Products of commutators of transvections over local rings

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Abstract

Let $R$ be a commutative local ring, and $M$ the maximal ideal of $R$. We prove that if $|R/M| > 3$, then every matrix in $\text{SL}_n R$ $(n \geq 2)$ is a product of at most $\lfloor n/2 \rfloor + 2$ commutators of transvections and the length $\lfloor \text{res} \ A/2 \rfloor + 2$ cannot be reduced for $n = 2$.

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1. Introduction

Let $S$ be a set of special generators of a group $G$. It is instructive to determine the smallest integer $k \geq 0$ (in general, denote $k$ by $l(G)$) such that every element of $G$ is a product of at most $k$ elements in $S$. Of particular interest has been the case when $G$ is the special linear group over a field and $S$ is the set of transvections. If $A$ is an element in $\text{SL}_n F$, then the minimal number, $l(A)$, of transvections needed to express $A$ is given by $\text{res} \ A + 1$ (see [1]).

Another type of generators is commutator. It was considered by Shoda [2], Thompson [3], Vaserstein and Wheland [4]. One can ask how many commutators of a special nature are needed to represent a matrix in $\text{SL}_n F$ $(F \neq F_2, F_3)$, the problems
have been studied in [7,8]. In [8], the authors proved that every element $A$ in $SL_n F$, where $F$ is a field with at least three elements, is the product of at most $\lfloor \text{res} A/2 \rfloor + 2$ commutators of transvections for all $n \geq 2$. In this paper, we extend the result to commutative local rings.

Let $R$ be a commutative local ring, and $M$ the maximal ideal of $R$. Let $R^n$ denote the $R$-module of $n$-columns over $R$. We shall identify $\text{Aut}(R^n)$ with $GL_n R$ and denote the subgroup of $GL_n R$ which consists of the matrices with determinant $1$ by $SL_n R$. The natural map $\pi : R \to R/M$ will induce the natural map $R^n \to (R/M)^n$ (still denote the map by $\pi$), and the group homomorphism $\lambda : GL_n R \to GL_n(R/M)$, $SL_n R \to SL_n(R/M)$. Sometimes we write $a$ for the image of an element $a$ of $R$ in $R/M$. For any matrix $A$ in $GL_n R$, write $\text{res} A = n - \dim(\pi P)$, where $P = \{v \in R^n \mid Av = v\}$ is the fixed submodule of $A$ (here, the definition of $\text{res} A$ is equivalent to the $u$-rank of $A - I_n$ defined in [5]). An invertible matrix $A$ over $R$ is a transvection, if $\text{res} A = 1$ and $\det A = 1$.

Here is the main result of this paper.

**Theorem.** Assume that $|R/M| > 3$. Then every matrix $A$ in $SL_n R$ is the product of at most $\lfloor \text{res} A/2 \rfloor + 2$ commutators of transvections ($\lfloor \cdot \rfloor$ denotes the integer part of a rational number).

2. Basic lemmas

We write $A \sim B$ to mean that $A$ and $B$ are similar matrices (in $GL_n R$). Sometimes we still denote a matrix which is similar to $A$ by $A$.

**Lemma 1.** Let $A \in SL_n R \ (n \geq 2)$, and $\lambda(A)$ be not in the center of $SL_n(R/M)$. Then $A$ is similar to a matrix whose first column has $1$ for the second entry and zeros for the other entries.

**Proof.** (1) If $A$ has a unit entry $c$ off the main diagonal, say $c$ in the $ji$th ($j \neq i$) place. Conjugating $A$ by permutation matrices, we get that $A$ has a unit entry $c$ in the second place of the first column. In fact we may take $c = 1$ (use diagonal matrix to conjugate $A$). Then the matrix

$$\prod_{i \neq 2} T_{i2}(-a_{11}) A \prod_{i \neq 2} T_{i2}(a_{11}),$$

where $T_{i2}(a)(i \neq 2)$ is the matrix with $a$ in the $i2$th entry and coinciding with $I_n$ in every other entry, is required.

