Some new characterizations of Dirichlet type spaces on the unit ball of \( \mathbb{C}^n \)

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Received 8 November 2005
Available online 3 February 2006
Submitted by R. Curto

Abstract

The Dirichlet type spaces on the unit ball of \( \mathbb{C}^n \) are characterized by some oscillations in the Bergman metric in this paper. As an application of these new characterizations of Dirichlet type spaces, we prove that composition operators induced by biholomorphic maps are bounded on \( D_p \) \((-1 \leq p < n)\).

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Keywords: Dirichlet type spaces; Oscillation; Composition operator

1. Introduction

Let \( B = \{ z \in \mathbb{C}^n : |z| < 1 \} \) be the open unit ball of \( \mathbb{C}^n \), \( S = \{ z \in \mathbb{C}^n : |z| = 1 \} \) be its boundary. \( dv \) denotes the normalized Lebesgue measure of \( B \), i.e. \( v(B) = 1 \). \( d\sigma \) denotes the normalized rotation invariant Lebesgue measure of \( S \) satisfying \( \sigma(S) = 1 \). Let \( d\lambda (z) = (1 - |z|^2)^{-n-1} \, dv(z) \). Then \( d\lambda (z) \) is Möbius invariant, that is for any \( \psi \in \text{Aut}(B) \), \( f \in L^1(B) \),

\[
\int_B f(z) \, d\lambda (z) = \int_B f \circ \psi(z) \, d\lambda (z).
\] (1)

We denote the class of all holomorphic functions on the unit ball by \( H(B) \).

For \( f \in H(B) \), \( z \in B \), set
\[ Q_f(z) = \sup \left\{ \frac{|\langle \nabla f(z), x \rangle|}{(H_z(x,x))^{1/2}} : 0 \neq x \in \mathbb{C}^n \right\}, \] (2)

where \( \nabla f(z) = \left( \frac{\partial f}{\partial z_1}(z), \ldots, \frac{\partial f}{\partial z_n}(z) \right) \) is the complex gradient of \( f \) and \( H_z(x,x) \) is the Bergman metric on \( B \), i.e.

\[ H_z(x,x) = \frac{n + 1}{2} \frac{(1 - |z|^2)|x|^2 + |\langle x, z \rangle|^2}{(1 - |z|^2)^2}. \]

For \( 0 < p < \infty \), the Bergman space \( L^p_\alpha(B) \) and the Hardy space \( H^p \) are defined respectively as

\[ L^p_\alpha(B) = \left\{ f : f \in H(B), \| f \|_{L^p_\alpha} = \int_B |f(z)|^p \, dv(z) < \infty \right\} \]

and

\[ H^p = \left\{ f : f \in H(B), \| f \|_{H^p} = \sup_{0 < r < 1} \int_S |f(r\xi)|^p \, d\sigma(\xi) < \infty \right\}. \]

Let \( f \in H(B) \) with Taylor expansion \( f(z) = \sum_{|\alpha| \geq 0} a_\alpha z^\alpha, p \in \mathbb{R} \). Recall that the Dirichlet type spaces \( D_p \) consist of all holomorphic functions such that (see [2]):

\[ \| f \|_{D_p} = \left( \sum_{|\alpha| \geq 0} (n + |\alpha|)^p |b_\alpha|^2 \omega_\alpha \right)^{1/2} < \infty, \] (3)

where

\[ \omega_\alpha = \int_{\partial B} |\xi^\alpha|^2 \, d\sigma(\xi) = \frac{(n - 1)!|\alpha|!}{(n + |\alpha| - 1)!}, \]

\[ \alpha = (\alpha_1, \ldots, \alpha_n), \quad |\alpha| = \alpha_1 + \cdots + \alpha_n, \quad \alpha! = \alpha_1! \cdots \alpha_n!, \quad z^\alpha = z_1^{\alpha_1} \cdots z_n^{\alpha_n}. \]

It is clear that \( D_p \) is a Hilbert space with the inner product \( \langle f, g \rangle \) defined by

\[ \langle f, g \rangle = \sum_{|\alpha| \geq 0} (n + |\alpha|)^p \omega_\alpha a_\alpha b_\alpha, \]

where \( f(z) = \sum_{|\alpha| \geq 0} a_\alpha z^\alpha, g(z) = \sum_{|\alpha| \geq 0} b_\alpha z^\alpha \). The spaces \( D_0 \) and \( D_{-1} \) are just the Hardy space \( H^2 \) and the Bergman space \( L^2_\alpha \), respectively. \( D_n \) is the Dirichlet space.

