Available online at www.sciencedirect.com

ScienceDirect

Computers and Mathematics with Applications 53 (2007) 803–824

An International Journal
computers & mathematics with applications

www.elsevier.com/locate/camwa

Hybrid fixed point theory for strictly monotone increasing multi-valued mappings with applications

B.C. Dhage

Kasubai, Gurukul Colony, Ahmedpur-413 515, Dist: Latur, Maharashtra, India

Received 8 January 2006; received in revised form 2 September 2006; accepted 11 October 2006

Abstract

In this paper, a general hybrid fixed point theorem for the strict monotone increasing multi-valued mappings in ordered Banach spaces is proved via measure of noncompactness and it is further applied to perturbed functional nonconvex differential inclusions for proving the existence results for the extremal solutions under mixed Lipschitz, compactness and strict monotonic conditions.

Keywords: Ordered Banach space; Hybrid fixed point theorem; Differential inclusion; Existence theorem

1. Introduction

Throughout this paper, unless otherwise mentioned, let X denote a Banach space with norm $\|\cdot\|$ and let $\mathcal{P}_p(X)$ denote the class of all nonempty subsets of X with property p. Here, p may be p = closed (in short cl) or p = convex (in short cv) or p = bounded (in short bd) or p = compact (in short cp). Thus $\mathcal{P}_{cl}(X)$, $\mathcal{P}_{cv}(X)$, $\mathcal{P}_{bd}(X)$ and $\mathcal{P}_{cp}(X)$ denote, respectively, the classes of all closed, convex, bounded and compact subsets of X. Similarly, $\mathcal{P}_{cl,bd}(X)$ and $\mathcal{P}_{cp,cv}(X)$ denote, respectively, the classes of closed–bounded and compact–convex subsets of X.

For $x \in X$ and $Y, Z \in \mathcal{P}_{bd,cl}(X)$ we denote by $D(x, Y) = \inf\{\|x - y\| \mid y \in Y\}$, and $\rho(Y, Z) = \sup_{a \in Y} D(a, Z)$. Define a function $d_H : \mathcal{P}_{cl}(X) \times \mathcal{P}_{cl}(X) \to \mathbb{R}^+$ by

$$d_H(Y, Z) = \max\{\rho(Y, Z), \rho(Z, Y)\}. \tag{1.1}$$

The function d_H is called a Hausdorff metric on X. Note that $\|Y\|_{\mathcal{P}} = d_H(Y, \{0\})$.

A correspondence $T : X \to \mathcal{P}_p(X)$ is called a multi-valued operator or mapping on X. A point $x_0 \in X$ is called a fixed point of the multi-valued operator $T : X \to \mathcal{P}_p(X)$ if $x_0 \in T(x_0)$. The fixed points set of T in X will be denoted by $\mathcal{F}_T$.

Definition 1.1 (*Dhage [1]*). A multi-valued operator $T : X \to \mathcal{P}_{cl}(X)$ is called *$\mathcal{D}$-Lipschitz* if there exists a continuous and nondecreasing function $\psi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$d_H(Tx, Ty) \le \psi(\|x - y\|) \tag{1.2}$$

E-mail address: bcd20012001@yahoo.co.in.

0898-1221/$ - see front matter
doi:10.1016/j.camwa.2006.10.020

for all $x, y \in X$, where $\psi(0) = 0$. The function ψ is called a $\mathcal{D}$-function of T on X. If $\psi(r) = kr$ for some $k > 0$, then T is called a Lipschitz on X with the Lipschitz constant k. Further if $k < 1$, then T is called a multi-valued contraction on X with contraction constant k. Finally, if $\psi(r) < r$ for $r > 0$, then T is called a nonlinear $\mathcal{D}$-contraction on X.

Let X be a metric space. A multi-valued mapping $T : X \to \mathcal{P}_{cl}(X)$ is called *lower semi-continuous* (resp. *upper semi-continuous*) if G is any open subset of X then $\{x \in X \mid Tx \cap G \neq \emptyset\}$ (resp. $\{x \in X \mid Tx \subset G\}$) is an open subset of X. The multi-valued mapping T is called *totally compact* if $\overline{T(S)}$, the closure of $T(S)$, is a compact subset of X for any $S \subset X$. T is called *compact* if $\overline{T(S)}$ is a compact subset of X for all bounded subsets S of X. Again T is called *totally bounded* if for any bounded subset S of X, $T(S)$ is a totally bounded subset of X. A multi-valued mapping $T : X \to \mathcal{P}_{cp}(X)$ is called *completely continuous* if it is upper semi-continuous and compact on X. Every compact multi-valued mapping is totally bounded but the converse may not be true. However, these two notions are equivalent on bounded subsets of a complete metric space X. The exhaustive treatment of these notions appears in Granas and Dugundji [2].

Let X be an ordered metric space with an order relation $\leq$. Let $a, b \in X$ be such that $a \leq b$. Then an order interval $[a, b]$ is a set in X defined by

$$[a, b] = \{x \in X \mid a \leq x \leq b\}.$$

When X is an ordered Banach space, the order relation "$\leq$" in X is defined by the cone K, which is a nonempty closed set in X satisfying (i) $K + K \subset K$, (ii) $\lambda K \subset K$ for all $\lambda \in \mathbb{R}^+$, and (iii) $\{-K\} \bigcap K = 0$, where 0 is the zero element of X. A cone K in a Banach space X is called normal, if the norm $\|\cdot\|$ is semi-monotone on K. It is known that if the cone K is normal, then every order-bounded set is bounded in norm. The details of cones and their properties appear in Guo and Lakshmikantham [3] and Heikkilä and Lakshmikantham [4]. In the following, we define an order relation in $\mathcal{P}_p(X)$ which is useful in the sequel.

Let $A, B \in \mathcal{P}_p(X)$. Then we define

$$A \pm B = \{a \pm b \mid a \in A \text{ and } b \in B\}$$
$$\lambda A = \{\lambda a \mid a \in A\}$$

for $\lambda \in \mathbb{R}$. Define an order relation $\leq$ in $\mathcal{P}_p(X)$ by

$$A \leq B \Leftrightarrow a \leq b \quad \text{for all } a \in A \text{ and } b \in B. \tag{1.3}$$

The order relation (1.3) defined in $\mathcal{P}_p(X)$ has been used in Dhage [5,14,7], Dhage and O'Regan [8] and Agarwal et al. [9] in the study of extremal solutions for differential and integral inclusions.

Definition 1.2. A single-valued mapping $Q : X \to X$ is called *monotone increasing* or *nondecreasing* if $x \leq y$, then $Qx \leq Qy$ for all $x, y \in X$.

Definition 1.3. A multi-valued mapping $Q : X \to \mathcal{P}_p(X)$ is called *strictly monotone increasing* if $x < y$, that is, $x \leq y$ and $x \neq y$, then $Qx \leq Qy$ for all $x, y \in X$.

Notice that if $Qx = \{fx\}$, f a single-valued mapping, then the notion of strictly monotone increasing multi-valued mappings is equivalent to the monotone increasing mappings on X.

The Hausdorff measure of noncompactness of a bounded subset S of X is a nonnegative real number $\beta(S)$ defined by

$$\beta(S) = \inf\left\{r > 0 : S \subset \bigcup_{i=1}^{n} \mathcal{B}_i(x_i, r), \text{ for some } x_i \in X\right\}, \tag{1.4}$$

where $\mathcal{B}_i(x_i, r) = \{x \in X \mid d(x, x_i) < r\}$.

The measure of noncompactness β enjoys the following properties:

(β_1) $\beta(A) = 0 \Longleftrightarrow \overline{A}$ is compact.
(β_2) $\beta(A) = \beta(-A) = \beta(\overline{co}A)$, where $\overline{co}A$ is the closed convex hull of A.
(β_3) $A \subset B \Rightarrow \beta(A) \leq \beta(B)$.

(β_4) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$.
(β_5) $\beta(\lambda A) = |\lambda|\beta(A), \forall \lambda \in \mathbb{R}$.
(β_6) $\beta(A + B) \leq \beta(A) + \beta(B)$.

The details of Hausdorff measure of noncompactness and its properties appear in Deimling [10], Zeidler [11] and the references therein.

Definition 1.4. A multi-valued mapping $T : X \to \mathcal{P}_{cp}(X)$ is called condensing (resp. countably condensing) if for any bounded (resp. bounded and countable) subset S of X, $T(S)$ is bounded and $\beta(T(S)) < \beta(S)$ for $\beta(S) > 0$.

Note that every condensing multi-valued mapping is countably condensing, but the converse may not be true. It is known that multi-valued contraction mappings and completely continuous multi-valued mappings are condensing. See Dhage [6], Petruşel [12] and the references therein. Some details of condensing multi-valued mappings and fixed points may be found in Gorniewicz [13] and the references cited therein.

A fixed point theorem for strictly monotone increasing multi-valued countably condensing mapping is

Theorem 1.1 (*Dhage [7]*). *Let $[\overline{x}, \overline{y}]$ be norm-bounded order interval in the ordered normed linear space X and let $T : [\overline{x}, \overline{y}] \to \mathcal{P}_{cl}([\overline{x}, \overline{y}])$ be upper semi-continuous and countably condensing. Further if T is strictly monotone increasing, then T has the least and the greatest fixed point in $[\overline{x}, \overline{y}]$.*

An improvement in the multi-valued analogue of Tarski's fixed point theorem of Dhage and O'Regan [8] is embodied in the following fixed point theorem in ordered metric spaces.

Theorem 1.2 (*Dhage [8]*). *Let $[a, b]$ be an order interval in a subset Y of the ordered Banach space X and let $Q : [a, b] \to \mathcal{P}_{cp}([a, b])$ be a strictly monotone increasing multi-valued mapping. If every monotone sequence $\{y_n\} \subset \bigcup Q([a, b])$ defined by $y_n \in Qx_n, n \in \mathbb{N}$ converges in Y, whenever $\{x_n\}$ is a monotone sequence in $[a, b]$, then Q has the least and the greatest fixed point.*

In the following section, we combine Theorems 1.1 and 1.2 to obtain a general hybrid fixed point theorem for multi-valued mappings on ordered Banach spaces.