(2) If $\lambda(A)$ is a diagonal matrix $\text{diag}(\overline{a}_{11}, \ldots, \overline{a}_{jj}, \ldots, \overline{a}_{nn})$ with $\overline{a}_{ii} \neq \overline{a}_{jj}$ (in $R/M$), then $T_{jj}(1) A T_{jj}(-1)$ has an unit $c$ ($\overline{c} = \overline{a}_{jj} - \overline{a}_{ii} \neq 0$) in the $ji$th ($j \neq i$) place. Return it to the case (1). □
We say that a matrix $A \in SL_n R$ is quasicentral if there is $u \in R^*$ (the set of units in $R$) such that $\text{rank}[\lambda(A - uI_n)] \leq 1$.

We write an $n$ by $n$ matrix $A$ in block form with all blocks 2 by 2 as follows:

$$
\begin{bmatrix}
A_{11} & A_{12} & \cdots & B_1 \\
A_{21} & A_{22} & \cdots & B_2 \\
\vdots & \vdots & \ddots & \vdots \\
C_1 & C_2 & \cdots & d
\end{bmatrix}
$$

(1)

When $n = 2k + 1$ is odd, $B_i$ is a 2 by 1 matrix, $C_i$ is a 1 by 2 matrix, and $d$ is a scalar.

**Lemma 2.** Assume that $n \geq 4$ and $A \in SL_n R$. Then $A$ is similar to a matrix whose (2, 1) block $A_{21}$ is invertible (see (1)) if and only if $A$ is not quasicentral.

**Proof.** The proof is similar to Lemma 2 in [6], although we proved the same result only for any field in [6]. Here, we need only to point out that a 2 by 2 matrix $A$ over $R$ is invertible if and only if $\lambda(A)$ is invertible over $R/M$. □

**Remark 1.** If the block $A_{21}$ is invertible, we may take $A_{21}$ as any invertible 2 by 2 matrix such as $I_2$ or $-I_2$, since the two matrices are similar.

An element $C \in SL_n R$ is a commutator of transvections (or transvection commutator) if $C = [A_1, A_2] = A_1A_2A_1^{-1}A_2^{-1}$ with $A_1$ and $A_2$ transvections.

**Lemma 3.** Let $A \in SL_2 R$ and $\lambda(A)$ be not in the center of $SL_2(R/M)$. If $\text{Tr}(A) - 2 \in R^*$, where $\text{Tr}$ denotes the trace of a matrix, then $A$ is a transvection commutator.

**Proof.** By the assumption and Lemma 1, we can assume

$$
A = \begin{bmatrix}
0 & -1 \\
1 & x
\end{bmatrix}.
$$

Since $\text{Tr}(A) - 2 \in R^*$, there exists $k \in R^*$ such that $x = k^2 + 2$. We have $A = [A_1, A_2]$, where

$$
A_1 = \begin{bmatrix}
0 & -1 \\
1 & 2
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
2 - k^{-1} & (k - 1)(k^{-1} - 1) \\
k^{-1} & k^{-1}
\end{bmatrix}
$$

both are transvections. □

**Lemma 4.** Let $n \geq 3$, and $A \in SL_n R$. If

$$
A \sim \begin{bmatrix}
0 & -1 \\
1 & 2 + k^2
\end{bmatrix} \otimes I_{n-2}, \quad \text{where } k \in R^* \cup \{0\},
$$

We say that a matrix $A \in SL_n R$ is quasicentral if there is $u \in R^*$ (the set of units in $R$) such that $\text{rank}[\lambda(A - uI_n)] \leq 1$.
then $A$ is a transvection commutator. In particular, if $A$ is a transvection and $\lambda(A)$ is not in the center of $SL_n(R/M)$, then $A$ is a transvection commutator.

**Proof.** When $k \in R^*$, see Lemma 3. If $k = 0$, $A$ is a transvection. Since $\lambda(A)$ is not in the center of $SL_n(R/M)$, so

$$A \sim \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \oplus I_{n-2} = [T_{13}(1), T_{32}(1)].$$

□

**Lemma 5.** Let $A \in SL_2 R$ be a transvection commutator and $A - I_2$ invertible. For any $L \in R^{2 \times 1}$, $M \in R^{2 \times 2}$ the matrices

\begin{align*}
\begin{bmatrix} A + I_2 & -I_2 \\ A & 0 \end{bmatrix}, & \begin{bmatrix} A + I_2 & -I_2 & L \\ A & 0 & L \\ 0 & 0 & 1 \end{bmatrix}_{5 \times 5} \\
\begin{bmatrix} A + I_2 & -I_2 & M \\ A & 0 & M \\ 0 & 0 & I_2 \end{bmatrix}_{6 \times 6}
\end{align*}

are transvection commutators.

