnegative solutions of (3.2) must have values in \([0, M]\), uniformly for \( \theta_i \in [0, 1], \) \( i = 1, 2, 3, 4 \).

For any \( t > 0 \), let \( E(t) := \{ u \in C(\overline{\Omega}) : |u| < t \} \), and \( \overline{E(t)} \) denotes its closure. For \( \theta_i \in (0, 1), \) \( i = 1, 2, 3, 4 \) and \( P > 0 \), define the operator \( A_{\theta_1 \theta_2 \theta_3 \theta_4} : [C(\overline{\Omega})]^4 \cap \overline{E(M)}^4 \rightarrow [C(\overline{\Omega})]^4 \) by

\[
A_{\theta_1 \theta_2 \theta_3 \theta_4} (u_1, \ldots, u_4) = (v_1, \ldots, v_4),
\]

where

\[v_i := (-\Delta + P)^{-1} \left( \theta_i u_i \left[ e_i + \sum_{j=1}^{4} a_{ij} u_j \right] + Pu_i \right),\]

Here, the inverse operator is taken with homogeneous Dirichlet boundary condition on \( \partial \Omega \).

We can take \( P \) sufficiently large so that the operator \( A_{\theta_1 \theta_2 \theta_3 \theta_4} \) is positive, compact and Frechet differentiable on \([C^+(\overline{\Omega}))]^4 \cap \overline{E(M)}^4 \). Let \( D := \{ C^+(\overline{\Omega}))]^4 \cap \overline{E(M)}^4 \), the bound on the solutions implies that these operators has no fixed point on \( \partial D \) (in the relative topology). We can further use a familiar cut-off procedure to extend \( A_{\theta_1 \theta_2 \theta_3 \theta_4} \) to be defined outside \( D \) as a compact positive mapping from the cone \( K := \{ C^+(\overline{\Omega}))]^4 \) into itself (cf. [11,22]). For convenience, we will denote \( i(A_{\theta_1 \theta_2 \theta_3 \theta_4}, y) = \text{index}(A_{\theta_1 \theta_2 \theta_3 \theta_4}, y, K) \) for a fixed point \( y \) of the map in \( K \), and denote \( i(A_{\theta_1 \theta_2 \theta_3 \theta_4}, D) = \text{index}(A_{\theta_1 \theta_2 \theta_3 \theta_4}, K, D) \).

Let \( A := A_{1111} \), we obtain by homotopy invariance that the cone indices of the mappings satisfy \( i(A, D) = i(A_{1111}, D) = i(A_{0000}, D) \). From definition, the \( i \)th component of \( A_{0000}(u_1, \ldots, u_4) \) is \((\Delta + P)^{-1}(Pu_i)\). One readily verifies that \( A_{0000}(u) \neq \lambda u \) for every \( u \in \partial D \) and every \( \lambda \geq 1 \). Hence by [1, Lemma 12.1], we conclude that \( i(A, D) = i(A_{0000}, D) = 1 \).
We will next define various subsets of $D$ containing solutions with some components identically zero, and then proceed to calculate the indices of the mapping $A$ on these subsets. For $i = 1, \ldots, 4, \epsilon \in (0, M)$, let $D^\epsilon_i := \{ u = (u_1, \ldots, u_4) : u \in K, 0 \leq u_i < \epsilon, 0 \leq u_j < M \text{ for } j \neq i \}$, which is a “slice” in $D$ containing all fixed point of $A$ with small $i$th component. Let $D^\epsilon_{i,j} := D_i^\epsilon \cup D_j^\epsilon$, and $\bar{D}_{i,j}^\epsilon := D_i^\epsilon \cap D_j^\epsilon$. Note that

$$\partial D_{i,j}^\epsilon = \{ u \in D : \min\{\|u_i\|, \|u_j\|\} = \epsilon, \text{ or } \min\{\|u_i\|, \|u_j\|\} \leq \epsilon \text{ and } \max\{|u_k| : 1 \leq k \leq 4\} = M \},$$

and

$$\partial \bar{D}_{i,j}^\epsilon = \{ u \in D : \max\{\|u_i\|, \|u_j\|\} = \epsilon, \text{ or } \max\{\|u_i\|, \|u_j\|\} \leq \epsilon \text{ and } \max\{|u_k| : k \neq i \text{ and } j \} = M \}.$$  

For convenience, we will use the notation

$$f_i(u_1, u_2, u_3, u_4) = e_i + \sum_{j=1}^4 a_{ij} u_j \quad \text{for } i = 1, 2, 3, 4. \quad (3.4)$$

Consider the mapping $A_{0110}$ in $D$. Suppose there exists a sequence of fixed points $(u^1_n, u^2_n, u^3_n, u^4_n)$, $n = 1, 2, 3, \ldots$ of the map $A_{0110}$, with $u^i_0 \neq 0$ or $u^i_0 = 0$. We have $\Delta u^1_n + u^2_n f_2(u^1_n, u^2_n, u^3_n, u^4_n) = 0$ in $\Omega$, $u^1_n = 0$ on $\partial \Omega$, $n = 1, 2, 3$. If both $u^1_n$ and $u^2_n \to 0$ in $C(\overline{\Omega})$, the equation for $u^4_n$ implies that $u^4_n \leq [e_4/|u_{44}|] + \delta$ for some small $\delta > 0$, provided $n$ is sufficiently large. (Note that $a_{41}$ and $a_{42}$ are $\leq 0$.) The equations for $u^1_n$, $u^2_n$ and Lemma 3.1 then imply that $u^1_n \leq B^1_1 + \delta$, $u^2_n \leq B^2_2 + \delta$ for any small $\delta > 0$, provided $n$ is sufficiently large. Thus for $\delta > 0$ sufficiently small, we have

$$f_3(u^1_n, u^2_n, u^3_n, u^4_n) > e_3 - \|a_{33}\|B^2_1 + |a_{32}|B^2_2 - \delta$$

for $n$ sufficiently large. By assumption (2.3), we find $\lambda_1(\Delta + f_3(u^1_n, u^2_n, u^3_n, u^4_n)) > 0$, and the equation for $u^4_n$, which is in $K$, implies that $u^4_n \equiv 0$ for all large $n$. (Note that (2.3) also implies that $e_3 > (|a_{33}|B^2_1 + |a_{32}|B^2_2)$.) Then, we have $\Delta u^1_n + u^2_n f_2(u^1_n, u^2_n, u^3_n, u^4_n) = 0$ in $\Omega$, and the second condition in (2.2) implies that $\lambda_1(\Delta + f_2(u^1_n, u^2_n, u^3_n, u^4_n)) > 0$ for $n$ sufficiently large. Thus we have $u^3_n \equiv 0$ for large $n$ too. This contradicts the assumptions above on $u^3_n$ and $u^4_n$. Consequently, the number

$$t := \inf\{\max\{\|u_2\|, \|u_3\|\} : (u_2, u_3) \neq (0, 0), \text{ where } (u_1, u_2, u_3, u_4) \text{ is a fixed point of } A_{0110} \text{ in } D \text{ for some } \theta \in [0, 1]\}$$

must satisfy $t > 0$. Choosing $\epsilon \in (0, t)$, we obtain by Lemma 4.3 in Section 4 below that $i(A, \bar{D}^\epsilon_{2,3}) = 0$. (The lemma indicates that all the fixed points of $A_{0110}$ in $\bar{D}^\epsilon_{2,3}$ have both their 2nd and 3rd components identically zero, and $A = A_{1111}$ can be deformed into $A_{0110}$ by homotopy in $\bar{D}^\epsilon_{2,3}$. It then shows that the only fixed point of $A_{0110}$ in $\bar{D}^\epsilon_{2,3}$ is $(0, 0, 0, 0)$, whose index is 0.)

