A₀-stability of variable stepsize BDF methods

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Abstract

In this paper the A₀-stability of interpolatory, variable coefficient and fixed leading coefficient BDF formulas is considered. A comparison of the stability properties of the three methods based on their spectral radius is presented. In addition, we study the effect of some strategies of stepsize variation, employed in usual codes, on the stability of these methods.

Keywords: Numerical solution ODEs; stiff systems; stability; variable stepsize.

1. Introduction

Most popular codes to solve stiff differential systems are based on Backward Differentiation Formulas (BDFs). Among them we may mention DIFSUB [5] and the LSODE family [7] which use Interpolatory BDF methods (INT), EPISODE which uses Variable Coefficient BDF methods (VC) and more recently VODE [2] which uses the so-called Fixed Leading Coefficient BDF methods (FLC).

Even though the absolute stability properties of BDF methods with fixed stepsize are well known from the earlier papers of Dahlquist, only a few results with limited practical value are available for the case of variable stepsizes. Thus, in [1,9] the absolute stability of VC methods is studied, mainly for orders ≤ 3, by choosing suitable norms which allow any product of propagation matrices of these methods to be bounded. Although the results given by these authors are theoretically almost optimal, since they have been derived under very general assumptions on the stepsize variation, the bounds on the stepsize ratios which ensure the stability are so sharp that they have a very limited practical interest. Also, it is worth remarking that in [8] the stability of INT, VC and two types of FLC methods is compared, for a special sequence of stepsizes, showing that VC is clearly superior while INT and the usual FLC method have a similar stability behaviour.

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On the other hand, practical experience as well as some theoretical studies of a number of authors [3,4,6,8] suggest that the VC are more stable than FLC and INT methods, but the relative merits of the last two methods are yet unclear. Thus, numerical experiments performed with LSODE and VODE show that there are stiff systems which can be solved only with VODE, and this fact is often interpreted in the sense that FLC formulas are more stable than INT formulas, but we have found that the way in which instability problems manifest themselves is largely dependent on other factors of the code such as error estimation or stepsize selection.

In this paper a comparison of the A_0-stability of VC, INT and FLC methods is presented. Such a comparison is based on the spectral radius of (products of) propagation matrices of the underlying methods. Although the stability derived from the spectral radius condition does not guarantee the boundedness of products of propagation matrices, it turns out to be a realistic criterion to compare the stability of different methods. Briefly, the paper is organized as follows. In Section 2 we formulate the methods in an adequate matrix form. In Section 3 we study the spectral radius of the propagation matrices and products of them, obtaining conditions on the stepsize ratios which ensure that the spectral radius is less than one. Special treatment is given to the ∞-point. Finally, in Section 4, we analyze the effect of some particular strategies of stepsize variation, employed in usual codes, on the stability of these methods.

2. Formulation of methods

Assume that approximations \( y_{n-j} \) to the solution of the IVP \( y' = f(t, y), \quad y(t_0) = y_0 \) have been computed at time steps \( t_{n-j}, \quad j = 1, \ldots, k \). In the \( k \)-step VC method the approximation \( y_n \) to the solution of the IVP at \( t_n \) is defined as the solution of the difference equation

\[
\sum_{j=0}^{k} \alpha_{j,n} y_{n-j} = h_n f(t_n, y_n),
\]

(2.1)

where \( h_n = t_n - t_{n-1} \) is the current stepsize and the coefficients \( \alpha_{j,n} \) are calculated so that (2.1) has order \( k \). It is easy to see that this condition determines uniquely the coefficients \( \alpha_{j,n} \) as rational functions of the stepsize ratios \( r_j = h_j / h_{j-1}, \quad j = n - k + 2, \ldots, n \).