In the setting of disk, the Dirichlet type space \( D_p \) was defined by Taylor (see [12]). Dirichlet type space was studied in many different aspects, such as integral characterizations, multipliers and Carleson measure (see [5,8,10,12]).

In the setting of the unit ball, the Dirichlet type space \( D_p \) has been studied by many authors. In [2], Hu and Shi gave the characterizations of the multiplication operator of \( D_p \). In [1], Feng obtained the integral characterizations of \( D_p \) with respect to invariant gradient, complex gradient and radial derivative. In [4], a BMO-type characterization of \( D_p \) was given by Hu and Zhang, namely the following theorem:

**Theorem (A).** Let \( f(z) = \sum_{|\alpha| \geq 0} a_\alpha z^\alpha \in H(B), p \leq n, \delta, \tau > -1 \) and \( \min(\delta, \tau) + p > -1 \). Then
\[ \int \int_{B} |f(z) - f(w)|^2 \left( \frac{1}{1 - \langle z, w \rangle} \right)^{n+2+\delta+p} \left( 1 - |z|^2 \right)^{n-1} d\nu(z) d\nu(w) \]

\approx \sum_{|\alpha| \geq 0} (n + |\alpha|)^p |a_{\alpha}|^2 \omega_{\alpha}. \quad (4)

In this paper, we add some characterizations of Dirichlet type space. For example, Dirichlet type spaces can now be characterized in terms of the invariant mean oscillation and mean oscillation in the Bergman metric. As an application of the new characterization of Dirichlet type space, we prove that the composition operators induced by biholomorphic maps are bounded on \( D_p \) \((-1 \leq p < n)\).

Throughout the paper, \( C \) will denote positive constant which is not necessarily the same at different occurrences. \( a \lesssim b \) means that there is a positive constant \( C \) such that \( a \leq Cb \). Moreover, if both \( a \lesssim b \) and \( b \lesssim a \) hold, then one says that \( a \approx b \).

2. Main result and proof

Let \( \delta + p = n, \tau = 0 \). By Theorem A, we see that if \(-1 < p < n\), then \( f \in D_p \) if and only if

\[ \int \int_{B} |f(z) - f(w)|^2 \left( \frac{1}{1 - \langle z, w \rangle} \right)^{n+2+\delta+p} \left( 1 - |z|^2 \right)^{n-1} d\nu(z) d\nu(w) < +\infty. \]  \quad (5)

According to above characterization of \( D_p \), we obtain the following characterization of Dirichlet type space \( D_p \).

**Theorem 1.** Suppose \(-1 < p < n\). Then \( f \in D_p \) if and only if

\[ \int_{B} Q_f^2(z) \left( 1 - |z|^2 \right)^{n-p} d\lambda(z) < \infty. \]  \quad (6)

**Proof.** Assume that (6) holds. Making a change of variables \( w = \varphi_z(u) \), by [3, Lemma 3.6],

\[ \int \int_{B} |f(z) - f(w)|^2 \left( \frac{1}{1 - \langle z, w \rangle} \right)^{n+2+\delta+p} \left( 1 - |z|^2 \right)^{n-1} d\nu(z) d\nu(w) \]

\[ = \int \int_{B} |f(z) - f(\varphi_z(u))|^2 \left( \frac{1}{1 - \langle z, \varphi_z(u) \rangle} \right)^{n+2+\delta+p} \left( 1 - |\varphi_z(u)|^2 \right)^{n-1} d\nu(z) d\nu(u) \]

\[ = \int \int_{B} |f(z) - f(\varphi_z(u))|^2 \left( \frac{1}{1 - \langle z, u \rangle} \right)^{n+2+\delta+p} \left( 1 - |z|^2 \right)^{n-1} d\nu(z) d\nu(u) \]

\[ \leq C \int_{B} Q_f^2(z) \left( 1 - |z|^2 \right)^{n-p} d\lambda(z). \]