**Proof.** Let

$$T = \begin{bmatrix} I_2 & I_2 \\ I_2 & A \end{bmatrix}.$$ 

Since $A - I_2$ is invertible, so is $T$. Then

$$T^{-1} \begin{bmatrix} A + I_2 & -I_2 \\ A & 0 \end{bmatrix} T = \begin{bmatrix} A & 0 \\ 0 & I \end{bmatrix}.$$ 

\begin{align*}
\begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_2 & L \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} A + I_2 & -I_2 & 0 \\ A & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} A + I_2 & -I_2 & L \\ A & 0 & L \\ 0 & 0 & 1 \end{bmatrix} \\
\begin{bmatrix} I_2 & 0 & 0 \\ 0 & I_2 & M \\ 0 & 0 & I_2 \end{bmatrix} & \begin{bmatrix} A + I_2 & -I_2 & 0 \\ A & 0 & 0 \\ 0 & 0 & I_2 \end{bmatrix}^{-1} = \begin{bmatrix} A + I_2 & -I_2 & M \\ A & 0 & M \\ 0 & 0 & I_2 \end{bmatrix}.
\end{align*}
3. Initial observations

Lemma 6. Assume that $|R/M| > 3$. Let $A \in SL_2(R)$. If $\lambda(A)$ is not in the center of $SL_2(R/M)$, then $A$ can be written as a product of at most two transvection commutators.

Proof. By Lemma 1 we can assume
\[
A = \begin{bmatrix}
0 & -1 \\
1 & x
\end{bmatrix}.
\]
Since $|R/M| > 3$, there exists $t \in R^*$ such that $t^2 - 1 \in R^*$. Further, there exists $s \in R^*$ such that $(t^2 - 1)s - 2 + x \in R^{*2}$, as $|(R/M)^{*2}| > 1$. Set $k^2 = (t^2 - 1)s - 2 + x$, where $k \in R^*$, and let
\[
A_1 = \begin{bmatrix}
-s & -1 \\
(s(2 + s + k^2) + 1 & 2 + s + k^2
\end{bmatrix}, \quad A_2 = \begin{bmatrix}
1 & x - (2 + s + k^2) \\
-ts & (ts)^2 + 1
\end{bmatrix}.
\]
Then $A = A_1A_2$. By Lemma 3 we know that $A_1$ and $A_2$ both are transvection commutator. □

Lemma 7. Assume that $|R/M| > 3$ and $n \geq 2$. Let $A \in SL_n(R)$ with $\text{res} A = 1$. Then $A$ can be written as a product of two transvection commutators.

Proof. When $n = 2$, we can assume
\[
A = \begin{bmatrix}
1 & 0 \\
a & 1
\end{bmatrix}, \quad a \in R.
\]
Since $|R/M| > 3$, there exists $x \in R^*$ such that $(\pi x)^2 \neq 1$, so $x - (1/x) \in R^*$. Take $b \in R^*$ such that $b + ax^2 \in R^*$ and let
\[
B = \begin{bmatrix}
x^2 & 0 \\
b & x^{-2}
\end{bmatrix}, \quad C = \begin{bmatrix}
x^2 & 0 \\
b + ax^2 & x^{-2}
\end{bmatrix}.
\]
It is obvious that $\lambda(B)$ and $\lambda(C)$ both are not in the center of $SL_2(R/M)$ and $\text{Tr}(B) - 2 = \text{Tr}(C) - 2 = x^2 + x^{-2} - 2 = (x - (1/x))^2 \in R^{*2}$. By Lemma 3, $B$ and $C$ both are transvection commutator. We have
\[
A = \begin{bmatrix}
1 & 0 \\
a & 1
\end{bmatrix} = CB^{-1},
\]
i.e., $A$ is a product of two transvection commutators. When $n \geq 3$, we can assume
\[
A = \begin{bmatrix}
1 & 0 \\
a & I_{n-1}
\end{bmatrix}, \quad \text{where } a \in R^{n-1}.
\]
If $\lambda(A)$ is not in the center of $SL_n(R/M)$, $A$ is a transvection commutator by Lemma 4. If $\lambda(A)$ is in the center of $SL_n(R/M)$, set
\[ \beta = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \in \mathbb{R}^{n-1}, \]