For the scalar test equation \( y' = \lambda y \), (2.1) is a linear difference equation which may be rewritten equivalently in the matrix form

\[
U_n = \Omega_n^{VC} U_{n-1},
\]

(2.2)

where \( U_n = (y_n, y_{n-1}, \ldots, y_{n-k+1})^T \in \mathbb{R}^k \), associated to the gridpoint \( t_n \), collects the information of the \( k \)-step method and \( \Omega_n^{VC} \in \mathbb{R}^{k \times k} \) is the so-called propagation matrix (in the \( U_n \) variables) given by

\[
(\Omega_n^{VC})_{ij} = \begin{cases} 
-\frac{\alpha_{j,n}}{\alpha_{0,n} - z_n}, & \text{if } i = 1, \\
\delta_{i,j+1}, & \text{if } i > 2,
\end{cases}
\]

(2.3)

with \( z_n = h_n \lambda \).
In the \( k \)-step FLC method the difference equation which defines \( y_n \) is

\[
y_n + \sum_{j=1}^{k} \alpha_j \beta_{j,n} y_{n-j} = h_n \beta_0 f(t_n, y_n) + h_n \beta_{1,n} f(t_{n-1}, y_{n-1}),
\]

(2.4)

where \( \beta_0 \) is the leading coefficient of the BDF formula and the remaining coefficients \( \alpha_j, \beta_{1,n} \) are uniquely determined from the requirement that (2.4) has order \( k \). Proceeding in a similar way as in the case of the VC methods, we arrive at

\[
U_n = \Omega_n^{\text{FLC}} U_{n-1},
\]

(2.5)

where

\[
(\Omega_n^{\text{FLC}})_{ij} = \begin{cases} 
-\frac{\alpha_j \beta_{j,n} z_n}{1 - \beta_0 z_n}, & \text{if } i = 1, \\
\delta_{i,j+1}, & \text{if } i \geq 2,
\end{cases}
\]

and \( \beta_{j,n} = 0, j = 2, \ldots, k \).

To give the matrix formulation of INT methods we introduce the Nordsieck vector of order \( k \), \( N_n = (y_n, h_n y_n', \ldots, h_n^k y_n^{(k)}/k!)^T \) associated with the polynomial approximation to the solution at the gridpoint \( t_n \). In terms of the Nordsieck vector, the equations of the INT method are

\[
N_{n,0} = PD(r_n) N_{n-1},
\]

(2.71)

\[
y_n = y_{n,0} + l^{-1} \left[ h_n f(t_n, y_n) - h_n y_{n,0} \right],
\]

(2.72)

\[
N_n = N_{n,0} + (y_n - y_{n,0}) I,
\]

(2.73)

where \( l = (1, l_1, \ldots, l_k)^T \) is a constant vector, \( D = \text{diag}(1, r_n, \ldots, r_n^k) \), \( r_n = h_n/h_{n-1} \) and \( P \) the Pascal matrix. Note that in (2.71) a predicted value \( N_{n,0} \) of \( N_n \) is computed by extrapolation of the solution at \( t_{n-1} \). Then, the value of \( y_n \) is determined from the implicit equation (2.72) and in (2.73) all components of \( N_{n,0} \) are corrected to the final value \( N_n \).

When the INT method is applied to the test equation, the matrix equation of the method is given by

\[
N_n = \Omega_n^{\text{INT}} N_{n-1},
\]

(2.8)

with

\[
\Omega_n^{\text{INT}} = \left[ I + z_n (l_1 - z_n)^{-1} e_0^T - (l_1 - z_n)^{-1} e_1^T \right] PD(r_n),
\]

(2.9)

and \( e_j = (\delta_{j,i}) \).

Note that in INT methods the propagation matrix (2.9) depends only on \( z_n \) and the last stepsize ratio \( r_n \).

As remarked above, our stability analysis is based on the spectral radius of the propagation matrices of the methods. This implies that the results are independent of the vector \( (U_n \) or \( N_n \) employed to describe the method.

In the following sections we are interested in finding conditions on the stepsize ratios (or sequences of stepsize ratios) which ensure that the spectral radius of the propagation matrix (or products of them) are \( \leq 1 \) for all \( z_n = h \lambda_n \in \mathbb{R}^+ \). Besides the case \( z = 0 \) (0-stability), the behaviour of the method for \( z \to -\infty \) is of particular interest for stiff systems. This case will be referred to in the remainder of the paper as \( \infty \)-stability.
3. Some stability results

The 0-stability of BDF type methods have been extensively studied by several authors. In particular, in [3,4] conditions are given on the stepsize ratios which ensure the 0-stability of INT, VC and FLC methods. In Table 3.1 we include the values of the stepsize ratios for which the propagation matrices of these methods with \( k \ (\leq 5) \) steps have spectral radius smaller than 1.