2. Hybrid fixed point theory

Our main multi-valued hybrid fixed point theorem of this paper is

Theorem 2.1. *Let $[a, b]$ be a norm-bounded order interval in the ordered Banach space X and let $T : [a, b] \times [a, b] \to \mathcal{P}_{cp}([a, b])$ be a multi-valued mapping satisfying the following conditions.*

(a) *A multi-valued mapping $x \mapsto T(x, y)$ is upper semi-continuous, condensing and strictly monotone increasing uniformly for $y \in [a, b]$,*
(b) *A multi-valued mapping $y \mapsto T(x, y)$ is strictly monotone increasing for each $x \in [a, b]$, and*
(c) *every monotone sequence $\{z_n\} \subset \bigcup T([a, b] \times [a, b])$ defined by $z_n \in T(x, y_n), n \in \mathbb{N}$ converges for each $x \in [a, b]$, whenever $\{y_n\}$ is a monotone sequence in $[a, b]$.*

Then the inclusion $x \in T(x, x)$ has the least and the greatest solution in $[a, b]$.

Proof. Define a multi-valued operator $Q : [a, b] \to \mathcal{P}_p([a, b])$ by

$$Qy = \{x \in [a, b] \mid x \in T(x, y)\}. \tag{2.1}$$

Let $y \in [a, b]$ be fixed and define the mapping $T_y(x) : [a, b] \to \mathcal{P}_{cp}([a, b])$ by $T_y(x) = T(x, y)$. Then T_y is a countably condensing, upper semi-continuous and strictly monotone increasing multi-valued mapping which maps a closed convex and bounded subset $[a, b]$ of the Banach space X into itself. Therefore, an application of Theorem 1.1 yields that T_y has the least and the greatest fixed point in $[a, b]$ and consequently the set Qy is nonempty for each $y \in [a, b]$.

We assume that Qy is compact for each $y \in [a, b]$. It will be shown that Q satisfies all the conditions of Theorem 1.2. First, we show that Q is a strict monotone increasing multi-valued operator on $[a, b]$. Let $y_1, y_2 \in [a, b]$ be any two elements such that $y_1 < y_2$, that is, $y_1 \leq y_2$ and $y_1 \neq y_2$. Then, we have

$$Qy_1 = \{z \in [a, b] \mid z \in T(z, y_1) = T_{y_1}(z)\}$$

and

$$Qy_2 = \{z \in [a, b] \mid z \in T(z, y_2) = T_{y_2}(z)\}.$$

From the strict monotonicity of $T(x, y)$ in y, it follows that

$$T_{y_1}(x) = T(x, y_1) \leq T(x, y_2) = T_{y_2}(x)$$

for all $x \in [a, b]$. Note also that the multi-valued mappings T_{y_1} and T_{y_2} are strict monotone increasing on $[a, b]$.

If $z \in Qy_1 = T_{y_1}(z)$ be arbitrary, then we have $z \leq T_{y_2}(z)$. Since $T_{y_1} : [a, b] \to \mathcal{P}_{cp}([a, b])$ is upper semi-continuous and countably condensing multi-valued mapping, by Theorem 1.1, the fixed point set of T_{y_1} is nonempty and has the least and the greatest elements. Let $z^*(y_1)$ be the greatest fixed point of of T_{y_1} in $[a, b]$. Then we have $z^*(y_1) \in T_{y_1}(z^*(y_1)) \leq T_{y_2}(z^*(y_1))$. Now consider the order interval $[z^*(y_1), b]$ in $[a, b]$. Then T_{y_2} defines an upper semi-continuous, countably condensing and strict monotone increasing multi-valued mapping $T_{y_2} : [z^*(y_1), b] \to \mathcal{P}_{cp}([a, b])$ and hence by Theorem 1.1, has the least fixed point $z_*(y_2)$ and the greatest fixed point $z^*(y_2)$ in $[z^*(y_1), b]$. Thus $z^*(y_1) \leq z_*(y_2)$ and so $Qy_1 \leq Qy_2$. Hence Q is defined as a strict monotone increasing multi-valued operator $Q : [a, b] \to \mathcal{P}_{cp}([a, b])$ (see also Dhage [5,13] and the references therein).

Next, let $\{y_n\}$ be a monotone sequence in $[a, b]$. We will show that the sequence $\{Qy_n\}$ converges. By definition of Q, there is a monotone increasing sequence $\{z_n\}$ in $[a, b]$ such that $z_n \in T(z_n, y_n), n \in \mathbb{N}$. Let $S = \{z_n\}$. Then S is a bounded and countable subset of $[a, b]$ such that $S \subseteq \bigcup_{n\in\mathbb{N}} T(S, y_n)$. Since the multi-valued $x \mapsto T(x, y)$ is countably condensing for each $y \in [a, b]$, one has

$$\beta(S) \leq \beta(\cup_{n\in\mathbb{N}} T(S, y_n)) = \sup_{n\in\mathbb{N}} \beta(T(S, y_n)) < \beta(S)$$

for each $n \in \mathbb{N}$. If $\beta(S) \neq 0$, then we get a contradiction. As a result, $\beta(S) = 0$ and that $\overline{S}$ is compact. Hence the sequence $\{z_n\}$ converges to a point, say z in $[a, b]$. By upper semi-continuity of $T(x, y)$ in x uniformly for y, there exists an $n_0 \in \mathbb{N}$ such that $z_n \in T(z, y_n)$ for all $n \geq n_0$. Now, by hypothesis (c), every sequence $\{z_n\}$ in $\{T(z, y_n)\}$ converges. As a result, the sequence $\{z_n\} \subseteq \bigcup Q([a, b])$ defined by $z_n \in Qy_n$ for each $n \in \mathbb{N}$ converges, whenever $\{y_n\}$ is a monotone increasing sequence in $[a, b]$.

Thus the multi-valued operator Q satisfies all the conditions of Theorem 1.2 on $[a, b]$ and hence on application it yields that Q has the least and the greatest fixed point. Next, we define the operator $\widehat{Q}$ on $[a, b]$ by

$$\widehat{Q}y = \{z\}$$

where, $z \in T(z, y)$ and z is the greatest element in $[a, b]$. Clearly, the above operator is well defined in view of Theorem 1.1 applied to the multi-valued mapping T_y on $[a, b]$. Then following the above arguments it is proved that $\widehat{Q}$ has the least and the greatest fixed point and the greatest fixed point of $\widehat{Q}$ is the greatest solution of the inclusion $x \in T(x, x)$ in $[a, b]$. Similarly, define the operator $\widehat{P} : [a, b] \to \mathcal{P}_{cp}([a, b])$ by

$$\widehat{P}y = \{z \in [a, b] \mid z \in T(z, y) \text{ and } z \text{ is the least element}\}. \tag{2.2}$$

Clearly the operator $\widehat{P}$ is well defined. It can be shown by using similar arguments that $\widehat{P}$ has the least fixed point in $[a, b]$ which is also the least solution of the inclusion $x \in T(x, x)$. As a result, the inclusion $x \in T(x, x)$ has the least and the greatest solution in $[a, b]$. This completes the proof. □

As a consequence of Theorem 2.1 we obtain

Corollary 2.2. *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach space X and let $T : [a, b] \times [a, b] \to \mathcal{P}_{cp}([a, b])$ be a multi-valued mapping satisfying*

(a) $x \mapsto T(x, y)$ *is an upper semi-continuous, countably condensing and strictly monotone increasing uniformly for $y \in [a, b]$, and*
(b) $y \mapsto T(x, y)$ *is strictly monotone increasing for each $x \in [a, b]$.*

Then the inclusion $x \in T(x, x)$ has the least and the greatest solution if any one of the following conditions is satisfied.

(a) *$[a, b]$ is norm-bounded and T is compact.*
(b) *The cone K in X is normal and $y \mapsto T(x, y)$ is compact for each $x \in [a, b]$.*
(c) *The cone K is regular.*

The origin of hybrid fixed point theorems involving the sum of two operators in a Banach space lies in the works of the Russian mathematician Krasnoselskii [15]. The multi-valued versions of Krasnoselskii fixed point appear in Dhage [6] and Petruşel [12]. In this case, one operator happens to be a contraction and another one happens to be a completely continuous on the domains of their definitions. Since every contraction is continuous, both the operators in such theorems are continuous. Below we prove a hybrid fixed point theorem involving the sum of three multi-valued operators in Banach spaces and relax the continuity of one of the operators in such hybrid fixed point theorems, instead we assume the monotonicity and prove a fixed point theorem on ordered Banach spaces.

To prove the main results in this direction, we need the following useful lemma.

Lemma 2.1. *Let $A, B : X \to \mathcal{P}_{cp}(X)$ be two multi-valued operators satisfying*

(a) *A is a multi-valued nonlinear $\mathcal{D}$-contraction, and*
(b) *B is completely continuous.*

Then the multi-valued operator $T : X \to \mathcal{P}_{cp}(X)$ defined by $Tx = Ax + Bx$ is an upper semi-continuous and condensing on X.

Proof. The proof appears in Dhage [6]. See also Petruşel [12] for the details. □

Theorem 2.3. *Let $[a, b]$ be an order interval in the ordered Banach space X. Let $A, B, C : [a, b] \to \mathcal{P}_{cp}(X)$ be three strictly monotone increasing multi-valued operators satisfying*

(a) *A is a multi-valued nonlinear $\mathcal{D}$-contraction,*
(b) *B is completely continuous*
(c) *every monotone sequence $\{z_n\} \subset \bigcup C([a, b])$ defined by $z_n \in C(y_n), n \in \mathbb{N}$ converges, whenever $\{y_n\}$ is a monotone sequence in $[a, b]$, and*
(d) *$Ax + By + Cz \subset [a, b]$ for all $x, y, z \in [a, b]$.*

Further if the cone K in X is normal, then the operator inclusion $x \in Ax + Bx + Cx$ has the least and the greatest solution in $[a, b]$.