and

\[ B = \begin{bmatrix} 1 & 0 \\ \alpha - \beta & I_{n-1} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ \beta & I_{n-1} \end{bmatrix}. \]

By Lemma 4, \( B \) and \( C \) both are transvection commutator. We have \( A = BC \), i.e., \( A \) can be written as a product of two transvection commutators. □

Let \( A \in \text{SL}_n \mathbb{R} \) and \( \text{res} \ A = r \). Then \( A \) is similar to a matrix of the form

\[ \begin{bmatrix} D^{(r)}_A & 0 \\ \ast & \cdots & \ast \\ \vdots & \vdots & I_{n-r} \end{bmatrix}. \tag{3} \]

We call (3) a normal form of \( A \).

**Lemma 8.** Assume that \( |R/M| > 3 \) and \( n \geq 2 \). Let \( A \in \text{SL}_n \mathbb{R} \) with \( \text{res} \ A = 2 \). Then \( A \) is a product of at most three transvection commutators. If \( \lambda(D^{(2)}_A) \) in the normal form of \( \lambda(A) \) is not in the center of \( \text{SL}_2(R/M) \), \( A \) can be written as a product of two transvection commutators.

**Proof.** (1) When \( \lambda(D^{(2)}_A) \) (see (3)) is not in the center of \( \text{SL}_2(R/M) \), \( A \) is similar to the form of the matrix

\[ \begin{bmatrix} 0 & -1 & 0 \\ 1 & b_2 & 0 \\ \vdots & \vdots & I_{n-2} \\ 0 & b_n \end{bmatrix}. \]

If \( n = 2 \), the consequence follows from Lemma 6. Let \( n \geq 3 \) and let

\[ A_1 = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \oplus I_{n-2}. \]

\( A_1 \) is a transvection commutator by Lemma 4, and

\[ A_1A = \begin{bmatrix} 1 & b_2 - 2 & 0 \\ 0 & 1 & 0 \\ 0 & b_3 & I_{n-2} \\ \vdots & \vdots & \vdots \\ 0 & b_n & I_{n-2} \end{bmatrix}. \]
If $b_2 - 2 \in R^*$, then $A_1 A$ is a transvection commutator by Lemma 4. If $b_2 - 2 \in M$, replacing $A_1$ by

$$A_2 = \begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix} \oplus I_{n-2},$$

we have

$$A_2 A = \begin{bmatrix} 1 & b_2 - 3 & 0 \\ 0 & 1 & 0 \\ 0 & b_3 & I_{n-2} \\ \vdots & \vdots & \vdots \\ 0 & b_n \\ & & \end{bmatrix}.$$ 

Since $b_2 - 3 = (b_2 - 2) + 1 \in R^*$, we return it to the above case.

(2) If $\lambda(D_A^{(3)})$ is in the center of $SL_2(R/M)$, we multiply $A$ on the right by

$$\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \oplus I_{n-2},$$

which is a product of two transvection commutators. Hence, $A$ is a product of three transvection commutators. □

**Lemma 9.** Let $A \in SL_3 R$ be not quasicentral. Then

$$A \sim \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & b_2 \\ 0 & 1 & b_3 \end{bmatrix}.$$ 

**Proof.** Let $f(x)$ be the characteristic polynomial of $\lambda(A)$. When $f(x)$ is also the minimal polynomial of $\lambda(A)$, then there exists $u \in R^3$ such that $(\pi(u), \pi(Au), \pi(A^2u))$ is linearly independent over $R/M$. Then $\{u, Au, A^2u\}$ forms a base in the free module $R^3$. Let $T = (u, Au, A^2u)$. We have

$$T^{-1} A T = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & b_2 \\ 0 & 1 & b_3 \end{bmatrix}.$$ 