As we can see, there are no great differences between the upper bounds of \( r \) for the three families of methods. However, it has been observed in practice that VC methods are more stable than INT methods. This fact will be explained analyzing the propagation matrices at the \( \infty \)-point.

**Theorem 3.1.** Let \( \rho(A) \) be the spectral radius of matrix \( A \). Then,

(i) for the \( k \)-step VC method, the propagation matrix at the \( \infty \)-point is constant and

\[
\rho(\Omega^\text{VC}_n(r_{n}, \ldots, r_{n-k+1}; \infty)) = 0, \quad \forall r_{n}, \ldots, r_{n-k+2};
\]

(ii) for the \( k \)-step FLC method, the spectral radius of the propagation matrix at the \( \infty \)-point is

\[
|\beta_{1,n}|, \quad \text{and therefore}
\]

\[
\rho(\Omega^\text{FLC}_n(r_{n}, \ldots, r_{n-k+1}; \infty)) \leq 1 \iff |\beta_{1,n}| \leq 1;
\]

(iii) for the \( k \)-step INT method,

\[
\rho(\Omega^\text{INT}(r; \infty)) \leq 1 \iff r \in \left[0, \frac{k}{k-1}\right].
\]

**Proof.** (i) It follows at once from the fact that for \( z \to - \infty \) the propagation matrix of the \( k \)-step VC method reduces to a constant matrix whose characteristic polynomial is \( \sigma(\zeta) = \zeta^k \).

(ii) It is clear from (2.6) that for \( z \to - \infty \) the only eigenvalue of the propagation matrix of the \( k \)-step FLC method is the coefficient \( \beta_{1,n} \) in the right-hand side of (2.4). This implies the statement (ii).

(iii) We consider first the two-step INT method. Its propagation matrix for \( z \to - \infty \) becomes

\[
\Omega^\text{INT}(r; \infty) = \begin{pmatrix}
0 & 0 & 0 \\
-\frac{3}{2} & -\frac{1}{2}r & \frac{1}{2}r^2 \\
-\frac{1}{2} & -\frac{1}{2}r & \frac{1}{2}r^2
\end{pmatrix}
\]

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<td>Bounds for 0-stability sets</td>
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<td>Bounds for ( \infty )-stability sets</td>
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and its characteristic polynomial is
\[ \sigma_2(\zeta, r) = -\zeta^2(\zeta - \frac{1}{2}r(r - 1)). \]
Therefore the spectral radius is \( \leq 1 \) if and only if \( r \in [0, 2] \).

For \( k = 3, 4, 5 \), after some computations with the help of an algebraic manipulator, the characteristic polynomials \( \sigma_k(\zeta) \) of the propagation matrices at the \( \infty \)-point are

\[ \sigma_3(\zeta, r) = \frac{1}{6}(6\zeta^4 + 5r(1 - r^2)\zeta^3 + r^3(2r^2 - 3r + 1)\zeta^2), \]
\[ \sigma_4(\zeta, r) = -\frac{1}{24}(24\zeta^5 - r(23r^3 + 14r^2 - 11r - 26)\zeta^4 + r^3(17r^4 - 6r^3 - 14r^2 - 6r + 9)\zeta^3 - r^6(6r^3 - 11r^2 + 6r - 1)\zeta^2), \]
\[ \sigma_5(\zeta, r) = \frac{1}{120}(120\zeta^6 - r(119r^4 + 105r^3 + 35r^2 - 105r - 154)\zeta^5 + r^5(109r^6 + 44r^5 - 16r^4 - 180r^3 - 44r^2 + 16r + 71)\zeta^4 + r^6(-74r^6 + 46r^5 + 66r^4 - 20r^3 - 6r^2 - 26r + 14)\zeta^3 + r^8(24r^4 - 50r^3 + 35r^2 - 10r + 1)\zeta^2). \]

Finally, by using Schur's conditions and again with the help of an algebraic manipulator it is seen that all the roots of \( \sigma_k \) are in modulus \( \leq 1 \) if and only if \( r \in (0, k/(k - 1)) \). \( \square \)

For comparison we have computed the set of values \( r \geq 0 \) such that the propagation matrix of the \( k \)-step FLC method at the \( \infty \)-point, with all the stepsize ratios \( r_j \) equal to \( r \), have the spectral radius \( \leq 1 \):

\[ I_k^{\text{FLC}} = \{ r \in \mathbb{R}^+ \mid \beta_{1,n}(r, \ldots, r) \leq 1 \}. \]

These sets, together with the corresponding sets for the interpolatory case \( I_k^{\text{INT}} \), are presented in Table 3.2.