Proof. Define a mapping T on $[a, b] \times [a, b]$ by $T(x, y) = Ax + Bx + Cy$. From hypothesis (d) it follows that T defines a multi-valued mapping $T : [a, b] \times [a, b] \to \mathcal{P}_{cp}([a, b])$. From Lemma 3.1, it follows that the multi-valued $x \mapsto T(x, y)$ is condensing, upper semi-continuous and strictly monotone increasing on $[a, b]$. Now the desired conclusion follows by an application of Theorem 2.1. □

Remark 2.1. Note that hypothesis (d) above holds if the elements a and b in X satisfy $a \leq Aa + Ba + Ca$ and $Ab + Bb + Cb \leq b$. Note also that the hypothesis (c) above holds if the multi-valued operator C is a compact on $[a, b]$.

The Kuratowskii measure α of noncompactness in a Banach space is a nonnegative real number $\alpha(S)$ defined by

$$\alpha(S) = \inf \left\{ r > 0 : S \subset \bigcup_{i=1}^{n} S_i, \text{ and } \operatorname{diam}(S_i) \leq r, \forall i \right\} \tag{2.3}$$

for all bounded subsets S of X.

It is known that the Kuratowskii measure α of noncompactness has all the properties (β_1) through (β_6) of the Hausdorff measure of noncompactness on X. The following result appears in Akhmerov et al. [16].

Lemma 2.2 ([16, page 7]). *Let α and β be respectively the Kuratowskii and Hausdorff measure of noncompactness in a Banach space X, then for any bounded set S in X we have*

$$\alpha(S) \leq 2\beta(S).$$

Lemma 2.3. *If $A : X \to X$ is a single-valued Lipschitz operator with the Lipschitz constant k, that is,*

$$\|Ax - Ay\| \le k\|x - y\|$$

for all $x, y \in X$, where the real number $k > 0$, then we have $\alpha(A(S)) \le k\alpha(S)$ for any bounded subset S of X.

When A is a single-valued mapping, we obtain

Corollary 2.4. *Let $[a, b]$ be an order interval in the ordered Banach space X. Let $B, C : [a, b] \to \mathcal{P}_{cp}(X)$ be two strictly monotone increasing and $A : [a, b] \to X$ be a nondecreasing operator satisfying*

(a) *A is a single-valued contraction with contraction $k < 1/2$,*
(b) *B is completely continuous,*
(c) *C is compact, and*
(d) *$Ax + By + Cz \subset [a, b]$ for all $x, y, z \in [a, b]$.*

Furthermore, if the cone K in X is normal, then the operator inclusion $x \in Ax + Bx + Cx$ has the least and the greatest solution in $[a, b]$.

Proof. Define a mapping $T : [a, b] \times [a, b] \to \mathcal{P}_{cp}([a, b])$ by

$$T(x, y) = Ax + Bx + Cy.$$

We shall show that the mapping $T_y(\cdot) = T(\cdot, y)$ is a condensing on $[a, b]$. Since the order cone K in X is normal, the order interval $[a, b]$ is a norm-bounded set in X. Now for any subset S in $[a, b]$ one has

$$T_y(S) \subset A(S) + B(S) + Cy.$$

Hence from Lemmas 2.2 and 2.3, it follows that

$$\begin{aligned}\beta(T_y(S)) &= \beta(A(S) + B(S) + Cy)\\ &\le \beta(A(S)) + \beta(B(S)) + \beta(Cy)\\ &\le \alpha(A(S))\\ &\le k\alpha(S)\\ &\le 2k\beta(S)\\ &< \beta(S)\end{aligned}$$

provided $\beta(S) > 0$. The rest of the proof is similar to Theorem 2.3. □

When A, B and C are single-valued operators, Theorem 2.3 reduces to

Corollary 2.5. *Let $[a, b]$ be an order interval in an ordered Banach space X. Let $A, B, C : [a, b] \to X$ be three nondecreasing single-valued operators satisfying*

(a) *A is contraction with a contraction constant $k < 1/2$,*
(b) *B is completely continuous,*
(c) *C is compact, and*
(d) *$Ax + By + Cz \in [a, b]$ for all $x, y, z \in [a, b]$.*

Furthermore, if the cone K in X is normal, then the operator inclusion $Ax + Bx + Cx = x$ has the least and the greatest solution in $[a, b]$.

Remark 2.2. Note that hypothesis (d) above holds if the elements a and b in X satisfy $a \le Aa + Ba + Ca$ and $Ab + Bb + Cb \le b$.

The hybrid fixed point theory involving the product of two multi-valued operators in a Banach algebra is initiated by the present author in [5] and developed further in the various directions in due course of time. See Dhage [17,6,14] and the references therein. The main feature of these fixed point theorems is again that both the operators are continuous on their domains of definition. Below we remove the continuity of one of the operators and prove a fixed point theorem involving the product of two operators in a Banach algebra. We need the following preliminaries in the sequel.

A cone K in a Banach algebra X is called positive if

(iv) $K \circ K \subseteq K$, where "$\circ$" is a multiplicative composition in X.

Let X be an ordered Banach algebra. Then for any $A, B \in \mathcal{P}_p(X)$, we denote

$$AB = \{ab \in X \mid a \in A \text{ and } b \in B\}.$$

We need the following results in the sequel.

Lemma 2.4 (*Dhage [14]*). *Let K be a positive cone in the Banach algebra X. If $u_1, u_2, v_1, v_2 \in K$ are such that $u_1 \le v_1$ and $u_2 \le v_2$, then $u_1u_2 \le v_1v_2$.*

Lemma 2.5 (*Dhage [6]*). *For any $A, B, C \in \mathcal{P}_p(X)$,*

$$d_H(AC, BC) \le d_H(C, 0)d_H(A, B) = \|C\|_{\mathcal{P}} d_H(A, B).$$

Lemma 2.6 (*Banas and Lecko [18]*). *If $A, B \in \mathcal{P}_{\mathrm{bd}}(X)$, then*

$$\beta(AB) \le \|A\|_{\mathcal{P}}\beta(B) + \|B\|_{\mathcal{P}}\beta(A).$$

Lemma 2.7. *Let S be a closed convex and bounded subset of a Banach algebra X and let $A, B : X \to \mathcal{P}_{\mathrm{cp}}(X)$ be two multi-valued operators such that*

(a) *A is a $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ,*
(b) *B is completely continuous, and*
(c) *$M\psi(r) < r$ for $r > 0$, where $M = \|B(S)\|_{\mathcal{P}} = \sup\{\|Bx\| \mid x \in S\}$.*

Then the multi-valued operator $T : X \to \mathcal{P}_{\mathrm{cp}}(X)$ defined by $Tx = AxBx$ is upper semi-continuous and condensing on X.

Proof. The proof appears in Dhage [17,6]. □

Theorem 2.6. *Let $[a, b]$ be an order interval in the ordered Banach algebra X and let $A, B : [a, b] \to \mathcal{P}_{\mathrm{cp}}(K)$ and $C : [a, b] \to \mathcal{P}_{\mathrm{cp}}(X)$ be three strictly monotone increasing multi-valued operators satisfying*

(a) *A is a $\mathcal{D}$-Lipschitz with the $\mathcal{D}$-function ψ,*
(b) *B is completely continuous,*
(c) *every monotone sequence $\{z_n\} \subset \bigcup C([a, b])$ defined by $z_n \in C(y_n), n \in \mathbb{N}$ converges, whenever $\{y_n\}$ is a monotone sequence in $[a, b]$, and*
(d) *$AxBy + Cz \in \mathcal{P}_{\mathrm{cp}}([a, b])$ for all $x, y, z \in [a, b]$.*

Furthermore, if the cone K in X is positive and normal, then the operator inclusion $x \in AxBx + Cx$ has the least and the greatest solution in $[a, b]$ whenever $M\psi(r) < r$ for $r > 0$, where $M = \|B([a, b])\|_{\mathcal{P}} = \sup\{\|Bx\|_{\mathcal{P}} : x \in [a, b]\}$.

Proof. Define a mapping T on $[a, b] \times [a, b]$ by $T(x, y) = AxBx + Cy$. From hypothesis (d), it follows that T defines a multi-valued mapping $T : [a, b] \times [a, b] \to \mathcal{P}_{\mathrm{cp}}([a, b])$. We show that the multi-valued map $x \mapsto T(x, y) = T_y(x)$ is upper semi-continuous, condensing and strictly monotone increasing on $[a, b]$. First we show that it is condensing on $[a, b]$. Let S be a subset of $[\overline{x}, \overline{y}]$. Since the cone K in X is normal, the order interval $[\overline{x}, \overline{y}]$ and consequently the set S is norm-bounded in X. Then by property (β_6),

$$\begin{aligned}\beta(T_y(S)) &\le \beta(A(S)B(S)) + \beta(C(y)) \\ &\le \|B(S)\|_{\mathcal{P}}\beta(A(S)) + \|B(S)\|_{\mathcal{P}}\beta(B(S)) + \beta(C(y)).\end{aligned}$$

As B is completely continuous and C is compact-valued, we have that $\beta(B(S)) = 0$ and $\beta(C(y)) = 0$. Again from Lemma 2.3, it follows that $\beta(A(S)) \le \psi(\beta(S))$. Hence we have

$$\beta(T_y(S)) = \|B(S)\|_{\mathcal{P}}\beta(A(S)) + \beta(C(S)) \le M\psi(\beta(S)) < \beta(S)$$

for all sets S in $[\overline{x}, \overline{y}]$ with $\beta(S) > 0$. This shows that the map $x \mapsto T(x, y)$ is condensing on $[a, b]$.