Consider the mapping $A_{1100}$ in $D$. Suppose that there exists a sequence of fixed points $(u^1_n, u^2_n, v^3_n, v^4_n)$, $n = 1, 2, 3, \ldots$ of the map $A_{1100}$, with both $u^i_n \neq 0$ and $v^i_n \neq 0$. The signs of $a_{1j}$ implies that $\lambda_1(\Delta + f_1(0, u^1_n, v^2_n, v^4_n)) > \lambda_1(\Delta + e_1)$ which is $> 0$.
by assumption (2.2). We can obtain by comparison from the equation satisfied by \( v_n^1 \) that they are uniformly bounded away from zero by a positive function, and thus cannot tend to zero as \( n \) tends to infinity. Similarly, we find \( \lambda_1(\Delta + f_2(u_1^1, 0, v_1^2, v_2^2)) > \lambda_1(\Delta + e_2) > 0 \), and deduce \( v_n^2 \) also cannot tend to zero as \( n \) tends to infinity. Consequently, the number

\[
t^* := \inf \{ ||u_1||, ||u_2|| : \text{ both } u_1 \not= 0 \text{ and } u_2 \not= 0, \text{ where } (u_1, u_2, u_3, u_4) \text{ is a fixed point of } A_{1100} \text{ in } D, \text{ some } \theta \in [0, 1] \}
\]

must satisfy \( t^* > 0 \). Further, the signs of \( a_{1j}, a_{2j} \) and assumptions (2.2) imply that \( \lambda_1(\Delta + f_1(0, u_1^1, 0, 0)) > 0 \) and \( \lambda_1(\Delta + f_2(u_1^1, 0, 0, 0)) > 0 \). Choosing \( \epsilon \in (0, t^*) \), we can thus obtain by Lemma 4.2 below that \( i(A, D_{1,2}^\epsilon) = 0 \). (Lemma 4.2 shows that \( A \) can be deformed in \( D_{1,2}^\epsilon \) to \( A_{1100} \) which has in \( D_{1,2}^\epsilon \) three fixed points, all with index zero.)

Consider the mapping \( A_{0011} \) in \( D \). Suppose there exists a sequence of fixed points \((w_n^1, w_n^2, w_n^3, w_n^4)\), \( n = 1, 2, 3, \ldots \), of the map \( A_{0011} \) in \( D \), \( \theta_0 \in [0, 1] \), with both \( w_n^3 \not= 0 \) and \( w_n^4 \not= 0 \). If \( w_n^3 \rightarrow 0 \), the equation for \( w_n^3 \) implies that \( w_n^4 \leq e_4/|u_{44}| + \delta \) for any small \( \delta > 0 \) provided that \( n \) is large enough. The equations for \( w_n^3 \) and \( w_n^4 \) and Lemma 3.1 then imply that \( w_n^3 \leq B_1^4 + \delta, \ w_n^4 \leq B_2^4 + \delta \) for any small \( \delta \) provided \( n \) is large enough. (Note that \( B_4^1 \) is a bound for \( u_1, i = 1, 2, \) with \( u_4 \) as the only prey, i.e., with \( u_3 \) absent.) Thus for \( \delta > 0 \) sufficiently small, we have

\[
f_3(w_n^1, \ldots, w_n^4) > e_3 - (|a_{11}|B_1^4 + |a_{22}|B_2^4) - \delta > 0,
\]

for \( n \) sufficiently large. Hence we obtain \( \lambda_1(\Delta + f_3(w_n^1, w_n^2, w_n^3, w_n^4)) > 0 \) by assumption (2.3), and the equation for \( w_n^3 \) implies that \( w_n^3 \equiv 0 \) for all large \( n \). This contradicts \( w_n^3 \not= 0 \), and thus \( w_n^3 \) cannot tend to zero as \( n \) tends to infinity. On the other hand, if \( w_n^4 \rightarrow 0 \), the equation for \( w_n^3 \) implies that \( w_n^3 \leq e_3/|u_{33}| + \delta \) for small \( \delta > 0 \) and \( n \) large enough. We continue to deduce in a symmetric way that \( \lambda_1(\Delta + f_4(w_n^1, w_n^2, w_n^3, w_n^4)) > 0 \) by assumption (2.4), leading to \( w_n^3 \equiv 0 \) for all large \( n \). We again conclude by contradiction that \( w_n^3 \) cannot tend to zero. Consequently, the number

\[
t^{**} := \inf \{ ||u_3||, ||u_4|| : \text{ both } u_3 \not= 0 \text{ and } u_4 \not= 0, \text{ where } (u_1, u_2, u_3, u_4) \text{ is a fixed point of } A_{0011} \text{ in } D, \text{ some } \theta \in [0, 1] \}
\]

must satisfy \( t^{**} > 0 \). Further, the signs of \( a_{3j}, a_{4j} \) and assumptions (2.3), (2.4) imply that \( \lambda_1(\Delta + f_3(0, 0, 0, u_3^0)) > 0 \) and \( \lambda_1(\Delta + f_4(0, 0, u_3^0, 0)) > 0 \). Choosing \( \epsilon \in (0, t^{**}) \), we obtain by Lemma 4.2 again that \( i(A, D_{3,4}^\epsilon) = 0 \).

Under the conditions of this part, the above paragraphs show that for sufficiently small \( \epsilon > 0 \), all the fixed points of \( A \) in \( D_{2,3}^\epsilon \) must have both the 2nd and 3rd components identically zero. Because of the symmetry of relations between \( u_1 \) and \( u_2 \) with respect to \( u_3 \) and \( u_4 \), analogous property can be obtained for fixed points of \( A \) in \( D_{1,3}^\epsilon, D_{1,4}^\epsilon, \) and \( D_{2,4}^\epsilon \) (here 1, 2 can be interchanged and 3, 4 can be interchanged). Also, the above paragraphs show that for sufficiently small \( \epsilon > 0 \), all the fixed points of \( A \) in \( D_{1,2}^\epsilon \) must have either the 1st or 2nd components identically zero; and those in \( D_{3,4}^\epsilon \) must have either the 3rd or 4th component identically zero. Let \( \hat{D} := \{ u \in D : \text{ each component of } u \text{ is } \not= 0 \} \). Thus we obtain \( i(A, \hat{D}) = i(A, \hat{D}) - i(A, D_{3,4}^\epsilon \cup D_{1,2}^\epsilon) \) for sufficiently small \( \epsilon > 0 \). Moreover,

\[
i(A, D_{3,4}^\epsilon \cup D_{1,2}^\epsilon) = i(A, D_{3,4}^\epsilon) + i(A, D_{1,2}^\epsilon) - i(A, D_{3,4}^\epsilon \cap D_{1,2}^\epsilon).
\]
Deducing from Venn diagrams, using the additive property of the indices of a map on disjoint open sets (cf. [1]), we also have

\[
i(A, D_{1,3}^\epsilon \cap D_{1,2}^\epsilon) = i(A, \tilde{D}_{1,3}^- \cup \tilde{D}_{1,4}^\epsilon \cup \tilde{D}_{2,3}^- \cup \tilde{D}_{2,4}^-)
\]

\[
= i(A, \tilde{D}_{1,3}^-) + i(A, \tilde{D}_{1,4}^\epsilon) + i(A, \tilde{D}_{2,3}^-) - i(A, y_2) - i(A, y_1) - i(A, y_3) - 3i(A, (0, 0, 0, 0)).
\]

Note that each \(y_i, i = 1, 2, 3, 4\), is inside exactly two of the sets \(\tilde{D}_{1,3}^\epsilon, \tilde{D}_{2,3}^\epsilon, \tilde{D}_{1,4}^\epsilon, \tilde{D}_{2,4}^\epsilon\) and \((0, 0, 0, 0)\) is inside all the four sets.