By (5), \( f \in D_p \).
Conversely, we assume that $f \in D_p$, i.e.
\[
\int_B \int_B |f(z) - f(w)|^2 \frac{1}{|1 - \langle z, w \rangle|^{2n+2}} \frac{1}{1 - |z|^2}^{n-p} d\lambda(z) d\lambda(w) < +\infty.
\]

By the mean value properties and integration in polar coordinates,
\[
f(z) = \frac{1}{v(rB)} \int_{rB} f \circ \varphi_z d\lambda(z) = \frac{1}{v(rB)} \int_{rB} f(w) \left( \frac{1 - |z|^2}{|1 - \langle z, w \rangle|^{n+1}} \right) d\lambda(w).
\]

Differentiate under the integral sign yields
\[
||\nabla f(0), \vec{v}|| \leq C \int_{rB} |f| d\lambda.
\]

By the definition of $Q_f$ and Jensen inequality, we have
\[
Q^2_f(0) \leq C \int_{rB} |f| d\lambda.
\]

Replacing $f$ by $f \circ \varphi_z - f(z)$ and using the change of variable formula,
\[
Q^2_f(z) \leq C \int_{E(z,r)} |f \circ \varphi_z - f(z)|^2 d\lambda(w) = C \int_{E(z,r)} |f(w) - f(z)|^2 \frac{(1 - |z|^2)^{n+1}}{|1 - \langle w, z \rangle|^{2(n+1)}} d\lambda(w).
\]

Write $E(z, r) = \{w \in B: |\varphi_z(w)| < r\}$. For $w \in E(z, r)$, we have (see [6])
\[
\frac{1}{1+r} \leq \frac{|1 - \langle z, w \rangle|}{1 - |w|^2} \leq \frac{2}{1-r^2}, 
\]
\[
\frac{1}{2} \leq \frac{|1 - \langle z, w \rangle|}{1 - |z|^2} \leq \frac{2}{1-r}.
\]

It follows that
\[
Q^2_f(z) \leq C \int_{E(z,r)} |f(w) - f(z)|^2 \frac{1}{|1 - \langle w, z \rangle|^{n+1}} d\lambda(w).
\]

By (8),
\[
\int_B Q^2_f(z)(1 - |z|^2)^{n-p} d\lambda(z)
\]
\[
\leq C \int_B \int_{E(z,r)} \frac{|f(w) - f(z)|^2}{|1 - \langle w, z \rangle|^{n+1}} \frac{1}{1 - |z|^2}^{n-p} d\lambda(z) d\lambda(w)
\]
\[
\leq C \int_B \int_{E(z,r)} \frac{|f(w) - f(z)|^2}{|1 - \langle w, z \rangle|^{n+1}} (1 - |z|^2)^{-1-p} d\lambda(w) d\lambda(z)
\]
\[
\leq C \int_B \int_{E(z,r)} \frac{|f(w) - f(z)|^2}{|1 - \langle w, z \rangle|^{2n+2}} (1 - |z|^2)^{n-p} d\lambda(w) d\lambda(z)
\]
\[ \leq C \int_B \int_B \frac{|f(w) - f(z)|^2}{|1 - \langle w, z \rangle|^{2n+2}} \left(1 - |z|^2\right)^{n-p} dv(w) dv(z). \]

Therefore \( f \in D_p \) implies (6). \( \square \)