If $f(x)$ is not equal to the minimal polynomial $m(x)$ of $\lambda(A)$, since $m(x)$ has degree 2 or 1, the invariant factors of $\lambda(A)$ are 1, $x - \overline{a}$, $(x - \overline{a})(x - \overline{b})$, or $x - \overline{a}, x - \overline{a}, x - \overline{a}$, where $a, b \in R^*$. So in any case above, there exists $a \in R^*$ such that $\text{rank} [\lambda(A - aI_3)] \leq 1$, which is contradictory to $A$ being not quasicentral. □

**Lemma 10.** Assume that $|R/M| > 3$ and $n \geq 3$. Let $A \in SL_n R$ with $\text{res } A = 3$. Then $A$ is a product of at most three transvection commutators. If $D_A^{(3)}$ in the normal form of $A$ is not quasicentral, then $A$ can be written as a product of two transvection commutators.
Proof. When $D^{(3)}_A$ is not quasicentral, by Lemma 9 we can assume

$$A = \begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & b_2 \\
0 & 1 & b_3 \\
0 & 0 & b_4 \\
\vdots & \vdots & I_{n-3} \\
0 & 0 & b_n
\end{bmatrix}.$$  

Let

$$H_1 = \begin{bmatrix}
4 & 1 & 0 \\
-4 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix} \oplus I_{n-3}.$$  

Since

$$\begin{bmatrix}
2 & 5 & 1 \\
-3 & -7 & -3 \\
1 & 2 & 1
\end{bmatrix}^{-1} \begin{bmatrix}
4 & 1 & 0 \\
-4 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
2 & 5 & 1 \\
-3 & -7 & -3 \\
1 & 2 & 1
\end{bmatrix} = \begin{bmatrix}
0 & -1 & 0 \\
1 & 3 & 0 \\
0 & 0 & 1
\end{bmatrix}.$$  

$H_1$ is a transvection commutator by Lemma 4.

Multiplying $A$ on the left by $H_1$, we have

$$H_1 A = \begin{bmatrix}
1 & 0 & 4 + b_2 \\
0 & 1 & -4 + b_3 \\
0 & 0 & 1 \\
0 & 0 & b_4 \\
\vdots & \vdots & I_{n-3} \\
0 & 0 & b_n
\end{bmatrix}.$$  

If $4 + b_2 \in R^{*}$, $H_1 A$ is a transvection commutator by Lemma 4, and the consequence follows. If $4 + b_2 \in M$, since $|R/M| > 3$, there exists $k \in R^{*}$ such that $\pi(k^2 - 1) \neq 0$ and $\pi(4 + k^2) \neq 0$. Let

$$H_2 = \begin{bmatrix}
3 + k^2 & 1 & 0 \\
-3 - k^2 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix} \oplus I_{n-3}.$$  

Since

$$\begin{bmatrix}
2 + k^2 & 1 & 1 \\
-4 - k^2 & 0 & -2 - k^2 \\
2 & -1 & 1
\end{bmatrix}^{-1} \begin{bmatrix}
3 + k^2 & 1 & 0 \\
-3 - k^2 & 0 & 1 \\
1 & 0 & 0
\end{bmatrix} \begin{bmatrix}
2 + k^2 & 1 & 1 \\
-4 - k^2 & 0 & -2 - k^2 \\
2 & -1 & 1
\end{bmatrix} = \begin{bmatrix}
1 + k^2 & 1 & 0 \\
k^2 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.$$  

$H_2$ is a transvection commutator by Lemma 3. Replacing $H_1$ by $H_2$, we have
\[ H_2A = \begin{bmatrix}
1 & 0 & 3 + k^2 + b_2 & 0 \\
0 & 1 & -3 - k^2 + b_3 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & b_4 & I_{n-3}
\end{bmatrix}. \]

Note that \(3 + k^2 + b_2 = (k^2 - 1) + (4 + b_2) \in \mathbb{R}^*,\) we have showed the result.