To show that the multi-valued $x \mapsto T(x, y)$ is upper semi-continuous, let $\{x_n\}$ be a sequence in $[a, b]$ converging to a point x^*. Let $\{y_n\}$ be a sequence in $Ax_nBx_n + Cy$ such that $y_n \to y^*$. It suffices to show that $y^* \in Ax^*Bx^* + Cy$. Now,

$$\begin{aligned} D(y^*, Ax^*Bx^* + Cy) &= \lim_{n\to\infty} D(y_n, Ax^*Bx^* + Cy) \\ &\le \lim_{n\to\infty} d_H(Ax_nBx_n + Cy, Ax^*Bx^* + Cy) \\ &\le \lim_{n\to\infty} d_H(Ax_nBx_n, Ax^*Bx^*) \\ &\le \lim_{n\to\infty} d_H(Ax_nBx_n, Ax^*Bx_n) + \lim_{n\to\infty} d_H(Ax^*Bx_n, Ax^*Bx^*) \\ &\le \lim_{n\to\infty} [d_H(Ax_n, Ax^*)d_H(0, Bx_n)] + \lim_{n\to\infty} [d_H(0, Ax^*)d_H(Bx_n, Bx^*)] \\ &\le M\psi(\lim_{n\to\infty} \|x_n - x^*\|) + \|Ax^*\|_{\mathcal{P}} \lim_{n\to\infty} d_H(Bx_n, Bx^*) \\ &\to 0 \quad \text{as } n \to \infty. \end{aligned}$$

This shows that $y^* \in Ax^*Bx^* + Cy$, and therefore, the multi-valued map $x \mapsto AxBx + Cy$ is upper semi-continuous on $[a, b]$. Now the desired conclusion follows by an application of Theorem 2.1. □

Remark 2.3. Note that hypothesis (d) above holds if the elements a and b in X satisfy $a \le AaBa + Ca$ and $AbBb + Cb \le b$. Again, the hypothesis (c) above holds if the multi-valued operator C is a compact on $[a, b]$.

Theorem 2.7. *Let $[a, b]$ be an order interval in an ordered Banach algebra X. Let $A : [a, b] \to K$, $C : [a, b] \to X$ be two nondecreasing single-valued operators and $B : [a, b] \to \mathcal{P}_{cp}(K)$, a strictly monotone increasing multi-valued operator satisfying*

(a) *A is Lipschitz with the Lipschitz constant k,*
(b) *B is completely continuous,*
(c) *C is compact, and*
(d) *$AxBy + Cz \in \mathcal{P}_{cp}([a, b])$ for all $x, y, z \in [a, b]$.*

Furthermore, if the cone K in X is positive and normal, then the operator inclusion $x \in AxBx + Cx$ has the least and the greatest solution in $[a, b]$ whenever $2Mk < 1$, where $M = \|B([a, b])\|_{\mathcal{P}} = \sup\{\|Bx\| : x \in [a, b]\}$.

Proof. Define a mapping $T : [a, b] \times [a, b] \to \mathcal{P}_{cp}([a, b])$ by

$$T(x, y) = AxBx + Cy.$$

We shall show that the mapping $T_y(\cdot) = T(\cdot, y)$ is a condensing on $[a, b]$. Since the order cone K in X is normal, the order interval $[a, b]$ is a norm-bounded set in X. Now for any subset S in $[a, b]$ one has

$$T_y(S) \subset A(S)B(S) + Cy.$$

Hence, from Lemmas 3.1 and 3.2, it follows that

$$\begin{aligned} \beta(T_y(S)) &= \beta(A(S)B(S) + Cy) \\ &\le \|B(S)\|_{\mathcal{P}}\beta(A(S)) + \|A(S)\|_{\mathcal{P}}\beta(B(S)) + \beta(Cy) \\ &\le \|B(S)\|_{\mathcal{P}}\alpha(A(S)) \\ &\le Mk\alpha(S) \\ &\le 2Mk\beta(S) \\ &< \beta(S) \end{aligned}$$

provided $\beta(S) > 0$. The rest of the proof is similar to Theorem 2.6. □

When A, B and C are single-valued operators, Theorem 2.6 reduces to

Corollary 2.8. *Let $[a, b]$ be an order interval in an ordered Banach algebra X. Let $A, B : [a, b] \to K$ and $C : [a, b] \to X$ be three nondecreasing single-valued operators satisfying*

(a) *A is Lipschitz with the Lipschitz constant k,*
(b) *B is completely continuous,*
(c) *C is compact, and*
(d) *$AxBy + Cz \in [a, b]$ for all $x, y, z \in [a, b]$.*

Furthermore, if the cone K in X is positive and normal, then the operator inclusion $AxBx + Cx = x$ has the least and the greatest solution in $[a, b]$, whenever $2Mk < 1$, where $M = \|B([a, b])\|_{\mathcal{P}} = \sup\{\|Bx\| : x \in [a, b]\}$.

Theorem 2.9. *Let $[a, b]$ be an order interval in an ordered Banach algebra X. Let $A, B : [a, b] \to \mathcal{P}_{cp}(K)$ and $C : [a, b] \to \mathcal{P}_{cp}(X)$ be three strictly monotone increasing multi-valued operators satisfying*

(a) *A is Lipschitz with the Lipschitz constant k,*
(b) *B is bounded and every monotone sequence $\{z_n\} \subset \bigcup B([a, b])$ defined by , $z_n \in B(y_n), n \in \mathbb{N}$ converges, whenever $\{y_n\}$ is a monotone sequence in $[a, b]$,*
(c) *C is completely continuous, and*
(d) *$AxBy + Cz \in \mathcal{P}_{cp}([a, b])$ for all $x, y, z \in [a, b]$.*

Furthermore, if the cone K in X is positive and normal, then the operator inclusion $x \in AxBx + Cx$ has the least and the greatest solution in $[a, b]$ whenever $Mk < 1$, where $M = \|B([a, b])\|_{\mathcal{P}} = \sup\{\|Bx\|_{\mathcal{P}} : x \in [a, b]\}$.

Proof. Define a mapping T on $[a, b] \times \mathcal{P}_{cv}(X)$ by $T(x, y) = AxBy + Cx$. From hypothesis (d), it follows that T defines a multi-valued mapping $T : [a, b] \times [a, b] \to \mathcal{P}_{cp}([a, b])$. It can be shown as in the proof of Theorem 2.3 with appropriate modifications that the multi-valued $x \mapsto T(x, y)$ is condensing and upper semi-continuous on $[a, b]$. Now the desired conclusion follows by an application of Theorem 2.1. $\square$

Remark 2.4. Note that hypothesis (d) above holds if the elements a and b in X satisfy $a \leq AaBa + Ca$ and $AbBb + Cb \leq b$. Again, the hypothesis (b) above holds if the multi-valued operator B is compact on $[a, b]$.

When A, B and C are single-valued operators, Theorem 2.6 reduces to

Corollary 2.10. *Let $[a, b]$ be an order interval in a subset Y of an ordered Banach algebra X. Let $A, B : [a, b] \to K$ and $C : [a, b] \to X$ be three nondecreasing single-valued operators satisfying*

(a) *A is Lipschitz with the Lipschitz constant k,*
(b) *B is compact,*
(c) *C is completely continuous, and*
(d) *$AxBy + Cz \in [a, b]$ for all $x, y, z \in [a, b]$.*

Furthermore, if the cone K in X is positive and normal, then the operator equation $AxBx + Cx = x$ has the least and the greatest solution in $[a, b]$ whenever $2Mk < 1$, where $M = \|B([a, b])\|_{\mathcal{P}} = \sup\{\|Bx\| : x \in [a, b]\}$.

Proof. Define a mapping $T : [a, b] \times [a, b] \to [a, b]$ by

$$T(x, y) = AxBy + Cx.$$

We shall show that the mapping $T_y(\cdot) = T(\cdot, y)$ is condensing on $[a, b]$. Since the order cone K in X is normal, the order interval $[a, b]$ is a norm-bounded set in X. Now for any subset S in $[a, b]$ one has

$$T_y(S) \subset A(S)By + C(S).$$

Hence, from Lemmas 2.3 and 2.4, it follows that

$$\begin{aligned}
\beta(T_y(S)) &= \beta(A(S)By + Cy) \\
&\leq \|By\|_{\mathcal{P}}\beta(A(S)) + \|A(S)\|_{\mathcal{P}}\beta(By) + \beta(C(S)) \\
&\leq M\alpha(A(S)) + \alpha(C(S)) \\
&\leq Mk\alpha(S) \\
&\leq 2Mk\beta(S) \\
&< \beta(S)
\end{aligned}$$

provided $\beta(S) > 0$. The rest of the proof is similar to Theorem 2.7. $\square$

Remark 2.5. Note that hypothesis (d) above holds if the elements a and b in X satisfy $a \leq AaBa + Ca$ and $AbBb + Cb \leq b$.

Theorem 2.11. *Let $[a, b]$ be an order interval in an ordered Banach algebra X with a cone K. Let $A, B : [a, b] \to \mathcal{P}_{cp}(K)$ and $C : [a, b] \to \mathcal{P}_{cp}(X)$ be three strictly monotone increasing multi-valued operators satisfying*

(a) *every monotone sequence $\{z_n\} \subset \bigcup A([a, b])$ defined by $z_n \in A(y_n), n \in \mathbb{N}$ converges, whenever $\{y_n\}$ is a monotone sequence in $[a, b]$,*
(b) *B is completely continuous,*
(c) *C is multi-valued contraction, and*
(d) *$AxBy + Cz \in \mathcal{P}_{cp}([a, b])$ for all $x, y, z \in [a, b]$.*

Furthermore, if the cone K in X is positive and normal, then the operator inclusion $x \in AxBx + Cx$ has the least and the greatest solution in $[a, b]$.