Combining the above formulas, we find

\[
i(A, \tilde{D}) = i(A, D) - i(A, D_{1,3}^\epsilon) - i(A, D_{1,2}^\epsilon) + i(A, \tilde{D}_{1,3}^-) + i(A, \tilde{D}_{2,3}^-)
\]

\[
+ i(A, \tilde{D}_{1,4}^\epsilon) + i(A, \tilde{D}_{2,4}^-) - \sum_{i=1}^{4} i(A, y_i) - 3i(A, (0, 0, 0, 0)). \tag{3.5}
\]

We have proved \(i(A, \tilde{D}_{1,3}^\epsilon) = 0\) above. Using the symmetry of the relations between \(u_1\) and \(u_2\) with respect to \(u_3\) and \(u_4\), we can interchange the role of \(u_1\) with \(u_2\) and the role of \(u_3\) with \(u_4\) to deduce

\[
i(A, \tilde{D}_{1,4}^-) = i(A, \tilde{D}_{1,4}^\epsilon) = i(A, \tilde{D}_{2,4}^-) = 0.
\]

In order to use (3.5), it remains to find \(i(A, y_i)\) and \(i(A, (0, 0, 0, 0))\). For convenience, we let \(\mu_i = \lambda_i(\Delta + f_i(0, 0, 0, 0)), i = 1, \ldots, 4\). The first part of (2.2) asserts that \(\mu_1\) is positive. The equation for \(u_1^0\) implies \(u_1^0 \leq e_1/a_{11} < B_1^-\); thus (2.3) implies \(\lambda_1(\Delta + f_1(y_1)) > 0\). Similarly, we have \(u_4^0 < B_4^-\) and (2.4) implies \(\lambda_4(\Delta + f_4(y_1)) > 0\). The second part of (2.2) and \(a_{21} \geq 0\) lead to \(\lambda_1(\Delta + f_2(y_1)) > 0\). By Lemma 4.1(ii), we obtain \(i(A, y_1) = 0\). The symmetry of the relations among \(u_1\) and \(u_2\) would readily lead to \(i(A, y_2) = 0\).

Condition (2.3) and comparison show that \(\mu_3\) is positive. Condition (2.2) and \(a_{31} \geq 0\) for \(i = 1, 2\) imply that \(\lambda_3(\Delta + f_3(y_1)) > 0\) for \(i = 1, 2\). Condition (2.2) and \(a_{32} \geq 0\) lead to \(\lambda_3(\Delta + f_3(y_2)) > 0\). By Lemma 4.1(ii), we obtain \(i(A, y_3) = 0\). We deduce in a symmetric way that \(i(A, y_4) = 0\). Finally, conditions (2.2) to (2.4) imply \(\mu_i > 0\) for each \(i\). We conclude that \(i(A, (0, 0, 0, 0)) = 0\) from Lemma 4.1(ii).

Finally, we apply index formula (3.5). The above paragraphs show that every term on the right of the formula is equal to zero, except \(i(A, D) = 1\). Consequently, we obtain \(i(A, \tilde{D}) = 1\). That is there must exist at least one positive solution for problem (1.1). This completes the proof of Theorem 2.1(i).

The proof of Theorem 2.1(ii) is similar to that of Theorem 2.1(i). The details will thus be omitted.

**Proof of Theorem 2.2(i).** Assume [C1], [A3], and (2.9) to (2.13). Consider the non-negative solutions of problem (3.2) in the present conditions. Since (2.9) is the same as the second part of (2.1), we can apply Lemma 3.1 to obtain inequalities (3.3). Note that \(\tilde{u}_3 = K_3\) and \(\tilde{u}_4 = K_4\) by definition. We then compare the first equation of (3.2) with the
scalar problem $-\Delta z = \theta \iota z [e_1 + a_{13} K_4 + a_{14} K_4]$ in $\Omega$, $z = 0$ on $\partial \Omega$ (using $a_{11}$ and $a_{12}$ are $\leq 0$), we readily obtain a bound for $u_1$ on $\partial \Omega$. Similarly, we deduce a bound for $u_2$.

Thus, as in the proof of Theorem 2.1, we obtain a constant $M > 0$ such that all components of all nonnegative solutions of (3.2) must have values in $[0, M]$, uniformly for $\theta \in [0, 1]$, $i = 1, 2, 3, 4$. We define sets $E(t), D, D_1^i, D_{i,j}^i, \hat{D}_{i,j}^i$, and the operators $A_{\theta, \theta, \theta} = \theta$ as compact positive mappings from the cone $K := [C_0^+(\Omega)]^4$ into itself, with no fixed point on $\partial D$, exactly as in the proof of Theorem 2.1. Also, we obtain in the same way that $i(A, D_1) = 1$.

Consider the mapping $A_{\theta_{110}}$ in $D$. Suppose there exists a sequence of fixed points $(u_{1, n}, u_{2, n}, u_{3, n}, u_{4, n})$, $n = 1, 2, 3, \ldots$, of the map $A_{\theta_{110}}$ in $D$, $\theta_n \in [0, 1]$, with $u_{2, n} \neq 0$ or $u_{3, n} \neq 0$. We have $\Delta u_{3, n}^2 + u_{3, n} f_3(u_{1, n}^2, \ldots, u_{4, n}^2) = 0$ in $\Omega$, $u_{3, n}^0 = 0$ on $\partial \Omega$, $i = 1, \ldots, 4$. If both $u_{2, n}^0$ and $u_{3, n}^0 \to 0$ in $C(\Omega)$, the equation for $u_{3, n}^0$ implies that $u_{3, n}^0 \leq |e_4/|a_{44}|| + \delta$ for any small $\delta > 0$, provided $n$ is sufficiently large. The equation for $u_{3, n}^0$ then implies that $u_{3, n}^0 \leq \hat{B}_1 + \delta$ any small $\delta > 0$, provided $n$ is sufficiently large. Similarly, we obtain $u_{2, n}^0 < \hat{B}_2 + \delta$. Thus, for $\delta > 0$ sufficiently small, we have

$$f_3(u_{2, n}^0, \ldots, u_{4, n}^0) = \epsilon_n - (a_{31} |\hat{B}_2^1| + |a_{32}| |\hat{B}_2^2|) - \delta > 0,$$

for $n$ sufficiently large. Thus we have $\lambda_1(\Delta + f_3(u_{1, n}^0, u_{2, n}^0, u_{3, n}^0, u_{4, n}^0)) > 0$ by assumption (2.12), and the equation for $u_{3, n}^0$ implies that $u_{3, n}^0 = 0$ for all large $n$. Then, we have $\Delta u_{3, n}^2 + u_{2, n}^0 f_2(u_{1, n}^2, u_{3, n}^0, u_{2, n}^0, u_{4, n}^0) = 0$ in $\Omega$. For any $\delta > 0$, we verify that

$$\lambda_1(\Delta + f_2(u_{1, n}^0, u_{2, n}^0, 0, u_{3, n}^0)) > \lambda_1(\Delta + \epsilon - |a_{21} a_{11}^{-1} |[e_1 + a_{14} K_4] + \delta)$$

for $n$ sufficiently large. Thus using (2.11), we obtain by comparison that $u_{2, n}^0 = 0$ for large $n$ too. This contradicts the assumptions above on $u_{2, n}^0$ and $u_{3, n}^0$. We deduce by contradiction as in the proof of Theorem 2.1 that $i(A, \hat{D}_{1, 3, 3}) = 0$ for $\epsilon > 0$ sufficiently small. By symmetry, we also obtain $i(A, \hat{D}_{1, 2, 2}) = i(A, \hat{D}_{2, 2, 2}) = 0$, as in the proof of Theorem 2.1.