When \(D_A^{(3)}\) is quasicentral, we can choose a transvection commutator \(H_1\) such that \(\text{res}\,(H_1A) \leq 4\) and \(D_{H_1A}^{(i)}(i \leq 3)\) is not quasicentral. By Lemmas 7, 8 and the above result, \(A\) can be written as a product of at most three transvection commutators.

**Lemma 11.** Assume that \(|R/M| > 3\) and \(n \geq 4.\) Let \(A \in SL_nR\) with \(\text{res}\,A = 4.\) Then \(A\) is a product of at most four transvection commutators. If \(D_A^{(4)}\) in the normal form of \(A\) is not quasicentral, then \(A\) can be written as a product of three transvection commutators.

**Proof.** (1) When \(D_A^{(3)}\) is not quasicentral, by Lemma 2 we can assume

\[
A = \begin{bmatrix}
0 & A_{12} & 0 \\
-\lambda & A_{22} & 0 \\
0 & * & I_{n-4}
\end{bmatrix}.
\]

Let

\[
H_1 = \begin{bmatrix}
C + I & -I \\
C & I
\end{bmatrix} \oplus I_{n-4},
\]

where

\[
C = \begin{bmatrix}
2 & 1 \\
1 & 1
\end{bmatrix}
\]

\((H_1\) is a transvection commutator). We have

\[
H_1A \sim \begin{bmatrix}
CA_{12} & 0 \\
* & I_{n-2}
\end{bmatrix}.
\]

If \(\lambda(CA_{12})\) is not in the center of \(SL_2(R/M),\) by Lemma 8 \(H_1A\) is a product of two transvection commutators, then \(A\) is a product of three transvection commutators. If \(\lambda(CA_{12})\) is in the center of \(SL_2(R/M),\) replacing \(C\) by

\[
C_1 = \begin{bmatrix}
1 + k^2 & 1 \\
k^2 & 1
\end{bmatrix},
\]

where \(k \in \mathbb{R}^*\) such that \(\pi k^2 \neq 1,\) we have \(\lambda(C_1A_{12})\) is not in the center of \(SL_2(R/M).\)

(2) When \(D_A^{(4)}\) is quasicentral, we can choose a transvection commutator \(H_1\) such that \(\text{res}\,(H_1A) \leq 4\) and \(D_{H_1A}^{(i)}(2 \leq i \leq 4)\) is not quasicentral. By Lemmas 7, 8, 10 and the above result, \(A\) is a product of at most four transvection commutators. \(\Box\)
4. Proof of the theorem

Lemma 12. Assume that $|R/M| > 3$. For every matrix

$$D = \begin{bmatrix}
0 & A_{11} & A_{12} & \cdots & A_{1k} \\
-I_2 & * & * & \cdots & * \\
0 & A_{21} & A_{22} & \cdots & A_{2k} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & A_{k1} & A_{k2} & \cdots & A_{kk}
\end{bmatrix} \in SL_r R,$$

where $r \geq 5$, and $A_{ij}$ is a $2$ by $2$ matrix (if $r$ is odd, $A_{kj}$ is a $1$ by $2$ matrix, $A_{ik}$ is a $2$ by $1$ matrix, $i, j = 1, 2, \ldots, k - 1$, and $A_{kk}$ is a scalar), there is a $2$ by $2$ transvection commutator $C$ with $C - I_2$ invertible, and a $2$ by $2$ matrix

$$L = \begin{bmatrix} s \\ t \\ 0 \\ 0 \end{bmatrix} \quad (if \ r = 5 \ L = \begin{bmatrix} s \\ t \end{bmatrix} \text{ is a } 2 \text{ by } 1 \text{ matrix})$$

such that the following matrix:

$$B = \begin{bmatrix}
CA_{11} + LA_{21} & CA_{12} + LA_{22} & \cdots & CA_{1k} + LA_{2k} \\
A_{21} & A_{22} & \cdots & A_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
A_{k1} & A_{k2} & \cdots & A_{kk}
\end{bmatrix}$$

is not quasicentral.