Proof. Define a mapping T on $[a, b] \times [a, b]$ by $T(x, y) = AyBx + Cx$. From hypothesis (d), it follows that T defines a multi-valued mapping $T : [a, b] \times [a, b] \to \mathcal{P}_{cp}([a, b])$. Now the desired conclusion follows by an application of Theorem 2.1. $\square$

Remark 2.6. Note that hypothesis (d) above holds if the elements a and b in X satisfy $a \leq AaBa + Ca$ and $AbBb + Cb \leq b$. Again, the hypothesis (a) above holds if the multi-valued operator A is compact on $[a, b]$.

When C is single-valued operator, Theorem 2.9 reduces to

Corollary 2.12. *Let $[a, b]$ be an order interval in a subset Y of the ordered Banach algebra X. Let $A, B : [a, b] \to \mathcal{P}_{cp}(K)$ be strictly monotone increasing and let $C : [a, b] \to X$ be a nondecreasing single-valued operator satisfying*

(a) *A is compact,*
(b) *B is completely continuous,*
(c) *C is contraction with a contraction constant $k < 1/2$, and*
(d) *$AxBy + Cz \in \mathcal{P}_{cp}([a, b])$ for all $x, y, z \in [a, b]$.*

Furthermore, if the cone K in X is positive and normal, then the operator inclusion $x \in AxBx + Cx$ has the least and the greatest solution in $[a, b]$.

Proof. Define a mapping $T : [a, b] \times [a, b] \to \mathcal{P}_{cp}([a, b])$ by

$$T(x, y) = AyBx + Cx.$$

We shall show that the mapping $T_y(\cdot) = T(\cdot, y)$ is condensing on $[a, b]$. Since the order cone K in X is normal, the order interval $[a, b]$ is a norm-bounded set in X. Now for any subset S in $[a, b]$ one has

$$T_y(S) \subset AyB(S) + C(S).$$

Hence from Lemmas 3.1 and 3.2, it follows that

$$\begin{aligned}
\beta(T_y(S)) &= \beta(AyB(S) + C(S)) \\
&\leq \|B(S)\|_{\mathcal{P}}\beta(Ay) + \|Ay\|_{\mathcal{P}}\beta(B(S)) + \beta(C(S)) \\
&\leq \alpha(C(S)) \\
&\leq k\alpha(S) \\
&\leq 2k\beta(S) \\
&< \beta(S),
\end{aligned}$$

provided $\beta(S) > 0$. The rest of the proof is similar to Theorem 2.9. $\square$

Remark 2.7. Note that hypothesis (d) above holds if the elements a and b in X satisfy $a \leq AaBa + Ca$ and $AbBb + Cb \leq b$.

Corollary 2.13. *Let $[a, b]$ be an order interval in a subset Y of the ordered Banach algebra X. Let $A : [a, b] \to K$, $C : [a, b] \to X$ be two nondecreasing and $B : [a, b] \to \mathcal{P}_{cp}(K)$ be a strictly monotone increasing multi-valued operator satisfying*

(a) *A is compact,*
(b) *B is completely continuous,*
(c) *C is a contraction with the contraction constant $k < 1/2$, and*
(d) *$AxBy + Cz \in \mathcal{P}_{cp}([a, b])$ for all $x, y, z \in [a, b]$.*

Furthermore, if the cone K in X is positive and normal, then the operator inclusion $x \in AxBx + Cx$ has the least and the greatest solution in $[a, b]$.

Corollary 2.14. *Let $[a, b]$ be an order interval in a subset Y of the ordered Banach algebra X. Let $A, B : [a, b] \to K$ and $C : [a, b] \to X$ be three nondecreasing single-valued mappings satisfying*

(a) *A is compact,*
(b) *B is completely continuous,*
(c) *C is a contraction with the contraction constant $k < 1/2$, and*
(d) *$AxBy + Cz \in [a, b]$ for all $x, y, z \in [a, b]$.*

Furthermore, if the cone K in X is positive and normal, then the operator equation $x = AxBx + Cx$ has the least and the greatest solution in $[a, b]$.

Proof. The proof is similar to Corollary 2.13 and so, we omit the details. □

Remark 2.8. Note that Remark 2.7 also remains true in case of Corollaries 2.13 and 2.14.

Note that Theorems 2.3 and 2.6 include the hybrid fixed point theorems proved in Dhage [19,20] for single as well as for multi-valued mappings in ordered Banach spaces and algebras as special cases. In the following section we prove the existence theorems for extremal solutions for perturbed discontinuous functional differential inclusions under the mixed Lipschitz, compactness and monotonic conditions of multi-valued functions involved in the multi-valued problems in question.

3. Discontinuous functional differential inclusions

The method of upper and lower solutions has been successfully applied to the problems of nonlinear differential equations and inclusions. For the first direction, we refer to Heikkilä and Lakshmikantham [4] and for the second direction we refer to Halidias and Papageorgiou [21]. In this section we apply the results of previous sections to the first order initial value problem of ordinary discontinuous differential inclusions for proving the existence of solutions between the given upper and lower solutions under monotonicity conditions.

3.1. Perturbed initial value problems

Let $\mathbb{R}$ denote the real line. Let $I_0 = [-\delta, 0]$ and $I = [0, T]$ be two closed and bounded intervals in $\mathbb{R}$ for some real numbers $\delta > 0$ and $T > 0$ and let $J = I_0 \cup I$. Let $\mathcal{C} = C(I_0, \mathbb{R})$ denote the Banach space of all continuous $\mathbb{R}$-valued functions on I_0 with the usual supremum norm $\|\cdot\|_{\mathcal{C}}$ given by

$$\|\phi\|_{\mathcal{C}} = \sup\{|\phi(\theta)| : -\delta \le \theta \le 0\}.$$

For any continuous function x defined on the interval J, where $J = [-\delta, T] = I_0 \bigcup I$, and for any $t \in I$ we denote by x_t the element of $\mathcal{C}$ defined by

$$x_t(\theta) = x(t + \theta), \quad -\delta \le \theta \le 0.$$

Given a function $\phi \in \mathcal{C}$, consider the perturbed functional differential inclusion (in short, FDI)

$$\left.\begin{array}{l} x'(t) \in F(t, x_t) + G(t, x_t) + H(t, x_t) \quad \text{a.e. } t \in J, \\ x_0 = \phi, \end{array}\right\} \tag{3.1}$$

where $F, G, H : I \times \mathcal{C} \to \mathcal{P}_p(\mathbb{R})$.

By a solution for the PDI (3.1) we mean a function $x \in C(J, \mathbb{R}) \cap AC(I, \mathbb{R})$ whose first derivative x' exists and is a member of $L^1(I, \mathbb{R})$ in $F(t, x_t) + G(t, x_t) + H(t, x_t)$, i.e. there exists a $v \in L^1(I, \mathbb{R})$ such that $v(t) \in F(t, x_t) + G(t, x_t) + H(t, x_t)$ a.e $t \in I$, and $x'(t) = v(t)$, $t \in I$ and $x_0 = \phi \in \mathcal{C}$, where $AC(I, \mathbb{R})$ is the space of all absolutely continuous real-valued functions on I and $C(J, \mathbb{R})$ is the space of continuous real-valued functions on J.

The special cases of PDI (3.1) have been discussed in the literature very extensively for different aspects of the solutions under different continuity conditions. See Deimling [10], Hale [23] and the references therein. But the study of PDI (3.1) or its special cases with discontinuous multi-valued functions have not been made so far in the literature for the existence results. In this section we shall prove the existence theorems for PDI (3.1) via functional theoretic approach embodied in Theorem 2.3 under the mixed Lipschitz, compactness and strict monotonic conditions.

We shall seek the solutions for the PDI (3.1) in the space $C(J, \mathbb{R})$ of continuous and real-valued functions on J. Define a norm $\|\cdot\|$ and an order relation "$\leq$" in $C(J, \mathbb{R})$ by

$$\|x\| = \sup_{t \in J} |x(t)| \tag{3.2}$$

and

$$x \leq y \iff x(t) \leq y(t) \quad \text{for all } t \in J. \tag{3.3}$$

Here the cone K in $C(J, \mathbb{R})$ is defined by

$$K = \{x \in C(J, \mathbb{R}) \mid x(t) \geq 0\},$$

which is obviously positive and normal. See Guo and Lakshmikantham [3] and Heikkilä and Lakshmikantham [4]. For any multi-valued mapping $\beta : I \times \mathcal{C} \to \mathcal{P}_{cp}(\mathbb{R})$, we denote

$$S^1_\beta(x) = \{v \in L^1(I, \mathbb{R}) \mid v(t) \in F(t, x_t) \text{ a.e. } t \in I\}$$

for some $x \in C(J, \mathbb{R})$. The integral of the multi-valued mapping β is defined as

$$\int_0^t \beta(s, x_s)\mathrm{d}s = \left\{\int_0^t v(s)\mathrm{d}s : v \in S^1_\beta(x)\right\}.$$

Definition 3.1. A multi-valued function $\beta : I \to \mathcal{P}_{cp}(\mathbb{R})$ is said to be measurable if for every $y \in \mathbb{R}$, the function $t \to d(y, \beta(t)) = \inf\{|y - x| : x \in \beta(t)\}$ is measurable.

Definition 3.2. A measurable multi-valued function $\beta : I \to \mathcal{P}_{cp}(\mathbb{R})$ is said to be integrably bounded if there exists a function $h \in L^1(I, \mathbb{R})$ such that $a|v| \leq h(t)$ a.e. $t \in I$ for all $v \in \beta(t)$.

Remark 3.1. It is known that if $\beta : I \to \mathcal{P}_{cp}(\mathbb{R})$ is an integrably bounded multi-valued function, then the set S^1_β of all Lebesgue integrable selections of β is closed and nonempty. See Hu and Papageorgiou [22] and the references therein.