Consider the mapping $A_{1100}$ in $D$. Suppose there exists a sequence of fixed points $(v_{1, n}^0, v_{2, n}^0, v_{3, n}^0, v_{4, n}^0)$, $n = 1, 2, 3, \ldots$, of the map $A_{1100}$ in $D$, $\theta_n \in [0, 1]$, with both $v_{1, n}^0 \neq 0$ and $v_{2, n}^0 \neq 0$. Using (3.3) to estimate $v_{3, n}^0$ and $v_{4, n}^0$, we then deduce from the second equation for $v_{2, n}^0$ that $v_{3, n}^0 \leq |a_{23}^{-1} [e_2 + a_{23} K_3 + a_{24} K_4]|$. Hence, we have

$$\lambda_1(\Delta + f_1(0, 0, v_{1, n}^0, v_{2, n}^0)) > \lambda_1(\Delta + \epsilon - |a_{12} a_{22}^{-1} |[e_2 + a_{23} K_3 + a_{24} K_4]) > 0,$$

by assumption (2.10). We can then compare with the equation satisfied by $v_{1, n}^0$ to deduce that $v_{1, n}^0$ cannot tend to zero as $n$ tends to infinity. Similarly, we deduce that $\lambda_1(\Delta + f_3(v_{1, n}^0, 0, v_{2, n}^0, v_{4, n}^0)) > 0$ by assumption (2.11), and find $v_{2, n}^0$ also cannot tend to zero as $n$ tends to infinity. Consequently, the number

$$t^* := \inf \{\|u_1\|, \|u_2\|: \text{ both } u_1 \neq 0 \text{ and } u_2 \neq 0, \text{ where } (u_1, u_2, u_3, u_4)$$

is a fixed point of $A_{1100}$ in $D$, some $\theta \in [0, 1]$]}

must satisfy $t^* > 0$. Further, assumptions (2.10) and (2.11) imply that $\lambda_1(\Delta + f_1(0, u_{1, n}^0, 0, 0)) > 0$ and $\lambda_1(\Delta + f_3(u_{1, n}^0, 0, 0, 0)) > 0$. Choosing $\epsilon \in (0, t^*)$, we can thus obtain by Lemma 4.2 below that $i(A, \hat{D}_{1, 3, 3}) = 0$.

Consider the mapping $A_{0011}$ in $D$. Suppose there exists a sequence of fixed points $(w_{1, n}^0, w_{2, n}^0, w_{3, n}^0, w_{4, n}^0)$, $n = 1, 2, 3, \ldots$, of the map $A_{0011}$ in $D$, $\theta \in [0, 1]$, with both $w_{4, n}^0 \neq 0$.
and \(w^n_4 \not\equiv 0\). If \(w^n_3 \to 0\), the equation for \(w^n_3\) implies that \(w^n_3 \leq e_4|a_{44}| + \delta\) for any small \(\delta > 0\) provided that \(n\) is large enough. The equation for \(w^n_1\) then imply that \(w^n_1 \leq \hat{B}_1^1 + \delta\) for any small \(\delta\) provided \(n\) is large enough. Similarly, we have \(w^n_2 \leq \hat{B}_2^2 + \delta\). Thus for \(\delta > 0\) sufficiently small, we have

\[
 f_3(w^n_1, \ldots, w^n_4) > \varepsilon_3 - (|a_{31}| \hat{B}_1^1 + |a_{32}| \hat{B}_2^2) - \delta > 0,
\]

for \(n\) sufficiently large. Hence we obtain \(\lambda_1(\Delta + f_3(w^n_1, w^n_2, w^n_3, w^n_4)) > 0\) by assumption (2.12), and the equation for \(w^n_3\) implies that \(w^n_3 \equiv 0\) for all large \(n\). This contradicts \(w^n_3 \not\equiv 0\), and thus \(w^n_3\) cannot tend to zero as \(n\) tends to infinity. On the other hand, if \(w^n_2 \to 0\), the equation for \(w^n_2\) implies that \(w^n_2 \leq e_1|a_{13}| + \delta\) for small \(\delta > 0\) and \(n\) large enough. We continue to deduce in a symmetric way that \(\lambda_1(\Delta + f_3(w^n_1, w^n_2, w^n_3, w^n_4)) > 0\) by assumption (2.13), leading to \(w^n_2 \equiv 0\) for all large \(n\). We again conclude by contradiction that \(w^n_2\) cannot tend to zero. Further, the assumptions (2.12), (2.13) imply that \(\lambda_1(\Delta + f_1(0, 0, 0, u^n_2)) > 0\) and \(\lambda_1(\Delta + f_4(0, 0, u^n_3, 0)) > 0\). Consequently, we obtain by Lemma 4.2 again that \(i(A, D^n_{v_i}) = 0\), for \(\epsilon > 0\) sufficiently small.

As in the proof of Theorem 2.1, we next use index formula (3.5). It remains to show that under the conditions of the present theorem, we still have \(i(A, (0, 0, 0, 0)) = i(A, v_1) = 0\), for \(i = 1, \ldots, 4\). They are all readily verified by applying Lemma 4.1, using (2.10) to (2.13). For evaluating \(i(A, y_1)\), we observe

\[
\begin{align*}
\lambda_1(\Delta + f_2(y_1)) &> \lambda_1(\Delta + e_2 - |a_{21}a_{11}^{-1}|e_1) > 0 \quad \text{by (2.11)}, \\
\lambda_1(\Delta + f_3(y_1)) &> \lambda_1(\Delta + e_3 - |a_{31}|e_1a_{11}^{-1}) > 0 \quad \text{by (2.12), and} \\
\lambda_1(\Delta + f_4(y_1)) &> \lambda_1(\Delta + e_4 - |a_{41}|e_1a_{11}^{-1}) > 0 \quad \text{by (2.13)}.
\end{align*}
\]

Then we apply Lemma 4.1(ii) to verify that \(i(A, y_1) = 0\). The other cases are similar or easier. Finally, applying formula (3.5), we obtain \(i(A, \hat{D}) = 1\), and complete the proof of Theorem 2.2(i). \(\square\)

The proof of Theorem 2.2(ii) is similar to the proof of the theorems above. The details will thus be omitted here.

4. **Lemmas for calculating indices**

In this section, we carefully justify the method for calculating the indices of the mappings used in the proofs in the last section. The methods described in Lemmas 4.1 and 4.2 are generalizations of results given in [19,22]. The technique of Lemma 4.3 is new. Consider the problem

\[-\Delta u_i = \theta_i u_i f_i(u_1, u_2, u_3, u_4) \quad \text{in } \Omega, \quad u_i = 0 \quad \text{on } \partial \Omega,\]

for \(i = 1, \ldots, 4, \theta_i \in [0, 1]\). For \(\theta_i \in [0, 1], \, i = 1, \ldots, 4\) and \(P > 0\), define the operator \(A_{\theta_1 \theta_2 \theta_3 \theta_4}: [C(\Omega)]^4 \to [C_0(\Omega)]^4\) by \(A_{\theta_1 \theta_2 \theta_3 \theta_4}(u_1, \ldots, u_4) = (v_1, v_2, v_3, v_4)\) where
Define an operator

\[ A = A_{1111}, \quad A_{ij}^k = A_{\theta_i \theta_j \theta_k \theta_4} \]

where \( \theta_i = \theta_j = 1, \ \theta_k = \theta \) for \( k \neq i \) or \( j \).

For convenience, we let \( \lambda = \mu_i \) represent the first eigenvalue of the problem

\[ \Delta u + f_i(0, 0, 0, 0)u = \lambda u \quad \text{in} \quad \Omega, \quad u = 0 \quad \text{on} \quad \partial \Omega. \tag{4.1} \]

**Remark 4.1.** In the proofs of the main theorems in Section 3, it is shown that under the hypotheses of the theorems, the positive solutions of (1.1), or the fixed points of \( A_{\theta_i \theta_j \theta_k \theta_4} \), are uniformly bounded for all \( \theta_i \in [0, 1] \). Let \( M \) be the uniform bound, \( E(M) = \{ u \in C(\Omega) : |u| < M \} \) and \( D = [C_0^r(\overline{\Omega})]^4 \cap [E(M)]^4 \). It is shown that \( A_{\theta_i \theta_j \theta_k \theta_4} \) has no fixed point on \( \partial D \), and can be extended to be defined as a compact, Frechet differentiable mapping from the cone \( K = [C_0^r(\overline{\Omega})]^4 \) into itself. We will assume these properties for all such operators in Lemmas 4.1 to 4.3 in this section.