Again, the corresponding sets for INT and FLC methods are not very different, but now the VC method shows stability properties clearly superior to those of the INT and FLC methods.

Next, let us consider the behaviour of the spectral radius of the propagation matrices for \( z \leq 0 \) and all stepsize ratios \( r_j = r \). We will study in detail the case of two-step methods and, for the sake of brevity, we will present the main results for \( k \geq 3 \).

In the two-step FLC method the characteristic polynomial of its propagation matrix is given by

\[ \rho(\zeta, r) = -\frac{1}{4}(3 - 2z + r^2 + 3 + z - rz)\zeta^2 - (3 - 2z - zr)\zeta + r^2]. \]

For \( z \leq 0 \) and \( r \geq 0 \) the roots of \( \rho(\zeta) \) have modulus \( \leq 1 \) if and only if \( r^2 \leq 3 - 2z, \quad z(3 - r) \leq 0, \)
and these conditions are equivalent to

\[ \begin{align*}
& r^2 \leq 3 - 2z, \quad \text{for } z \in [-3, 0], \\
& r \leq 3, \quad \text{for } z \leq -3. 
\end{align*} \] (3.1)
A similar result is obtained for the two-step INT method. In this case the characteristic polynomial of its propagation matrix is

\[ \rho(\xi, r) = -\frac{1}{2} \left[ (3 - 2z)\xi^2 - (r^2 + 3 + rz - rz^2)\xi + r^2 \right], \]

and from the Schur’s conditions we conclude that the roots of \( \rho(\xi, r) \) have modulus \( \leq 1 \) if and only if

\[
\begin{align*}
& r^2 \leq 3 - 2z, & \text{for } z \in \left[ -\frac{1}{2}, 0 \right], \\
& r \leq 2, & \text{for } z \leq -\frac{1}{2}.
\end{align*}
\]

Finally, the characteristic polynomial of the two-step VC method is given by

\[ \rho(\xi, r) = \frac{1}{1+r} \left[ (1 + 2r - z - zr)\xi^2 - (1 + r)^2 \xi + r^2 \right], \]

and its roots are in modulus \( \leq 1 \) if and only if

\[ z(1+r) \leq 1 + 2r - r^2. \]

The conditions (3.1)–(3.3) determine in the \((z, r)\)-plane \((r \in \mathbb{R}^+, z \leq 0)\) the regions where the spectral radius of the propagation matrices of the corresponding two-step methods is \( \leq 1 \). The boundaries of these regions are plotted in Fig. 3.1.

For \( k \)-step methods with \( k = 3, 4, 5 \), we have determined for \( r_n = r_{n-1} = \cdots = r \), in a similar way, the regions where \( \rho(\Omega_k) < 1 \). The boundaries of these regions are plotted in Figs. 3.2–3.4. Note that, as in the two-step case, the boundary of the stability region for VC methods goes to infinity when \( z \to -\infty \), while for FLC and INT methods there exist for all \( k \), values of \( \alpha \) and \( r^* \) such that for \( z \leq \alpha, \rho(\Omega_k) \leq 1 \) if and only if \( r \leq r^* \). In addition to that, for the FLC methods of orders 4 and 5, there is a region (the shaded area in the figures) with \( r < 1 \) and \( z < 0 \) where \( \rho(\Omega_{FLC}) > 1 \). This means that in the FLC methods consecutive reductions of stepsize could lead to instabilities.