Definition 3.3. A multi-valued mapping $\beta : I \times \mathcal{C} \to \mathcal{P}_{cp}(\mathbb{R})$ is said to be L^1-Carathéodory if

(i) $t \mapsto \beta(t, x)$ is measurable for each $x \in C(J, \mathbb{R})$,
(ii) $x \mapsto \beta(t, x)$ is upper semi-continuous for almost all $t \in I$, and
(iii) for each real number $k > 0$, there exists a function $h_k \in L^1(I, \mathbb{R})$ such that

$$\|\beta(t, x)\|_{\mathcal{P}} = \sup\{|u| : u \in \beta(t, x)\} \leq h_k(t), \quad \text{a.e. } t \in I$$

for all $x \in \mathcal{C}$ with $\|x\|_{\mathcal{C}} \leq k$.

Then we have the following lemmas due to Lasota and Opial [24].

Lemma 3.1. *Let E be a Banach space. If $\dim(E) < \infty$ and $\beta : J \times E \to \mathcal{P}_{cp}(E)$ is L^1-Carathéodory, then $S^1_G(x) \neq \emptyset$ for each $x \in E$.*

Lemma 3.2. *Let E be a Banach space, β an L^1-Carathéodory multi-valued mapping with $S^1_\beta \neq \emptyset$ and let $\mathcal{K} : L^1(I, \mathbb{R}) \to C(I, E)$ be a linear continuous mapping. Then the operator*

$$\mathcal{K} \circ S^1_\beta : C(I, E) \longrightarrow \mathcal{P}_{\mathrm{cp}}(C(I, E))$$

is a closed graph operator in $C(I, E) \times C(I, E)$.

Remark 3.2. It is known that a *compact* multi-valued map $T : E \to \mathcal{P}_{\mathrm{cp}}(E)$ is upper semi-continuous if and only if it has a *closed graph* in E, that is, if $\{x_n\}$ and $\{y_n\}$ are sequences in E such that $y_n \in T x_n$ for $n = 0, 1, \ldots$; and $x_n \to x^*$, $y_n \to y^*$, then $y^* \in T x^*$.

We need the following definitions in the sequel.

Definition 3.4. A multi-valued mapping $\beta(t, x)$ is called strictly monotone increasing in x almost everywhere for $t \in I$ if $\beta(t, x) \leq \beta(t, y)$, a.e. $t \in I$, for all $x, y \in \mathcal{C}$, whenever $x < y$.

Definition 3.5. A multi-valued mapping $\beta : I \times \mathcal{C} \to \mathcal{P}_{\mathrm{cp}}(\mathbb{R})$ is called L^1-Chandrabhan if

(i) $t \mapsto \beta(t, x_t)$ is measurable for each $x \in C(J, \mathbb{R})$,
(ii) $x \mapsto \beta(t, x)$ is strictly monotone increasing almost everywhere for $t \in I$, and
(iii) for each real number $r > 0$ there exists a function $h_r \in L^1(I, \mathbb{R})$ such that

$$\|\beta(t, x)\|_{\mathcal{P}} = \sup\{|u| : u \in \beta(t, x)\} \leq h_r(t) \quad \text{a.e. } t \in I$$

for all $x \in \mathcal{C}$ with $\|x\|_{\mathcal{C}} \leq r$.

Remark 3.3. Note that every L^1-Chandrabhan multi-valued mapping β on $I \times \mathcal{C}$ is integrably bounded and so, the set of all Lebesgue integrable selectors $S^1_\beta(x)$ of β is nonempty closed subset of $L^1(I, \mathbb{R})$.

Definition 3.6. A function $a \in C(J, \mathbb{R})$ is called a strict lower solution of PDI (3.1) if for all $v_1 \in S^1_F(a)$ with $v_2 \in S^1_G(a)$ and $v_3 \in S^1_H(a)$ we have that $a'(t) \leq v_1(t) + v_2(t) + v_3(t)$ a.e. $t \in I$ and $a_0 \leq \phi$. Similarly a function $b \in C(J, \mathbb{R})$ is called a strict upper solution of PDI (3.1) if for all $v_1 \in S^1_F(b)$ with $v_2 \in S^1_G(b)$ and $v_3 \in S^1_H(b)$ we have that $b'(t) \geq v_1(t) + v_2(t) + v_3(t)$ a.e. $t \in I$ and $b_0 \geq \phi$.

We now introduce the following hypotheses in the sequel.

(F_1) $F(t, x)$ is compact subset of $\mathbb{R}$ for each $t \in I$ and $x \in \mathcal{C}$.
(F_2) The multi-valued $t \mapsto F(t, x)$ is integrally bounded for all $x \in \mathcal{C}$.
(F_3) There exists a function $\ell \in L^1(I, \mathbb{R})$ such that

$$d_H(F(t, x), F(t, y)) \leq \ell(t)\|x - y\|_{\mathcal{C}} \quad \text{a.e. } t \in I,$$

for all $x, y \in \mathcal{C}$.
(F_4) $F(t, x)$ is strictly monotone increasing in x for almost everywhere $t \in I$.
(G_1) $G(t, x)$ is compact subset of $\mathbb{R}$ for each $t \in I$ and $x \in \mathcal{C}$.
(G_2) G is L^1-Carathéodory.
(G_3) $G(t, x)$ is strictly monotone increasing in x for almost everywhere $t \in I$.
(H_1) $H(t, x)$ is a compact subset of $\mathbb{R}$ for each $t \in I$ and $x \in \mathcal{C}$.
(H_2) $S^1_H(x) \neq \emptyset$ for all $x \in C(J, \mathbb{R})$.
(H_3) H is L^1-Chandrabhan.
(H_4) PDI (3.1) has a strict lower solution a and a strict upper solution b with $a \leq b$.

Theorem 3.1. *Assume that the hypotheses* (F_1)–(F_4), (G_1)–(G_3) *and* (H_1)–(H_4) *hold. Then the PDI (3.1) has the least and the greatest solution in* $[a, b]$.

Proof. Let $X = C(J, \mathbb{R})$ and define an order interval $[a, b]$ in $C(J, \mathbb{R})$ which does exist in view of hypothesis (H_4). Note that the cone K is normal in X, and therefore, the order interval $[a, b]$ is norm bounded in X. As a result there is a constant $r > 0$ such that $\|x\| \le r$ for all $x \in [a, b]$.

Now PDI (3.1) is equivalent to the integral inclusion

$$x(t) \in \phi(0) + \int_0^t F(s, x_s)\mathrm{d}s + \int_0^t G(t, x_s)\mathrm{d}s + \int_0^t H(t, x_s)\mathrm{d}s, \quad \text{if } t \in I, \tag{3.4}$$

satisfying

$$x(t) = \phi(t), \quad \text{if } t \in I_0. \tag{3.5}$$

Define three multi-valued operators $A, B, C : [a, b] \to \mathcal{P}_p(X)$ by

$$Ax(t) = \begin{cases} \int_0^t F(s, x_s)\mathrm{d}s, & \text{if } t \in I, \\ 0, & \text{if } t \in I_0, \end{cases} \tag{3.6}$$

$$Bx(t) = \begin{cases} \phi(0) + \int_0^t G(s, x_s)\mathrm{d}s, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0, \end{cases} \tag{3.7}$$

and

$$Cx(t) = \begin{cases} \int_0^t H(s, x_s)\mathrm{d}s, & \text{if } t \in I, \\ 0, & \text{if } t \in I_0. \end{cases} \tag{3.8}$$

Clearly the multi-valued operators A, B and C are well defined on $[a, b]$ in view of the hypotheses (F_2), (G_2) and (H_2) respectively. Now the FDI (3.1) is transformed into an operator inclusion as

$$x(t) \in Ax(t) + Bx(t) + Cx(t), \quad t \in J.$$

We shall show that A, B and C satisfy all the conditions of Theorem 2.3 on $[a, b]$.

Step I : First, we show that A has compact and convex values on $[a, b]$. Observe that the operator A is equivalent to

$$Ax(t) = \begin{cases} (\mathcal{L} \circ S_F^1)(x)(t) & \text{if } t \in I, \\ 0, & \text{if } t \in I_0, \end{cases} \tag{3.9}$$

where $\mathcal{L} : L^1(I, \mathbb{R}) \to X$ is a continuous operator defined by

$$\mathcal{L}v(t) = \int_0^t v(s)\mathrm{d}s, \quad \text{if } t \in I.$$

To show that A has compact values, it then suffices to prove that the composition operator $\mathcal{L} \circ S_F^1$ has compact values on $[a, b]$. Let $x \in [a, b]$ be arbitrary and let $\{v_n\}$ be a sequence in $S_F^1(x)$. Then, by the definition of S_F^1, $v_n(t) \in F(t, x_t)$ a.e. for $t \in J$. Since $F(t, x_t)$ is compact, there is a convergent subsequence of $v_n(t)$ (for simplicity call it $v_n(t)$ itself) that converges in measure to some $v(t)$, where $v(t) \in F(t, x_t)$ a.e. for $t \in J$. From the continuity of $\mathcal{L}$, it follows that $\mathcal{L}v_n(t) \to \mathcal{L}v(t)$ pointwise on I as $n \to \infty$. In order to show that the convergence is uniform, we first show that $\{\mathcal{L}v_n\}$ is an equi-continuous sequence. Let $t, \tau \in I$; then

$$\begin{aligned} |\mathcal{L}v_n(t) - \mathcal{L}v_n(\tau)| &\le \left| \int_0^t v_n(s)\mathrm{d}s - \int_0^\tau v_n(s)\mathrm{d}s \right| \\ &\le \left| \int_\tau^t |v_n(s)|\mathrm{d}s \right|. \end{aligned} \tag{3.10}$$

Now, $v_n \in L^1(I, \mathbb{R})$, so the right-hand side of (3.10) tends to 0 as $t \to \tau$. Hence, $\{\mathcal{L}v_n\}$ is equi-continuous, and an application of the Ascoli theorem implies that it has a uniformly convergent subsequence. We then have

$\mathcal{L}v_{n_j} \to \mathcal{L}v \in (\mathcal{L} \circ S_F^1)(x)$ as $j \to \infty$, and so $(\mathcal{L} \circ S_F^1)(x)$ is compact. Therefore, A is a compact-valued multi-valued operator on $[a, b]$.