Let \( j \) be an integer between 1 to 4. For simplicity, we denote \( y_j = (0, \ldots, u_j, \ldots, 0) \) where every component is the zero function except the \( j \)th component. Also, recall the definitions of the sets \( D_i^k, D_{i,j}, D_{i,j}^k \) given in the proof of Theorem 2.1(i) in Section 3.

**Lemma 4.1.**

(i) If \( \max \{ \mu_1, \ldots, \mu_4 \} > 0 \), and at most one of \( \mu_i \), \( i = 1, \ldots, 4 \), is zero, then \( i(A, (0, 0, 0, 0)) = 0 \).

(ii) Suppose that \( \mu_j > 0 \). If there exists \( k \neq j \) such that \( \lambda_i(\Delta + f_k(y_j)) > 0 \), and \( \lambda_i(\Delta + f_r(y_j)) \neq 0 \) for all \( r \neq j \) and \( k \), then \( i(A, y_j) = 0 \).

**Proof.** (i) The proof is the same as [22, Lemma 7]. We outline the main idea for convenience of the reader.

Suppose \( \mu_j > 0 \) and \( \mu_i \neq 0 \) for all other \( i \)'s. For \( y \in K \), define

\[ K_y := \{ p \in [C(\overline{\Omega})]^4 : y + sp \in K \text{ for some } s > 0 \} \quad \text{and} \]

\[ S_y := \{ p \in K_y : -p \in K_y \}, \]

as in [5] or [19]. We have \( \overline{K}_{(0,0,0,0)} = K \). \( S_{(0,0,0,0)} = \{ (0, 0, 0, 0) \} \). The \( k \)th component of \( A'(0, 0, 0, 0)u \) is \( (-\Delta + P)^{-1}f_k(0, 0, 0, 0) + P)u_k \). Hence \( [I - A'(0, 0, 0, 0)]u = 0 \) for \( u \in K \) implies that \( \Delta + f_k(0, 0, 0, 0)u_j = 0, u_j \in C_0^r(\overline{\Omega}) \). Thus the assumption \( \mu_j > 0 \) implies that \( u_j = 0 \). Similarly the assumption \( \mu_i \neq 0 \) implies that \( u_i = 0 \) for all other \( i \)'s. Further, the assumption \( \mu_j > 0 \) and the continuity in \( t \in [0, 1] \) for \( \lambda_i(\Delta + tf_j(0, \ldots, 0) + (t - 1)P) \) imply that there exists a nontrivial function \( \overline{\pi} \in C_0^r(\overline{\Omega}) \) such that \( (-\Delta + P)\overline{\pi} = t(f_j(0, 0, 0, 0) + P)\overline{\pi}, \) or \( \pi - t(-\Delta + P)^{-1}(f_j(0, 0, 0, 0) + P)\overline{\pi} = 0 \in (0, 0, 0, 0) \), for some \( t \in (0, 1) \). Thus it follows from [22, Theorem D], [19], or [5, Remark (2)] that \( i(A, (0, 0, 0, 0)) = 0 \).

Next, suppose \( \mu_j > 0 \), \( \mu_r = 0 \) for some \( 1 \leq j, r \leq 4 \), and \( \mu_i \neq 0 \) for all other \( i \)'s. Define an operator \( A_r \) by modifying the \( r \)th component of the operator \( A \) by changing \( f_r(u_1, u_2, u_3, u_4) \) to \( f_r(u_1, u_2, u_3, u_4) - t. \) From the last paragraph, we obtain
Let \( u = (u_1, \ldots, u_k) \) be any element in \([C(\partial)]^{4}\). For \( i \neq j \), the \( i \)th component of \( A'(y_j)u \) is \((-\Delta + P)^{-1}[f_i(y_j) + Pu_i] \); the \( j \)th component is

\[
(-\Delta + P)^{-1}\left\{ \sum_{i \neq j} u_j^0(\partial f_j / \partial u_i)(y_j)u_i + u_j^0(\partial f_j / \partial u_j)(y_j) + f_j(y_j) + Pu_j \right\}.
\]

One readily checks that \( \tilde{R}_j = C_0^+ (\partial) \oplus \cdots \oplus C_0^+ (\partial) \) and \( S_{y_j} = [0] \oplus \cdots \oplus C_0(\partial) \oplus \cdots \oplus C_0(\partial) \) where \( C_0(\partial) \) and \( C_0(\partial) \) are certain nontrivial subsets of \( C_0(\partial) \) without sign restriction, appearing in the \( j \)th component in both cases. Let \( [I - A'(y_j)] \tilde{u} = 0 \) for \( \tilde{u} \in \tilde{R}_j \). As in part (i), the assumption that \( \lambda_1(\Delta + f_i(y_j)) \neq 0 \) for \( i \neq j \) implies that \( \tilde{u}_i = 0 \). Thus the \( j \)th component can be written as \( \Delta \tilde{u}_j + (e_j + 2a_{j,j}u_j^0)\tilde{u}_j = 0 \).

Since \( \lambda_1(\Delta + (e_j + 2a_{j,j}u_j^0)) < 0 \) by comparison, we must have \( \tilde{u}_j = 0 \). Thus we have \( \tilde{u} = 0 \).

As in the proof of part (i), the assumption \( \lambda_1(\Delta + f_k(y_j)) > 0 \) implies that there exists a nontrivial function \( \tilde{w}_k \in C_0^+ (\partial) \) satisfying \((-\Delta + P)\tilde{w}_k - t(f_k(y_j) + Pu_k)\tilde{w}_k = 0 \). Let \( w \) be the column vector function on \( \partial \) with \( \tilde{w}_k \) as its \( k \)th component and zero function as all other components. Then \( w \) has the properties \( w \in \tilde{R}_j \setminus S_{y_j} \) and \([I - tA'(y_j)]w \in S_{y_j} \). Thus the operator \( A'(y_j) \) has the properties as described in [22, Theorem D] or [19], and we assert that \( i(A, y_j) = 0 \). This completes the proof of part (ii).

**Lemma 4.2.** Suppose \( i \neq j \) are integers with \( 1 \leq i, j \leq 4 \) with \( \mu_i > 0 \) and \( \mu_j > 0 \). Let

\[
t := \min \{ \|u_i\|, \|u_j\| \text{ both } u_i \neq 0 \text{ and } u_j \neq 0, \text{ where } \text{col}(u) = (u_1, u_2, u_3, u_4) \}
\]

be a fixed point of \( A_{ij}^0 \) in \( D \) with \( u_k \geq 0 \) for \( k = 1, 2, 3, 4 \), some \( \theta \in [0, 1] \).

Let \( y_k \in [C_0^+(\partial)]^4 \) with the \( k \)th component as \( u_k^0 \) and all other components as the trivial function. Assume \( t > 0 \), and further that

\[
\lambda_1(\Delta + f_i(y_j)) > 0, \quad \lambda_1(\Delta + f_j(y_i)) > 0;
\]

then for any \( \epsilon \in (0, t) \), we have \( i(A, D_{ij}^\epsilon) = 0 \).