Proof. Let

$$C_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad \text{and} \quad C_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}.$$

Then $C_1$ and $C_2$ both are transvection commutators by Lemma 3 and $C_1 - I_2$, $C_2 - I_2$ both are invertible. Write

$$Q = \begin{bmatrix}
C_1A_{11} & C_1A_{12} & \cdots & C_1A_{1k} \\
A_{21} & A_{22} & \cdots & A_{2k} \\
\vdots & \vdots & \ddots & \vdots \\
A_{k1} & A_{k2} & \cdots & A_{kk}
\end{bmatrix} = \begin{bmatrix}
q_{11} & q_{12} & \cdots & q_{1,r-2} \\
q_{21} & q_{22} & \cdots & q_{2,r-2} \\
\vdots & \vdots & \ddots & \vdots \\
q_{r-2,1} & q_{r-2,2} & \cdots & q_{r-2,r-2}
\end{bmatrix}.$$

(i) If $Q$ is not quasicentral, we take $s = 0, t = 0$, i.e.,

$$L = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$
(ii) If $Q$ is quasicentral, we can assume that $q_{12}$ or $q_{21}$ is invertible in $R$. Otherwise, replacing $C_1$ by $C_2$, we have

$$Q_1 = \begin{bmatrix} C_2C_1^{-1}A_{11} & C_2A_{12} & \cdots & C_2A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix} = \begin{bmatrix} q_{21} & q_{22} & q_{23} & \cdots & q'_{1,r-2} \\ 3q_{21} - q_{11} & 3q_{22} - q_{12} & q'_{13} & \cdots & q'_{2,r-2} \\ \vdots & \vdots & \ddots & \vdots \\ qr_{r-2,1} & qr_{r-2,2} & qr_{r-2,3} & \cdots & qr_{r-2,r-2} \end{bmatrix}.$$ 

If $q_{11}, q_{22}$ both lie in $M$ (note that $q_{12}, q_{21} \in M$), then $Q$ is not quasicentral. Replacing $Q$ by $Q_1$, we are done. Assume $q_{21} \neq M$, i.e., $q_{21} \neq 0$, which does not loss generality. For any $x \in R^*$, write

$$\alpha_1(x) = \begin{bmatrix} q_{11} - x \\ q_{21} \\ q_{31} \\ \vdots \\ qr_{r-2,1} \end{bmatrix},$$

$$\alpha_2(x) = \begin{bmatrix} q_{12} \\ q_{22} - x \\ q_{32} \\ \vdots \\ qr_{r-2,2} \end{bmatrix},$$

$$\alpha_3(x) = \begin{bmatrix} q_{13} \\ q_{23} \\ q_{33} \\ \vdots \\ qr_{r-2,3} \end{bmatrix} + x \begin{bmatrix} s \\ t \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$ 

If the set $\{\pi \alpha_1(x), \pi \alpha_2(x)\}$ is linearly dependent, then

$$\det \left[ \lambda \begin{bmatrix} q_{11} - x & q_{12} \\ q_{21} & q_{22} - x \end{bmatrix} \right] = 0.$$ 

That means there are $x_1, x_2 \in R^*$ such that $\tilde{x} = \tilde{x}_1$ or $\tilde{x} = \tilde{x}_2$ (note that $\tilde{x}^2 - (\tilde{q}_{11} + \tilde{q}_{22})\tilde{x} + \tilde{q}_{11}\tilde{q}_{22} - \tilde{q}_{12}\tilde{q}_{21} = 0$). Since $\tilde{q}_{21} \neq 0$, $\tilde{x}_1 \neq 0$, $\tilde{x}_2 \neq 0$ and $|R/M| > 3$, we may find $s \in R^*$ and take $t = 0$ such that...
\[
\det \left[ \lambda \begin{bmatrix} q_{11} - x_1 & q_{13} + x_2 \tilde{s} \\ q_{21} & q_{23} + x_1 \tilde{s} \end{bmatrix} \right] = \tilde{q}_{11} \tilde{q}_{23} - \tilde{x}_i \tilde{q}_{23} - \tilde{q}_{21} \tilde{x}_i \tilde{s} \neq 0
\]
for \( i = 1, 2 \), i.e., \( \{\pi \alpha_1(x_1), \pi \alpha_3(x_1)\} \) and \( \{\pi \alpha_1(x_2), \pi \alpha_3(x_2)\} \) are linearly independent sets. For any \( x \in \mathbb{R}^* \)