For \( f \in C^1(B) \), let \( \tilde{\nabla}f \) denote the invariant gradient on \( B \), i.e. \( \langle \tilde{\nabla}f \rangle(z) = \nabla(f \circ \varphi_z)(0) \). Analogously, for \( f \in C^2(B) \), the variant Laplacian \( \tilde{\Delta}f \) is defined by

\[ \langle \tilde{\Delta}f \rangle(z) = \Delta(f \circ \varphi_z)(0), \]

where \( \Delta \) is the ordinary Laplacian. Since

\[ \tilde{\Delta} f(z) = 4 \left(1 - |z|^2\right) \sum_{i,k=1}^n (\delta_{ik} - z_i \bar{z}_k) \frac{\partial^2 f}{\partial z_i \partial \bar{z}_k}(z), \]

where \( \delta_{ik} \) denotes the Kronecker delta (see [9]), we get

\[ \tilde{\Delta} |f|^2(z) = 4 \left(1 - |z|^2\right) \left( |\nabla f(z)|^2 - |Rf(z)|^2 \right) = 4|\tilde{\nabla} f(z)|^2. \]

**Theorem 2.** Suppose \(-1 < p < n\). Then \( f \in D_p \) if and only if

\[ \int_B \left| \tilde{\nabla} f(z) \right|^2 \left(1 - |z|^2\right)^{n-p} d\lambda(z) < \infty. \quad (9) \]

\( f \in D_p \) if and only if

\[ \int_B \tilde{\Delta} |f|^2(z) \left(1 - |z|^2\right)^{n-p} d\lambda(z) < \infty. \quad (10) \]

**Proof.** By [14, Theorem 3.1], we find that

\[ Q_f(z) \approx \sqrt{\tilde{\Delta} |f|^2(z)} \approx |\tilde{\nabla} f(z)|. \]

Therefore the result follows from Theorem 1. \( \square \)

**Remark.** We point out that the condition \( p \leq n \) in Theorem A (see [4]) should be replaced by \( p < n \). If \( p = n \), checking the proofs of Theorems 1 and 2, \( f \in D_n \) if and only if

\[ \int_B \tilde{\Delta} |f|^2(z) d\lambda(z) < \infty. \quad (11) \]

When \( n \geq 2 \), only constant holomorphic functions \( f \) in \( B \) satisfy (11) (see [13]). This is a contradiction.

Next, we give some characterizations of \( D_p \) by some oscillations. First, let us recall the definition of Berezin transform (see [14]). For any function \( f \in L^1(B, dA) \), we define a function

\[ Bf(z) = \int_B \frac{(1 - |z|^2)^{n+1}}{|1 - (z, w)|^{2(n+1)}} f(w) dv(w), \quad z \in B. \]
We call $Bf$ the Berezin transform of $f$. By a change of variables,

$$Bf(z) = \int_B f \circ \varphi_z(w) \, dv(w), \quad z \in B.$$ 

For $f \in L^2(B, dv)$, define

$$MO(f)(z) = \sqrt{B(|f|^2)(z) - |Bf(z)|^2}.$$ 

It is easy to see that

$$[MO(f)(z)]^2 = \int_B \left| f \circ \varphi_z(w) - Bf \right|^2 \, dv(w)$$

$$= \int_B \left| f(w) - Bf(z) \right|^2 \frac{(1 - |w|^2)^{n+1}}{|1 - \langle z, w \rangle|^{2(n+1)}} \, dv(w).$$

If the function $f$ is holomorphic, then $Bf = f$. Therefore

$$[MO(f)(z)]^2 = \int_B \left| f(w) - f(z) \right|^2 \frac{(1 - |w|^2)^{n+1}}{|1 - \langle z, w \rangle|^{2(n+1)}} \, dv(w). \quad (12)$$

Since

$$MO(f \circ \varphi)(z) = MO(f)(\varphi(z)),$$

where $\varphi \in Aut(B)$, we call $MO(f)(z)$ as the invariant mean oscillation of $f$ in the Bergman metric at the point $z$.