Next, we will study the spectral radius of products of propagation matrices of VC, FLC and INT methods. Since in many codes for solving stiff systems a change of order is contemplated only after running at order \( k \) at least for \( k + 1 \) steps, we will consider the spectral radius of products of \( \mu (\geq k + 1) \) consecutive propagation matrices.
In the two-step VC method we have a matrix product of type
\[
\Omega^{VC}(r_{n+\mu-1}; z_{n+\mu-1}) \cdots \Omega^{VC}(r_n; z_n),
\]
with \( z_{n+1} = h_{n+2}\lambda = r_{n+1}z_n \), and more generally,
\[
z_{n+j} = r_{n+j}r_{n+j-1} \cdots r_{n+1}z_n.
\]
Then the product (3.4) becomes
\[
\Omega^{VC}(r_{n+\mu-1}; r_{n+\mu-1}, \ldots, r_{n+1}z_n) \cdots \Omega^{VC}(r_n; z_n).
\]
Denoting by \( g_\mu(r, z) \) the spectral radius of (3.5) for
\[
r_{n+\mu-1} = r_{n+\mu-2} = \cdots = r_n = r, \quad z_n = z,
\]
we have determined numerically the sets of points
\[
\{(r, z); r > 0, z \leq 0, g_\mu(r, z) \leq 1\}
\]
for several values of \( \mu \geq 3 \) and they have been plotted in Fig. 3.5. Two conclusions can be drawn from this numerical study. First, denoting by \( r = \chi_\mu(z) \) the function which defines the boundary of (3.6), we have \( \chi_3(z) > \chi_1(z) \); therefore the effect of repeating for at least three consecutive steps improves clearly the stability of the two-step VC formula. Moreover, the stability properties also improve when \( z \to -\infty \). Secondly, due to the rapid increase of the function \( \chi_3(z) \) with \( -z \), the very stiff components allow almost arbitrary variations of the stepsize ratios.

The above study has been carried out for the three-, four- and five-step VC methods, and in Figs. 3.5–3.8 we have plotted the regions (3.6) for \( r_n = r_{n-1} = \cdots = r \). As can be seen from these graphs, the first conclusion given for the two-step VC method also holds for higher-order methods. However for \( k \geq 3 \), the boundary of the region (3.6) does not increase as rapidly with \( -z \) as in the case of the two-step VC methods; therefore suitable bounds of the upper bounds of \( r \) should be used to ensure their stability.

The improvement of the stability regions when we consider products of propagation matrices of VC methods can be partially explained if we take into account that for these methods,
3. Stability according to Figs. 3.1–3.4, the stability is better when \( z \to -\infty \) and that if \( r_n > 1 \), then \( z_{n+1} = r_n z_n < z_n \). However, for INT and FLC methods, the boundary of the stability region is a horizontal line for \( z \in (-\infty, z_0] \), for some value of \( z_0 < 0 \). Therefore, a great gain in the stability regions is not expected when we consider products of propagation matrices. In this sense, we have carried out, for INT and FLC, a study similar to that for VC methods and we have concluded that no significant gain is attained for these methods and even more, the stability regions for these methods are practically the same as those presented in Figs. 3.1–3.4 with the only difference in the value of \( z_0 \) mentioned above.

4. Stability with particular strategies of stepsize variation

In the last section we have found bounds on the stepsize ratios which ensure that the spectral radius of propagation matrices (or arbitrary products of them) remain \( \leq 1 \). However, it is unlikely that such a general situation holds in practice. Thus, for example, in LSODE [7], if the method being used has order \( k \), an increasing of the stepsize and/or a change of order is only allowed after \( k + 2 \) consecutive steps with constant size and order. This strategy is intended to improve the stability (recall that it ensures the O-stability for interpolatory Adams methods), and, moreover, for stiff problems it reduces the number of LU factorizations of the Jacobian matrix of \( f \).

In this section we examine the behaviour of the spectral radius of products of propagation matrices when the sequence of stepsize ratios satisfies the assumption used in LSODE.

**Theorem 4.1.** For the \( k \)-step FLC and INT methods we have:

(i) if \( r_m = \cdots = r_{n-k-2} = 1 \) and \( n \geq m + k - 1 \), then

\[
\prod_{j=m}^{n} \Omega_{FLC}^{PRO}(r_j, \ldots, r_{j-1-k+2}; \infty) = 0;
\]

(ii) \( \rho\left(\Omega_{INT}(r; \infty)(\Omega_{INT}(1, \infty))^{k-1}\right) = 0 \),
for all \( r < \infty \);

(iii) \( \Omega^{\text{INT}}(r; \infty)(\Omega^{\text{INT}}(1; \infty))^{k-1} \Omega^{\text{INT}}(r'; \infty) = 0 \),

for all \( r \) and \( r' < \infty \).