\[
B - xI_{(r-2)} = \begin{bmatrix} CA_{11} + LA_{21} & CA_{12} + LA_{22} & \ldots & CA_{1k} + LA_{2k} \\ A_{21} & A_{22} & \ldots & A_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ A_{k1} & A_{k2} & \ldots & A_{kk} \end{bmatrix} - xI_{(r-2)}
\]

\[
= \begin{bmatrix} I_2 & L & 0 \\ 0 & I_{(r-4)} & \end{bmatrix} \left( \begin{bmatrix} I_2 & -L \\ 0 & I_{(r-4)} \end{bmatrix} \right)
\]

Thus \( \text{rank} \left[ \lambda (B - xI_{(r-2)}) \right] = \text{rank} \left[ \lambda (\alpha_1(x), \alpha_2(x), \alpha_3(x), \ldots) \right] \geq 2 \). Hence \( B \) is not quasicentral.

Now let us finish the proof of the Theorem. \( \square \)

Let \( \text{res} \ A = r \). By Lemmas 7 to 11, we need only to show this for \( r \geq 5 \). Assume that the matrix \( D_A^{(r)} \) in the normal form of \( A \) is not quasicentral and is similar to the matrix

\[
D = \begin{bmatrix} 0 & A_{11} & A_{12} & \ldots & A_{1k} \\ -I_2 & * & * & \ldots & * \\ 0 & A_{21} & A_{22} & \ldots & A_{2k} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & A_{k1} & A_{k2} & \ldots & A_{kk} \end{bmatrix} \in SL_r \mathbb{R},
\]

where \( r \geq 5 \), and \( A_{ij} \) is a 2 by 2 matrix (if \( r \) is odd, then \( A_{ij} \) is a 1 by 2 matrix, \( A_{ik} \) is a 2 by 1 matrix, \( i, j = 1, 2, \ldots, k-1 \)). We multiply \( A \) on the left by

\[
H_1 = \begin{bmatrix} C + I_2 & -I_2 & L \\ C & 0 & L \\ 0 & 0 & I \end{bmatrix} \oplus I,
\]

where \( C \) (transvection commutator) and \( L \) are chosen according to the arguments in the Lemma 12 (by Lemma 5, \( H_1 \) is a transvection commutator). Then \( H_1 A \) is similar to the matrix
and the left-upper block in (4) is not quasicentral. Note that \( \text{res} (H_1 A) \) is at most \( r - 2 \). Continuing the procedure until \( \text{res} (H_{(r-4)/2} \cdots H_2 H_1 A) = r' \leq 4 \) if \( r \) is even or \( \text{res} (H_{(r-3)/2} \cdots H_1 A) = r' \leq 3 \) if \( r \) is odd, we may apply Lemmas 7, 8, 10 and 11 to \( H_k \cdots H_1 A \) \( (k = (r-4)/2 \) or \( (r-3)/2) \) to obtain that \( H_k \cdots H_1 A \) is a product of at most three or two transvection commutators (note that \( D^{(r')}_{H_k \cdots H_1 A} \) is not quasicentral), which depends on the parity of \( r \). So \( A \) is a product of at most \( \left\lceil \frac{r}{2} \right\rceil + 1 \) transvection commutators. When the \( D^{(r')}_{A} \) in the normal form of \( A \) is quasicentral, we need only to multiply \( A \) on the left by a suitable transvection commutator \( H \) such that the \( D^{(r')}_{HA} \) in the normal form of HA is not quasicentral and \( \text{res} HA \leq \frac{r}{2} \), then \( A \) is a product of at most \( \left\lceil \frac{r}{2} \right\rceil + 2 \) transvection commutators.

In [8], we proved that \(-I_2\) cannot be written as a product of two transvection commutators over the field of real numbers. So in general, the length \( \left\lceil \text{res} A/2 \right\rceil + 2 \) cannot be reduced.

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**References**