**Theorem 3.** Suppose $-1 < p < n$. Then $f \in D_p$ if and only if

$$\int_B [MO(f)(z)]^2 (1 - |z|^2)^{n-p} \, d\lambda(z) < \infty. \quad (13)$$

**Proof.** By (5), we consider the following integral

$$L_a(f) = \int_B \int_B \frac{|f(z) - f(w)|^2}{|1 - \langle z, w \rangle|^{2(n+1)}} (1 - |z|^2)^{n-p} \, dv(z) \, dv(w). \quad (14)$$

We rewrite (14) as an iterated integral

$$L_a(f) = \int_B (1 - |z|^2)^{n-p} \, d\lambda(z) \int_B \left| f(z) - f(w) \right|^2 \frac{(1 - |z|^2)^{n+1}}{|1 - \langle z, w \rangle|^{2(n+1)}} \, dv(w).$$

By (12),

$$L_a(f) = \int_B [MO(f)(z)]^2 (1 - |z|^2)^{n-p} \, d\lambda(z).$$

This proves the desired result. \square
Let \( \beta(z, w) \) denote the Bergman metric between two points \( z \) and \( w \) in \( B \). It is well known that
\[
\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}.
\]
For \( z \in B \) and \( r > 0 \), the set
\[
D(z, r) = \left\{ w \in B : \beta(z, w) < r \right\}, \quad z \in B,
\]
is a Bergman metric ball at \( z \) with radius \( r \).

Fix a positive \( r \) and denote by
\[
\hat{f}_r(z) = \frac{1}{|D(z, r)|} \int_{D(z, r)} f(w) \, dv(w)
\]
the average of \( f \) over the Bergman metric ball \( D(z, r) \). We define the mean oscillation of \( f \) at \( z \) in the Bergman metric as following (see [14]):
\[
MO_r(f)(z) = \left\{ \frac{1}{|D(z, r)|} \int_{D(z, r)} \left| f(w) - \hat{f}_r(z) \right|^2 \, dv(w) \right\}^{1/2}.
\] (15)

**Theorem 4.** Suppose \(-1 < p < n\). Then \( f \in D_p \) if and only if
\[
\int_B \left[ MO_r(f)(z) \right]^2 \left( 1 - |z|^2 \right)^{n-p} \, d\lambda(z) < \infty,
\] (16)
where \( r \) is any fixed positive radius.

**Proof.** First, we prove that there exists a positive constant \( C \) such that
\[
MO_r(f)(z) \leq CMO(f)(z), \quad z \in B.
\]

Since \((1 - |z|^2) \sim (1 - |w|^2)\) for \( w \in D(z, r) \), we get
\[
\frac{1}{|D(z, r)|} \int_{D(z, r)} \left| f(w) - \hat{f}_r(z) \right| \, dv(w)
\]
\[
\leq \sup_{w \in D(z, r)} \left| f(w) - \hat{f}_r(z) \right| \leq \sup_{w \in D(z, r)} \frac{1}{|D(z, r)|} \int_{D(z, r)} \left| f(w) - f(u) \right| \, dv(u)
\]
\[
\leq \sup_{w \in D(z, r)} \left( \sup_{u \in D(z, r)} \left| f(w) - f(u) \right| \right)
\]
\[
\leq \sup_{w \in D(z, r)} \sup_{u \in D(z, r)} \left( \left| f(w) - f(z) \right| + \left| f(z) - f(u) \right| \right)
\]
\[
\leq \sup_{w \in D(z, r)} \left| f(w) - f(z) \right| + \sup_{u \in D(z, r)} \left| f(z) - f(u) \right|
\]
\[
\leq C \sup_{w \in D(z, r)} \left| f(w) - f(z) \right|.
\]
Since $|g(u)|^2$ is subharmonic for all holomorphic functions $g$ on $B$, by [14, Lemma 2.24],

$$|g(w)|^2 \leq \frac{C}{|D(w, r)|} \int_{D(w, r)} |g(u)|^2 \, dv(u).$$

Replacing $g$ by $f - f(z)$,

$$|f(w) - f(z)|^2 \leq \frac{C}{|D(w, r)|} \int_{D(w, r)} |f(u) - f(z)|^2 \, dv(u).$$