**Proof.** Since \( 0 < \epsilon < t \), the operator \( A_{ij}^0 \) in \( D \) has no fixed point on \( \partial D_{ij}^\epsilon \). By homotopy invariance, \( i(A, D_{ij}^\epsilon) = i(A_{ij}^0, D_{ij}^\epsilon) \). Let \( (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3, \tilde{u}_4) \) be a nontrivial fixed point of \( A_{ij}^0 \) in \( D_{ij}^\epsilon \). Then \((-\Delta + P)^{-1}[P\tilde{u}_k] = 0 \), for \( k \neq i \) and \( j \), implies that such \( \tilde{u}_k = 0 \). The condition on \( \epsilon \) implies either \( \tilde{u}_i = 0 \) or \( \tilde{u}_j = 0 \). Thus \( A_{ij}^0 \) has three fixed points in \( D_{ij}^\epsilon \), namely, \( (0, 0, 0, 0), y_i \) and \( y_j \). Moreover, we find

\[
i(A_{ij}^0, D_{ij}^\epsilon) = i(A_{ij}^0, (0, 0, 0, 0)) + i(A_{ij}^0, y_i) + i(A_{ij}^0, y_j).
\]
Applying a natural modification of Lemma 4.1(i) for the operator \( A^i_{ij} \), we find \( i(A^i_{ij}, (0, 0, 0, 0)) = 0 \). Also, applying a natural modification of Lemma 4.1(ii) for the operator \( A^i_0 \), we obtain \( i(A^i_0, y_j) = 0 \) and \( i(A^i_0, y_i) = 0 \). This proves Lemma 4.2. \( \square \)

**Lemma 4.3.** Let \( i \neq j \) be integers, \( 1 \leq i, j \leq 4 \), and
\[
t := \inf \{ \max \{ \| u_1 \|, \| u_2 \|, \| u_3 \|, \| u_4 \| \} : (u_1, u_2, u_3, u_4) \text{ is a fixed point of } A^i_0 \text{ in } D \text{ with } u_k \geq 0 \text{ for } k = 1, 2, 3, 4, \text{ for some } \theta \in [0, 1] \}.
\]
Suppose that \( t > 0 \), and further assume either \( \mu_i > 0 \) or \( \mu_j > 0 \). Then for any \( \epsilon \in (0, t) \), we have \( i(A, \hat{D}^i_{ij}) = 0 \).

**Proof.** The assumption that \( 0 < \epsilon < t \) implies that all fixed points \( \hat{u} = (\hat{u}_1, \hat{u}_2, \hat{u}_3, \hat{u}_4) \) of \( A^i_0 \) in \( D \) contained in \( \hat{D}^i_{ij} \) must satisfy \( \hat{u}_i = \hat{u}_j = 0 \); and they are bounded away from the other nonnegative fixed points in \( D \) by \( \delta \hat{D}^i_{ij} \). Applying homotopy invariance principle on the set \( \hat{D}^i_{ij} \), we find \( i(A, \hat{D}^i_{ij}) = i(A^i_1, \hat{D}^i_{ij}) = i(A^i_0, \hat{D}^i_{ij}) \). Note that for \( k \neq i \) and \( j \), the \( k \)th component of \( A^i_{ij} (u_1, \ldots, u_4) \) is of the form \((\Delta + P)^{-1}(Pu_k)\); thus any fixed point of \( A^i_{ij} \) in \( \hat{D}^i_{ij} \) must have all the \( k \)th component identically zero for \( k \neq i \) and \( j \) too. Hence \((0, 0, 0, 0)\) is the only fixed point of \( A^i_{ij} \) in \( \hat{D}^i_{ij} \), and \( i(A^i_{ij}, \hat{D}^i_{ij}) = i(A^i_{ij}, (0, 0, 0, 0)) \). We then apply a modification of Lemma 4.1(i) to the operator \( A^i_0 \) with \( \mu_k \) changed to \( \hat{\mu}_k \), where \( \hat{\mu}_k = \lambda_1(\Delta) < 0 \) for \( k \neq i \) and \( j \), and \( \hat{\mu}_i = \mu_i, \hat{\mu}_j = \mu_j \). Since at least one of \( \mu_i \) and \( \mu_j \) is positive, we conclude as in Lemma 4.1(i) that \( i(A^i_0, (0, 0, 0, 0)) = 0 \). \( \square \)

### 5. Cooperative groups with competition or cooperation within the groups, other extensions

In this section we consider the coexistence of positive solutions for problem (1.1) when the two groups of species cooperate with each other. As before, there are two groups with a pair of species within each group. The cooperativeness between the groups means that the assumption \([C1]\) in Section 2 is replaced by

\[ [C2] \; a_{m3} \text{ and } a_{m4} \geq 0 \text{ for } m = 1, 2; a_{n1} \text{ and } a_{n2} \geq 0 \text{ for } n = 3, 4. \]

This will always be assumed in this section. Within each group, the species may compete or cooperate as expressed in \([A4]\) and \([A2]\) in Section 2. The case of \([A3]\) is the same as \([A2]\) if we interchange species 1 and 2 with 3 and 4. The case of \([C2]\) together with \([A1]\) would mean all species cooperate. It will then be unnecessary to classify the species into two groups for study.

**Theorem 5.1** (Cooperating groups with competition or cooperation within each group).

(i) Assume interaction relations \([C2]\) and \([A4]\). Suppose that
\[ |a_{mm}| > a_{m3} + a_{m4} \quad \text{for} \quad m = 1, 2, \quad \text{and} \]
\[ |a_{nn}| > a_{n1} + a_{n2} \quad \text{for} \quad n = 3, 4, \quad (5.1) \]
then the problem (1.1) has a positive solution if the following conditions are satisfied:
\[ \lambda_1(\Delta + e_1 + a_{12} \hat{Q}_2) > 0, \quad \text{where} \quad \hat{Q}_2 = \max \{ |e_2/k_2|, |e_3/k_2|, |e_4/k_2| \}, \]
\[ k_2 = \min \{ |a_{22}| - a_{23} - a_{24}, |a_{33}| - a_{32}, |a_{44}| - a_{42} \}, \quad (5.2) \]
\[ \lambda_1(\Delta + e_2 + a_{21} \hat{Q}_1) > 0, \quad \text{where} \quad \hat{Q}_1 = \max \{ |e_1/k_1|, |e_3/k_1|, |e_4/k_1| \}, \]
\[ k_1 = \min \{ |a_{11}| - a_{13} - a_{14}, |a_{33}| - a_{31}, |a_{44}| - a_{41} \}, \quad (5.3) \]
\[ \lambda_1(\Delta + e_3 + a_{34} \hat{Q}_4) > 0, \quad \text{where} \quad \hat{Q}_4 = \max \{ |e_4/k_4|, |e_1/k_4|, |e_2/k_4| \}, \]
\[ k_4 = \min \{ |a_{44}| - a_{41} - a_{42}, |a_{11}| - a_{14}, |a_{22}| - a_{24} \}, \quad (5.4) \]
\[ \lambda_1(\Delta + e_4 + a_{43} \hat{Q}_3) > 0, \quad \text{where} \quad \hat{Q}_3 = \max \{ |e_3/k_3|, |e_1/k_3|, |e_2/k_3| \}, \]
\[ k_3 = \min \{ |a_{33}| - a_{31} - a_{32}, |a_{11}| - a_{13}, |a_{22}| - a_{23} \}. \quad (5.5) \]

(ii) Assume interaction relations [C2] and [A2]. Suppose
\[ |a_{11}| > a_{12} + a_{13} + a_{14}, \quad |a_{22}| > a_{21} + a_{23} + a_{24}, \]
\[ |a_{nn}| > a_{n1} + a_{n2} \quad \text{for} \quad n = 3, 4, \quad (5.6) \]
then problem (1.1) has a positive solution if the following conditions are satisfied:
\[ \lambda_1(\Delta + e_1) > 0, \quad \lambda_1(\Delta + e_2) > 0, \quad (5.7) \]
\[ \lambda_1(\Delta + e_3 + a_{34} R_4) > 0, \quad \text{where} \quad R_4 = \max \{ |e_4/\rho_4|, |e_1/\rho_4|, |e_2/\rho_4| \}, \]
\[ \rho_4 = \min \{ |a_{44}| - a_{41} - a_{42}, |a_{11}| - a_{12} - a_{14}, |a_{22}| - a_{21} - a_{24} \}, \quad (5.8) \]
\[ \lambda_1(\Delta + e_4 + a_{43} R_3) > 0, \quad \text{where} \quad R_3 = \max \{ |e_3/\rho_3|, |e_1/\rho_3|, |e_2/\rho_3| \}, \]
\[ \rho_3 = \min \{ |a_{33}| - a_{31} - a_{32}, |a_{11}| - a_{12} - a_{13}, |a_{22}| - a_{21} - a_{23} \}. \quad (5.9) \]