Next, we show that A is a multi-valued contraction on $[a, b]$. Let $x, y \in [a, b]$ and let $u_1 \in A(x)$. Then $u_1 \in X$ and $u_1(t) = \int_0^t v_1(s)\mathrm{d}s$ for some $v_1 \in S_F^1(x)$. From (H_2), it follows that

$$d_H(F(t, x_t), F(t, y_t)) \le \ell(t)\|x_t - y_t\|_{\mathcal{C}},$$

so we obtain the existence of $w \in F(t, y_t)$ such that

$$|v_1(t) - w| \le \ell(t)\|x_t - y_t\|_{\mathcal{C}}.$$

Thus, the multi-valued operator U defined by $U(t) = S_F^1(y)(t) \cap K(t)$, $t \in I$, where

$$K(t) = \{w : |v_1(t) - w| \le \ell(t)\|x_t - y_t\|_{\mathcal{C}}\},$$

has nonempty values and is measurable. Let v_2 be a measurable selection for U which exists by the Kuratowski–Ryll–Nardzewski's selection theorem (see Deimling [10]). Then $v_2 \in F(t, y_t)$ and

$$|v_1(t) - v_2(t)| \le \ell(t)\|x_t - y_t\|_{\mathcal{C}} \quad \text{a.e. } t \in I.$$

Let

$$u_2(t) = \begin{cases} \displaystyle\int_0^t v_2(s)\mathrm{d}s, & \text{if } t \in I \\ 0, & \text{if } t \in I_0. \end{cases}$$

It follows that $u_2 \in A(y)$ and

$$\begin{aligned} |u_1(t) - u_2(t)| &\le \left| \int_0^t v_1(s)\mathrm{d}s - \int_0^t v_2(s)\mathrm{d}s \right| \\ &\le \int_0^t |v_1(s) - v_2(s)|\mathrm{d}s \\ &\le \int_0^t \ell(s)\|x_s - y_s\|_{\mathcal{C}}\mathrm{d}s \\ &\le \|\ell\|_{L^1}\|x - y\|. \end{aligned}$$

Taking the supremum over t, we obtain

$$\|u_1 - u_2\| \le \|\ell\|_{L^1}\|x - y\|.$$

From this, and the analogous inequality obtained by interchanging the roles of x and y, we obtain

$$d_H(Ax, Ay) \le \|\ell\|_{L^1}\|x - y\|,$$

for all $x, y \in X$. This shows that A is a multi-valued contraction since $\|\ell\|_{L^1} < 1$. Hence, condition (i) of Theorem 2.3 holds.

Next we show that A is a strictly monotone increasing multi-valued operator on $[a, b]$. Let $x, y \in [a, b]$ be such that $x < y$. Since $x \mapsto F(t, x)$ is strictly monotone increasing, one has $F(t, x) \le F(t, y)$. As a result we have $S_F^1(x) \le S_F^1(y)$. Hence $Ax \le Ay$ and consequently A is strictly monotone increasing on $[a, b]$.

Step II: Secondly we show that the multi-valued operator B satisfies all the conditions of Theorem 2.3. It can be proved as in the Step I that B is a compact-valued strictly monotone increasing operator on $[a, b]$. We only prove that it is completely continuous on $[a, b]$. First, we show that B maps bounded sets into bounded sets in X. If S is a bounded set in X, then there exists $r > 0$ such that $\|x\| \le r$ for all $x \in S$. Now for each $u \in Bx$, there exists a $v \in S_G^1(x)$ such that

$$u(t) = \begin{cases} \phi(0) + \displaystyle\int_0^t v(s)\mathrm{d}s, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases}$$

Then, for each $t \in J$,

$$\begin{aligned} |u(t)| &\leq \|\phi\|_{\mathcal{C}} + \int_0^t |v(s)|\mathrm{d}s \\ &\leq \|\phi\|_{\mathcal{C}} + \int_0^t h_r(s)\mathrm{d}s \\ &\leq \|\phi\|_{\mathcal{C}} + \|h_r\|_{L^1}. \end{aligned}$$

This further implies that

$$\|u\| \leq \|\phi\|_{\mathcal{C}} + \|h_r\|_{L^1}$$

for all $u \in Bx \subset \bigcup B(S)$. Hence, $\bigcup B(S)$ is bounded.

Next we show that B maps bounded sets into equicontinuous sets. Let S be, as above, a bounded set and $u \in Bx$ for some $x \in S$. Then there exists $v \in S_G^1(x)$ such that

$$u(t) = \begin{cases} \phi(0) + \int_0^t v(s)\mathrm{d}s, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I_0. \end{cases}$$

Then for any $t_1, t_2 \in I$ with $t_1 \leq t_2$, we have

$$\begin{aligned} |u(t_1) - u(t_2)| &\leq \left| \int_0^{t_1} v(s)\mathrm{d}s - \int_0^{t_2} v(s)\mathrm{d}s \right| \\ &= \int_{t_1}^{t_2} |v(s)|\mathrm{d}s \\ &\leq \int_{t_1}^{t_2} h_r(s)\mathrm{d}s. \end{aligned}$$

If $t_1, t_2 \in I_0$ then $|u(t_1) - u(t_2)| = |\phi(t_1) - \phi(t_2)|$. For the case where $t_1 \leq 0 \leq t_2$ we have that

$$\begin{aligned} |u(t_1) - u(t_2)| &\leq \left| \phi(t_1) - \phi(0) - \int_0^{t_2} v(s)\mathrm{d}s \right| \\ &\leq |\phi(t_1) - \phi(0)| + \int_0^{t_2} |v(s)|\mathrm{d}s \\ &\leq |\phi(t_1) - \phi(0)| + \int_0^{t_2} h_r(s)\mathrm{d}s. \end{aligned}$$

Hence, in all cases, we have

$$|u(t_1) - u(t_2)| \to 0 \quad \text{as } t_1 \to t_2.$$

As a result $\bigcup B(S)$ is an equicontinuous set in X. Now an application of Arzelá–Ascoli theorem yields that the multi B is totally bounded on X.

Step IV. Next we prove that B has a *closed graph* in X. Let $\{x_n\} \subset X$ be a sequence such that $x_n \to x_*$ and let $\{y_n\}$ be a sequence defined by $y_n \in Bx_n$ for each $n \in \mathbb{N}$ such that $y_n \to y_*$. We will show that $y_* \in Bx_*$. Since $y_n \in Bx_n$, there exists a $v_n \in S_G^1(x_n)$ such that

$$y_n(t) = \begin{cases} \phi(0) + \int_0^t v_n(s)\mathrm{d}s, & \text{if } t \in I, \\ \phi(t), & \text{if } t \in I. \end{cases}$$

Consider the linear and continuous operator $\mathcal{K} : L^1(I, \mathbb{R}) \to C(I, \mathbb{R})$ defined by

$$\mathcal{K}v(t) = \int_0^t v_n(s)\mathrm{d}s.$$

Now

$$\begin{aligned}|y_n(t) - \phi(0) - (y_*(t) - \phi(0))| &\le |y_n(t) - y_*(t)|\\ &\le \|y_n - y_*\| \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

From Lemma 2.2 it follows that $(\mathcal{K} \circ S_G^1)$ is a closed graph operator and from the definition of $\mathcal{K}$ one has

$$y_n(t) - \phi(0) \in (\mathcal{K} \circ S_F^1(x_n)).$$

As $x_n \to x_*$ and $y_n \to y_*$, there is a $v \in S_G^1(x_*)$ such that

$$y_*(t) = \begin{cases}\phi(0) + \displaystyle\int_0^t v_*(s)\mathrm{d}s, & \text{if } t \in I\\ \phi(t), & \text{if } t \in I_0.\end{cases}$$

Hence, B is an upper semi-continuous multi-valued operator on $[a, b]$.

Step V: Finally, we show that the multi-valued operator C satisfies all the conditions of Theorem 2.3. It can be proved as in the Step I that C is a compact-valued strictly monotone increasing map on $[a, b]$. We only prove that it is compact on $[a, b]$. First, we show that C maps bounded sets into bounded sets in X. If S is a set in $[a, b]$, then there exists $r > 0$ such that $\|x\| \le r$ for all $x \in S$, because the cone K is normal in X. Now for each $u \in Cx$, there exists a $v \in S_H^1(x)$ such that

$$u(t) = \begin{cases}\displaystyle\int_0^t v(s)\mathrm{d}s, & \text{if } t \in I,\\ 0, & \text{if } t \in I_0.\end{cases}$$

Since (H_3) holds, we have

$$\begin{aligned}|u(t)| &\le \int_0^t |v(s)|\mathrm{d}s\\ &\le \int_0^t h_r(s)\mathrm{d}s\\ &\le \|h_r\|_{L^1}\end{aligned}$$

for all $t \in J$. This implies that

$$\|u\| \le \|h_r\|_{L^1}$$

for all $u \in Cx \subset \bigcup C(S)$. Hence, $\bigcup C(S)$ is bounded.