Since $D(w, r) \subset D(z, 2r)$ for $w \in D(z, r)$, there is a constant $M > 0$ such that

$$\frac{C}{|D(w, r)|} \leq \frac{M}{|D(z, 2r)|}.$$

Then

$$\sup_{w \in D(z, r)} |f(w) - f(z)| \leq \left( \frac{C}{|D(z, 2r)|} \int_{D(z, 2r)} |f(w) - f(z)|^2 \, dv(w) \right)^{1/2} \leq C \left( \int_{D(z, 2r)} |f(w) - f(z)|^2 \, dv(w) \right)^{1/2} \leq C \left( \int_{D(z, 2r)} |f \circ \varphi_z(w) - f(z)|^2 \, dv(w) \right)^{1/2} \leq C \left( \int_B |f \circ \varphi_z(w) - f(z)|^2 \, dv(w) \right)^{1/2} = CMO(f)(z).$$

Therefore

$$MO_{r}(f)(z) \leq CMO(f)(z), \quad z \in B.$$

It follows that

$$\int_B \left[ MO_r(f)(z) \right]^2 (1 - |z|^2)^{n-p} \, d\lambda(z) \leq C \int_B \left[ MO(f)(z) \right]^2 (1 - |z|^2)^{n-p} \, d\lambda(z)$$

for each $a \in B$. By Theorem 3, it follows that $f \in D_p$ implies (16).

Conversely, by [14, Lemma 2.4], there exists a positive constant $C$ such that

$$|\nabla g(0)| \leq C \int_{D(0, r)} |g(w) - C| \, dv(w)$$

for all holomorphic $g$ in $B$. Applying $g = f \circ \varphi_z$, $C = \hat{f}_r(z)$, using Hölder inequality and the fact that

$$\frac{(1 - |z|^2)^{n+1}}{|1 - \langle z, w \rangle|^{2(n+1)}} \sim \frac{1}{(1 - |w|^2)^{n+1}} \sim \frac{1}{|D(z, r)|}$$
when \( w \in D(z, r) \), we get
\[
\left| \tilde{\nabla} f(z) \right| \leq C \int_{D(0, r)} \left| f \circ \varphi_z(w) - \tilde{f}_r(z) \right| dv(w)
\]
\[
\leq C \int_{D(z, r)} \left| f(w) - \tilde{f}_r(z) \right| \frac{(1 - |z|^2)^{n+1}}{|1 - \langle z, w \rangle|^{2(n+1)}} dv(w)
\]
\[
\leq \frac{C}{|D(z, r)|} \int_{D(z, r)} \left| f(w) - \tilde{f}_r(z) \right| dv(w) = CMO_r(f)(z).
\]
It follows that for each \( a \in B \),
\[
\int_B \left| \tilde{\nabla} f(z) \right|^2 (1 - |z|^2)^{n-p} d\lambda(z) \leq C \int_B \left[ MO_r(f)(z) \right]^2 (1 - |z|^2)^{n-p} d\lambda(z).
\]
By Theorem 2, (16) implies \( f \in D_p \). The proof is completed. \( \square \)

For a function \( f \) on \( B \), we define two oscillations of \( f \) at \( z \) in the Bergman metric as follows (see [14]):
\[
w_r(f)(z) = \sup_{w \in D(z, r)} \left| f(z) - f(w) \right|,
\]
and
\[
\hat{w}_r(f)(z) = \sup_{w \in D(z, r)} \left| \hat{f}_r(z) - f(w) \right|.
\]

From the proof of Theorem 4, we get the following characterizations of Dirichlet type space:

**Theorem 5.** Suppose \(-1 < p < n \) and \( r > 0 \). Then the following statements are equivalent.

1. \( f \in D_p \).
2. \( \int_B \left[ w_r(f)(z) \right]^2 (1 - |z|^2)^{n-p} d\lambda(z) < \infty. \)
3. \( \int_B \left[ \hat{w}_r(f)(z) \right]^2 (1 - |z|^2)^{n-p} d\lambda(z) < \infty. \)

**Remark.** Let \( p = 0 \). Then \( D_0 = H^2 \). By above all theorems, we have given some new characterizations of Hardy space \( H^2(B) \) on the unit ball. Some results are new even in the setting of the unit disk. Some related results have been obtained by Stoll in [11]. For example, Stoll proved that a holomorphic function \( f \) on \( B \) belongs to \( H^2(B) \) if and only if
\[
\int_B \left| \tilde{\nabla} f(z) \right|^2 (1 - |z|^2)^n d\lambda(z) < \infty.
\]
In [7], the authors proved the following result: A holomorphic function \( f \) in \( L^p_{\alpha}(B) \) \((0 < \alpha < \infty)\) if and only if
\[ \int_B \left| \nabla f(z) \right|^2 |f(z)|^{p-2}(1-|z|^2)^{n+1} d\lambda(z) < \infty. \]

Using this characterization and carefully checking the proofs of above theorems, we get the following new characterizations of Bergman space \( L^2_{a}(B) \).