**Lemma 5.1.** Let \((z_1(x), \ldots, z_n(x))\) with each component being a nonnegative \(C^2\) function on \(\overline{\Omega}\) be a solution of
\[ \Delta z_i + z_i \left[ k_i(x) + \sum_{j=1}^{n} d_{ij} z_j \right] = 0 \quad \text{in} \quad \Omega, \quad z_i = 0 \quad \text{on} \quad \partial \Omega, \quad i = 1, \ldots, n \]
where all \(d_{ij}\) are constants and nonnegative for \(i \neq j\) and \(k_i \in C(\overline{\Omega})\). Suppose \((w^\alpha_1(x), \ldots, w^\alpha_n(x))\) with each component being a nonnegative \(C^2\) function on \(\overline{\Omega}\), is a componentwise nondecreasing family of functions for \(\alpha \leq \delta \leq \beta\), which satisfies
\[ \Delta w^\alpha_i + w^\alpha_i \left[ k_i(x) + \sum_{j=1}^{n} d_{ij} w^\delta_j \right] \leq 0 \quad \text{in} \quad \Omega, \quad w^\delta_i \geq 0 \quad \text{on} \quad \partial \Omega, \]
and \(z_i(x) \neq w^\delta_i(x)\) in \(\overline{\Omega}\) for \(i = 1, \ldots, n, \alpha \leq \delta \leq \beta\). Suppose further that \(z_i(x) < w^\beta_i(x)\) for all \(x \in \Omega, \quad i = 1, \ldots, n\), then we can conclude that \(z_i(x) < w^\alpha_i(x)\) for all \(x \in \Omega, \quad i = 1, \ldots, n\).
Proof. The proof is the same as showing that the assumption \( u_i(x) < v_i^M \) in \( \Omega \) leads to the conclusion \( u_i(x) < v_i \) in \( \Omega \), for \( i = 1, 2 \), in Lemma 3.1. The essential condition is that \( d_{ij} > 0 \) for \( i \neq j \) here, as \( c_{ij} > 0 \) in Lemma 3.1. The details will be omitted. \( \square \)

Proof of Theorem 5.1(i). Assume conditions [C2], [A4], (5.1) to (5.5). Let \((\overline{u}_1(x), \overline{u}_2(x), \overline{u}_3(x), \overline{u}_4(x))\) be a given nonnegative solution of (3.2). Define \( h_1(x) = e_1 + a_13\overline{u}_3(x), h_2(x) = e_2 + a_23\overline{u}_1(x), h_3(x) = e_3 + a_34\overline{u}_4(x) \), and \( h_4(x) = e_4 + a_43\overline{u}_3(x) \) for \( x \in \Omega \). For \( \theta \in (0, 1) \), Consider the Dirichlet problem

\[
\begin{cases}
\Delta w_1 + \theta_1 w_1 h_1(x) + a_{11} w_1 + a_{13} w_3 + a_{14} w_4 = 0 & \text{in } \Omega, \\
\Delta w_2 + \theta_2 w_2 h_2(x) + a_{22} w_2 + a_{23} w_3 + a_{24} w_4 = 0 & \text{in } \Omega, \\
\Delta w_3 + \theta_3 w_3 h_3(x) + a_{33} w_3 + a_{31} w_1 + a_{32} w_2 = 0 & \text{in } \Omega, \\
\Delta w_4 + \theta_4 w_4 h_4(x) + a_{44} w_4 + a_{41} w_1 + a_{42} w_2 = 0 & \text{in } \Omega, \\
w_i(x) = 0, & \text{on } \partial \Omega.
\end{cases}
\]

(5.10)

The system in (5.10) is quasimonotone nondecreasing for \( w_i \geq 0 \), \( i = 1, \ldots, 4 \) (i.e., non-decreasing with respect to all other components). Assumption (5.1) implies that the constant

\[
k := \min \left\{ |a_{11}| - a_{13} - a_{14}, |a_{22}| - a_{23} - a_{24}, |a_{33}| - a_{31} - a_{32}, |a_{44}| - a_{41} - a_{42} \right\}
\]

is strictly \( > 0 \). Let \( Q := \max\{e_i / k : i = 1, \ldots, 4\} > 0 \). By [C2], [A4], and (5.1) we obtain for \( m = 1, 2, x \in \Omega \): \( h_m(x) + (a_{mn} + a_{m3} + a_{m4}) Q \leq e_m - k Q \leq 0 \). For \( n = 3, 4, x \in \Omega \), we obtain

\[
h_n(x) + (a_{nn} + a_{n1} + a_{n2}) Q \leq e_n - k Q \leq 0.
\]

Thus the constant function \((w_1, w_2, w_3, w_4) = (Q, Q, Q, Q)\) is an upper solution for the problem (5.10). Moreover, there exists a constant \( \lambda > 1 \) such that \( \overline{u}_i(x) \leq \overline{\lambda}Q \) in \( \overline{\Omega} \), for \( i = 1, \ldots, 4 \). We readily verify that the functions \((\lambda Q, \lambda Q, \lambda Q, \lambda Q)\), \( 1 \leq \lambda \leq \overline{\lambda} \), form a family of upper solutions for problem (5.10). Since \((w_1, w_2, w_3, w_4) = (Q, Q, Q, Q)\) is a solution for problem (5.10), we use Lemma 5.1 to assert that \( \overline{u}_i(x) < \overline{Q} \) in \( \overline{\Omega} \), for \( i = 1, \ldots, 4 \). We thus find an a priori bound for any nonnegative solution of problem (3.2). The same argument works even if some of the \( \theta_i \)’s are zero. We define sets \( E(i), D, D_i^e, D_i^e, D_i^{\overline{e}}, D_i^{\overline{e}} \) and the operators \( A_{\theta_1 \theta_2 \theta_3 \theta_4} \), as compact mapping from the cone \( K := [C_0^\infty(\overline{\Omega})]^4 \) into itself, with no fixed point on \( \partial D \), exactly as the proof of Theorem 2.1. Also we obtain in the same way that \( i(A, D) = 1 \).