**Proof.** (i) Taking into account (2.6), when \( z = -\infty \) and \( n > m + k - 1 \), it is straightforward to see that the product of \( n \) consecutive propagation matrices of the \( k \)-step FLC method becomes

\[
\prod_{j=m}^{n} \Omega^{\text{FLC}}(r_j, \ldots, r_{j-k+2}; \infty) = (q_{ij}),
\]

with

\[
q_{ij} = \begin{cases} 
\prod_{l=m}^{n-i+1} \beta_{1,j}, & \text{if } j = 1, \\
0, & \text{otherwise}.
\end{cases}
\]

Since for equal stepsizes the FLC formula reduces to the fixed stepsize formula with the same number of steps, under this assumption the additional coefficient \( \beta_{1,n} \) of (2.4) vanishes. This implies that \( q_{11} = 0 \) for \( n > m + k - 1 \) and (i) is satisfied.

(ii) For \( z = -\infty \), the propagation matrix (2.9) reduces to

\[
\Omega^{\text{INT}}(r, \infty) = (I - le_l^T)PD(r).
\]

The computation of the product

\[
\Omega^{\text{INT}}(r, \infty)(\Omega^{\text{INT}}(1, \infty))^{k-1}
\]

has been done by using an algebraic manipulator and we have found that the matrix obtained has spectral radius zero.

(iii) We have computed the product of those matrices and it turns out to be the null matrix.

\[\square\]

**Corollary 4.2.** Let \( \{r_j\} \geq 0 \) be a bounded stepsize ratios sequence. If there is a positive integer \( q \) such that in any set of \( q + 2k - 1 \) consecutive steps we can find at least \( k \) consecutive steps with the same size, then for the \( k \)-order FLC or INT method, any product of consecutive propagation matrices is uniformly bounded.

**Proof.** First, consider the FLC case. From the above assumptions, in a product of consecutive matrices \( T_{n,m} = \Omega_n \cdots \Omega_m \) with \( n > m + 2(q + 2k - 1) \), we can find a subproduct \( T_{i,s} = \Omega_i \cdots \Omega_s \) with \( r_i = \cdots = r_{l-k+2} = 1 \) and \( s > l + k - 1 \) and from Theorem 4.1(i) it follows that \( T_{i,s} = 0 \). Therefore, \( T_{n,m} = 0 \) for all \( n > m + 2(q + 2k - 1) \). Furthermore, for any norm \( \| \cdot \| \), if \( r_j \leq r^* \) for all \( j \), there exists a positive constant \( K \) such that \( \| T_{n,m} \| \leq K \) for \( n, m \) with \( n - m \leq 2(q + 2k - 2) \). Consequently, \( \| T_{n,m} \| \leq K \) for all \( n, m \).

The proof for INT methods follows the same pattern. \( \square \)

Note that the hypothesis on the stepsize ratios used in Corollary 4.2 is equivalent to the following strategy of stepsize variation. Given a fixed positive integer \( q \) and starting from a grid point, e.g., \( t = t_1 \), we take a sequence of \( k \) steps of constant size, and after that we may
take \( q_1 (\leq q) \) steps with arbitrary steplength. Then, after \( k_2 (\geq k) \) consecutive constant steps we can again take \( q_2 (\leq q) \) arbitrary steps and so on.

Finally, a general analysis of the spectral radius of products of propagation matrices for arbitrary values of \( z \in \mathbb{R}^- \) would be too complicated. Then we have fixed our attention to typical sequences of stepsize ratios which arise in LSODE. Let us assume a stepsize sequence of type

\[
1, \ldots, 1, r_i, 1, \ldots, 1, r_{j+1}, 1, \ldots .
\]

Now, to study the product of the corresponding propagation matrices, note that it can be decomposed in products of consecutive matrices of type

\[
\Omega(r, z) = \Omega(1, \ldots, 1; z)\Omega(1, \ldots, 1, r; z) \cdots \Omega(r, 1, \ldots, 1; z),
\]

for FLC and VC methods, and

\[
\Omega(r, z) = \Omega(1; z)^k \Omega(r; z),
\]

for INT methods.

The computation of the spectral radius of these matrices has been carried out numerically and the regions where it is \( \leq 1 \) have been plotted in Figs. 4.1–4.4.
A remarkable feature of these graphs is that the boundary of the regions go to $\infty$ when $z \to -\infty$ for the three types of methods. Once again, the stability regions for the VC methods are larger than those of INT and FLC methods. For the last two types of methods, they are similar.

References