**Proposition 6.** Let \( r > 0 \). Then the following statements are equivalent.

1. \( f \in L^2_{a}(B) \).
2. \[ \int_B \int_B \frac{|f(z) - f(w)|^2}{|1-\langle z, w \rangle|^2} (1-|z|^2)^{n+1} d\nu(z) d\nu(w) < \infty. \]
3. \[ \int_B Q^2_f(z) (1-|z|^2)^{n+1} d\lambda(z) = \int_B Q^2_f(z) d\nu(z) < \infty. \]
4. \[ \int_B \left| \nabla f(z) \right|^2 (1-|z|^2)^{n+1} d\lambda(z) = \int_B \left| \nabla f(z) \right|^2 d\nu(z). \]
5. \[ \int_B \Delta |f|^2(z) (1-|z|^2)^{n+1} d\lambda(z) = \int_B \Delta |f|^2(z) d\nu(z) < \infty. \]
6. \[ \int_B \left[ MO(f)(z) \right]^2 (1-|z|^2)^{n+1} d\lambda(z) = \int_B \left[ MO(f)(z) \right]^2 d\nu(z) < \infty. \]
7. \[ \int_B \left[ MO_r(f)(z) \right]^2 (1-|z|^2)^{n+1} d\lambda(z) = \int_B \left[ MO_r(f)(z) \right]^2 d\nu(z) < \infty. \]
8. \[ \int_B \left[ w_r(f)(z) \right]^2 (1-|z|^2)^{n+1} d\lambda(z) = \int_B \left[ w_r(f)(z) \right]^2 d\nu(z) < \infty. \]
9. \[ \int_B \left[ \hat{w}_r(f)(z) \right]^2 (1-|z|^2)^{n+1} d\lambda(z) = \int_B \left[ \hat{w}_r(f)(z) \right]^2 d\nu(z) < \infty. \]

Let \( \varphi = (\varphi_1, \ldots, \varphi_n) \) be a holomorphic self-map of \( B \). The composition operator \( C_{\varphi} \) on the space \( H(B) \) is defined by \( C_{\varphi} f = f \circ \varphi \). As an application of new characterizations of the Dirichlet type space, we obtain the following result.

**Theorem 7.** Suppose \(-1 \leq p < n\). If \( f \in D_p \), then \( C_{\varphi_a} f \in D_p \) for every \( \varphi_a \in Aut(B) \).

**Proof.** For \( \varphi_a \in Aut(B) \),
\[ \frac{1-|z|^2}{1-|\varphi_a|^2} = \frac{|1-\langle z, a \rangle|^2}{1-|a|^2} \leq \frac{2}{1-|a|^2}. \]
Let \( f \in D_p \). Since \( \nabla \) and \( d\lambda(z) \) are invariant under \( Aut(B) \). By Theorem 2 and making a change of variables \( w = \varphi_a(z) \),
\[ \|C_{\varphi_a} f\|_{D_p} \approx \int_B \left| \nabla (f \circ \varphi_a)(z) \right|^2 (1-|z|^2)^{n-p} d\lambda(z). \]
\[
\begin{align*}
&= \int_B |\tilde{\nabla}(f)(\varphi_a(z))|^2 (1 - |z|^2)^{n-p} d\lambda(z) \\
&= \int_B |\tilde{\nabla}(f)(w)|^2 (1 - |w|^2)^{n-p} \frac{(1 - |z|^2)^{n-p}}{(1 - |w|^2)^{n-p}} d\lambda(w) \\
&\leq C \left( \frac{2}{1 - |a|^2} \right)^{n-p} \|f\|_D^p .
\end{align*}
\]

References