Consider the mapping \( A_{\theta_1 \theta_2 \theta_3 \theta_4} \) in \( D \). Suppose there exists a sequence of fixed points \((u_1^n, u_2^n, u_3^n, u_4^n)\), \( n = 1, 2, 3, \ldots \) of the map \( A_{\theta_n \theta_n \theta_n \theta_n} \) in \( D \), \( \theta_n \in [0, 1] \), with \( u_2^n \neq 0 \) or \( u_3^n \neq 0 \). We have \( \Delta u_1^n + u_1^n \Delta u_3^n = 0 \) in \( \Omega \), \( u_1^n = 0 \) on \( \partial \Omega \), \( i = 1, \ldots, 4 \). Suppose both \( u_2^n \) and \( u_3^n \rightarrow 0 \) in \( C(\overline{\Omega}) \), consider the following system of equations for large integer \( n \):

\[
\begin{align*}
\Delta \omega_1 + \theta_4 \omega_1 \left[ e_1 + a_{12} u_2^n + a_{13} u_3^n + a_{14} u_4^n \right] + a_{11} \omega_1 + a_{14} \omega_4 &= 0 & \text{in } \Omega, \\
\Delta \omega_4 + \theta_4 \omega_4 \left[ e_4 + a_{42} u_2^n + a_{43} u_3^n + a_{44} u_4^n \right] + a_{41} \omega_1 + a_{44} \omega_4 &= 0 & \text{in } \Omega, \\
\omega_1 = \omega_4 &= 0 & \text{on } \partial \Omega.
\end{align*}
\]

(5.11)
The system is quasimonotone nondecreasing for $\omega_i \geq 0$, $i = 1, 4$. For any small $\delta > 0$, we have

$$(e_1 + a_{12}u^n_2 + a_{13}u^n_3) < e_1 + \delta, \quad (e_4 + a_{42}u^n_2 + a_{43}u^n_3) < e_4 + \delta$$

for $n$ sufficiently large. Since $(\omega_1, \omega_4) = (u^n_1, u^n_4)$ is a solution of problem (5.11), we can use Lemma 5.1 to deduce that $u^n_i < K + \delta$, $i = 1, 4$, for $n$ sufficiently large, where

$$K = \max\{e_1/p, e_4/p\}, \quad p = \min\{|a_{11}| - a_{14}, |a_{44}| - a_{41}\}.$$

Thus we find that for any $\delta > 0$, $f_3(u^n_1, u^n_2, 0, u^n_3) \geq e_3 + a_{34}K - \delta$ for $n$ sufficiently large. Since $K \leq \bar{Q}_4$ from definition, we can then use (5.4) to deduce that $u^n_4$ is uniformly bounded away from zero by a positive function, and thus cannot tend to zero as $n$ tends to infinity. In case $u^n_2 \equiv 0$ for large $n$, and $u^n_2 \to 0$, then from the estimate of $u^n_2$ above for system (5.11), we deduce that $f_2(u^n_1, 0, 0, u^n_3) \geq e_2 + a_{21}K - \delta$ for any small $\delta > 0$. Since $K \leq \bar{Q}_1$, we use (5.3) to find that $u^n_4$ cannot tend to 0 as $n$ tends to infinity. We can then apply Lemma 4.3 to obtain $i(A, D^i_{3/4}) = 0$. By the symmetry of the relationship of species 1, 2 with species 3, 4, we readily obtain

$$i(A, \bar{D}^i_{2,4}) = i(A, \bar{D}^i_{1,4}) = i(A, \bar{D}^i_{1,3}) = i(A, \bar{D}^i_{2,3}) = 0.$$

Consider the mapping $A_{0\theta_{11}}$ in $D$. Suppose there exists a sequence of fixed points $(u^n_1, u^n_2, u^n_3, u^n_4), n = 1, 2, 3, \ldots$, of the maps $A_{\theta_0,0,11}$ in $D$, $\theta \in [0, 1]$, with both $u^n_1 \not\equiv 0$ and $u^n_4 \not\equiv 0$. Suppose $u^n_3 \to 0$, consider the following system of equations for large integer $n$:

$$\begin{align*}
\Delta \omega_1 + \theta_0 \omega_1 \left(e_1 + a_{12}u^n_2 + a_{13}u^n_3\right) + a_{11} \omega_1 + a_{14} \omega_4 &= 0 \quad \text{in } \Omega, \\
\Delta \omega_2 + \theta_0 \omega_2 \left(e_2 + a_{21}u^n_1 + a_{23}u^n_3\right) + a_{22} \omega_2 + a_{24} \omega_4 &= 0 \quad \text{in } \Omega, \\
\Delta \omega_4 + \omega_4 \left(e_4 + a_{42}u^n_2 + a_{43}u^n_3\right) + a_{44} \omega_4 + a_{41} \omega_1 + a_{42} \omega_2 &= 0 \quad \text{in } \Omega, \\
\omega_1 &= \omega_2 = \omega_3 = 0 \quad \text{on } \partial \Omega.
\end{align*}$$

Let $\bar{Q}_4$ and $\bar{Q}_4$ be as defined in (5.4). We can use Lemma 5.1 to show that for any $\delta > 0$, we have $u^n_3 \leq \bar{Q}_4 + \delta$ for $n$ large enough. Thus $f_3(u^n_1, u^n_2, 0, u^n_3) > e_3 + a_{34}(\bar{Q}_4 + \delta)$, and we can use hypothesis (5.4) to deduce that $u^n_4$ are uniformly bounded away from zero by a positive function. This contradicts the assumption that $u^n_4$ tend to zero. Suppose $u^n_4 \to 0$, we deduce in an analogous way with the role of $u^n_3$ and $\omega_4$ replaced respectively by $u^n_4$ and $\omega_3$ in (5.12) that $u^n_3 \leq \bar{Q}_3 + \delta$ for $n$ large enough. Then we use (5.5) to obtain a contradiction for $u^n_4 \to 0$. Observe also that for $i = 3, 4$, we have

$$\bar{Q}_i \geq e_i/(|a_{ii}| - a_{1i} - a_{2i}) \geq e_i/|a_{ii}| \geq u^n_i.$$

Hence, we can use Lemma 4.2 and (5.4), (5.5) to obtain $i(A, D^i_{3/4}) = 0$, for $\epsilon > 0$ sufficiently small. The fact that $i(A, D^i_{3/2}) = 0$ can be proved in the analogous way by symmetry, with the role of hypotheses (5.4), (5.5) respectively replaced by (5.2) and (5.3).

The fact that $i((A, (0, 0, 0, 0))) = i(A, \gamma_i) = 0$, for $i = 1, \ldots, 4$ can be readily proved by using Lemma 4.1, hypotheses (5.2) to (5.5) and the relation $\bar{Q}_i \geq u^n_i$ for $i = 1, \ldots, 4$. Finally, applying index formula (3.5), we complete the proof of Theorem 5.1(ii). □

The proof of Theorem 5.1(ii) is similar to that of part (i). It is thus too lengthy to be included here.
Remark 5.1. In Section 2, we consider the situation when species in group I consists of predators with species in group II as preys. The species within each group are assumed to cooperate or compete with each other. We have omitted the case when there may be further prey-predator relationship within one group. For example, species in group I are cooperative and the species in group II form a prey-predator pair. More generalizations of Section 2 can also be done for prey-predator groups when prey-predator relations occur within each group, or prey-predator within one group and cooperating relation within another. Other cases can be treated similarly. Some of the theorems may conceivably be proved by other methods. However, if we consider the first case, i.e., Theorem 2.1(i), it does not seem that one can readily prove the theorem by other methods. Note that $B_{3i}^4$ or $B_{4i}^3$ may not be a bound for predator species $i$ when all the prey species 3 and 4 are present. Thus the condition in (2.3) and (2.4) may not be strong enough for proving the result by using other methods. Generalization of Section 5 is also possible for cooperative groups with prey-predator relations within each group. Since the methods are similar for these cases, the details will be omitted here. There is also the situation of a group of 3 interacting with a fourth species in the same way.

When there is a large number of $m$ species in group I, each of which competes with $n$ species in group II, existence of positive solutions is studied in [16] with bifurcation and upper–lower solutions methods. Within each group, there may be various types of structures. There are, however, limitations to the amount of interactions between the groups in order to prove the existence of positive solutions in [16]. In order to use the technique of this paper when there are groups of large numbers of $m$ and $n$ species, the methods in Section 4 have to be extended more systematically. More interesting results remain to be found.

Remark 5.2. Further research should also address the issue of time stability and persistence of the systems. Such problems are studied in many references in [2,3,13,17]. Under the hypotheses that the various related principle eigenvalues are positive, it should be possible to obtain some information about the dynamics when the boundary equilibria are repellors relative to the positive cone. Some sort of conclusions about persistence should be possible as in [2,3]. It would also be interesting to treat the cases where some of the principle eigenvalues are negative.

References