Next we show that C maps bounded sets into equicontinuous sets. Let S be, as above, a bounded set and $u \in Bx$ for some $x \in S$. Then there exists a $v \in S_H^1(x)$ such that

$$u(t) = \begin{cases}\displaystyle\int_0^t v(s)\mathrm{d}s, & \text{if } t \in I,\\ 0, & \text{if } t \in I_0.\end{cases}$$

Then for any $t_1, t_2 \in I$ with $t_1 \le t_2$ we have

$$\begin{aligned}|u(t_1) - u(t_2)| &\le \left|\int_0^{t_1} v(s)\mathrm{d}s - \int_0^{t_2} v(s)\mathrm{d}s\right|\\ &= \int_{t_1}^{t_2} |v(s)|\mathrm{d}s\\ &\le \int_{t_1}^{t_2} h_r(s)\mathrm{d}s.\end{aligned}$$

If $t_1, t_2 \in I_0$ then $|u(t_1) - u(t_2)| = 0$. For the case, where $t_1 \le 0 \le t_2$ we have that

$$|u(t_1) - u(t_2)| \le \left|\int_0^{t_2} v(s)\mathrm{d}s\right| \le |p(t_2) - p(0)|,$$

where $p(t) = \int_0^t h_r(s)\mathrm{d}s$.

Hence, in all cases, we have

$$|u(t_1) - u(t_2)| \to 0 \quad \text{as } t_1 \to t_2.$$

As a result $\bigcup C(S)$ is an equicontinuous set in X. Now an application of Arzelá–Ascoli theorem yields that C is totally bounded on $[a, b]$.

Thus all the conditions of Theorem 2.3 are satisfied and hence the operator inclusion $x \in Ax + Bx + Cx$ has the least and the greatest solution in $[a, b]$. This further implies that the differential inclusion (3.1) has a minimal and a maximal solution on J. $\square$

3.2. Differential inclusions in Banach algebras

Given a function $\phi \in C$, consider the perturbed functional differential inclusion (in short FDI)

$$\left.\begin{array}{l}\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{x(t) - h(t, x(t))}{f(t, x(t))}\right] \in G(t, x_t) \quad \text{a.e. } t \in I,\\ x_0 = \phi\end{array}\right\} \tag{3.11}$$

where $f : I \times \mathbb{R} \to \mathbb{R} \setminus \{0\}$, $h : I \times \mathbb{R} \to \mathbb{R}$ and $G : I \times \mathcal{C} \to \mathcal{P}_p(\mathbb{R})$.

By a solution of the FDI (3.11) we mean a function $x \in AC(J, \mathbb{R})$ such that

(i) the function $t \mapsto \frac{x(t)-h(t,x(t))}{f(t,x(t))}$ is differentiable and
(ii) there exists a $v \in L^1(I, \mathbb{R})$ with $v(t) \in G(t, x_t)$ a.e. $t \in I$ such that

$$\left(\frac{x(t) - h(t, x(t))}{f(t, x(t))}\right)' = v(t) \quad \text{for all } t \in I \text{ satisfying } x(t) = \phi(t) \text{ for } t \in I_0.$$

The special case of (3.11) in the form of differential equation (DE)

$$\left.\begin{array}{l}\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{x(t) - h(t, x(t))}{f(t, x(t))}\right] = g(t, x_t) \quad \text{a.e. } t \in I\\ x_0 = \phi\end{array}\right\} \tag{3.12}$$

is also new to the theory of functional differential equations and includes the functional differential equation,

$$\left.\begin{array}{l}\dfrac{\mathrm{d}}{\mathrm{d}t}\left[\dfrac{x(t)}{f(t, x(t))}\right] = g(t, x_t) \quad \text{a.e. } t \in I\\ x_0 = \phi\end{array}\right\} \tag{3.13}$$

which has been studied in Dhage et al. [25] for the existence results under the mixed conditions. Thus the study of FDI (3.11) or DE (3.12) has got importance in recent years in the field of nonlinear differential analysis. In this section we prove the existence results for the FDI (3.11) under the mixed Lipschitz, monotonic and a weaker Carathéodory condition of the multi-valued mapping G. Note that here we do not require any type of continuity condition on the multi-valued functions G.

We need the following definition in the sequel.

Definition 3.7. A function $a \in C(J, \mathbb{R})$ is called a strict lower solution of FDI (3.11) if $\frac{x(t)-h(t,x(t))}{f(t,x(t))}$ is absolutely continuous, and for all $v \in S^1_G(a)$, we have $\left(\frac{a(t)-h(t,a(t))}{f(t,a(t))}\right)' \leq v(t)$ for all $t \in I$ and $a_0 \leq \phi$. Similarly, a function $b \in C(J, \mathbb{R})$ is called a strict upper solution of the FDI (3.11) if the above inequalities hold with reverse sign.

We consider the following set of assumptions :

(f_1) f defines a continuous mapping $f : I \times \mathbb{R} \to \mathbb{R}^+ \setminus \{0\}$ with $f(0, x) = 1$ for all $x \in \mathbb{R}$.
(f_2) f defines a continuous mapping $f : I \times \mathbb{R} \to \mathbb{R}^+$ such that $f(0, x) = 1$ for all $x \in \mathbb{R}$, and there exists a bounded function $\ell_1 : I \to \mathbb{R}$ with bound $\|\ell_1\|$ such that

$$|f(t, x) - f(t, y)| \leq \ell_1(t)|x - y| \quad t \in I$$

for all $x, y \in \mathbb{R}$.

(f_3) $f(t,x)$ is monotone increasing in x almost everywhere for $t \in I$.
(h_1) h defines a continuous mapping $h : I \times \mathbb{R} \to \mathbb{R}^+$ with $h(0,x) = 0$ for all $x \in \mathbb{R}$.
(h_2) h defines a continuous mapping $h : I \times \mathbb{R} \to \mathbb{R}^+$ such that $h(0,x) = 0$ for all $x \in \mathbb{R}$, and there exists a bounded function $\ell_2 : I \to \mathbb{R}$ with bound $\|\ell_2\|$ such that

$$|h(t,x) - h(t,y)| \leq \ell_2(t)|x-y| \quad t \in I$$

for all $x, y \in \mathbb{R}$.
(h_3) $h(t,x)$ is monotone increasing in x almost everywhere for $t \in I$.
(G_4) G defines a multi-valued mapping $G : I \times \mathcal{C} \to \mathcal{P}_{cp}(\mathbb{R}^+)$.
(G_5) G is L^1-Chandrabhan.
(G_6) The FDI (3.11) has a strict lower solution a and a strict upper solution b on J with $a \leq b$.

Theorem 3.2. *Assume that the hypotheses* (f_1), (f_3), (h_2)–(h_3) *and* (G_4)–(G_6) *hold. Further, if* $\|\ell_2\| < 1/2$ *and* $\phi \geq 0$ *on* I_0*, then the FDI* (3.11) *has a minimal and a maximal solution on* J.

Proof. Let $X = C(J,\mathbb{R})$ and define a norm $\|\cdot\|$ and an order relation $\leq$ in X by Eqs. (3.2) and (3.3), respectively. Then X is an ordered Banach algebra with respect to the multiplication "$\cdot$" defined by $(x.y)(t) = x(t)y(t)$ for $t \in J$. Consider the order interval $[a,b]$ in X which does exist in view of hypothesis (B_4). Define three operators $A : [a,b] \to K$, $B : [a,b] \to \mathcal{P}_{cp}(K)$ and $C : [a,b] \to X$ by

$$Ax(t) = \begin{cases} f(t,x(t)), & \text{if } t \in I, \\ 1 \quad \text{if } t \in I_0, \end{cases} \tag{3.14}$$

$$Bx(t) = \begin{cases} \left\{ u(t) : u(t) = \phi(0) + \int_0^t v(s)\mathrm{d}s,\ v \in S_G^1(x) \right\}, & \text{if } t \in I \\ \phi(t), \quad \text{if } t \in I_0 \end{cases} \tag{3.15}$$

and

$$Cx(t) = \begin{cases} h(t,x(t)), & \text{if } t \in I, \\ 0, \quad \text{if } t \in I_0. \end{cases} \tag{3.16}$$

Then the FDI (3.11) is transformed into the operator inclusion

$$x(t) \in Ax(t)Bx(t) + Cx(t), \quad t \in J.$$

We shall show that the operators A, B and C satisfy all the conditions of Theorem 2.6 on $[a,b]$.

Step I: It follows from hypothesis (B_1) that A, B and C define the operators $A : [a,b] \to K$, $B : [a,b] \to \mathcal{P}_{cp}(K)$ and $C : [a,b] \to X$. Let $x, y \in [a,b]$ be such that $x \leq y$. Next we show that A and C are monotone increasing and and B is strictly monotone increasing on $[a,b]$. By (B_3),

$$\begin{aligned} Ax(t) &= \begin{cases} f(t,x(t)) & \text{if } t \in I \\ 1, \quad \text{if } t \in I_0 \end{cases} \\ &\leq \begin{cases} f(t,y(t)) & \text{if } t \in I \\ 1, \quad \text{if } t \in I_0 \end{cases} \\ &= Ay(t) \end{aligned}$$

for all $t \in J$. Hence $Ax \leq Ay$. Similarly, $Cx \leq Cy$. Next let $x, y \in [a,b]$ be such that $x \leq y$ with $x \neq y$. Since $G(t,x)$ is L^1-Chandrabhan, we have $S_G^1(x) \leq S_G^1(y)$. As a result we obtain $Bx \leq By$. Thus B is strictly monotone increasing on $[a,b]$. By (G_6), $a \leq AaBa + Ca$ and $AbBb + Cb \leq b$.

Step II: Next we show that A is completely continuous on $[a,b]$. Now the cone K in X is normal, so the order interval $[a,b]$ is norm-bounded. Hence there exists a constant $r > 0$ such that $\|x\| \leq r$ for all $x \in [a,b]$. As f is continuous on compact $J \times [-r,r]$, it attains its maximum, say M. Therefore, for any subset S of $[a,b]$ we have

$$\begin{aligned} \|A(S)\|_{\mathcal{P}} &= \max\{1, \sup\{\|Ax\| : x \in S\}\} \\ &= \sup\left\{ \sup_{t \in J} |f(t,x(t))| : x \in S \right\} + 1 \end{aligned}$$

$$\begin{aligned}&\leq \sup\left\{\sup_{t\in J}|f(t,x)| : x\in[-r,r]\right\}+1\\ &\leq M+1.\end{aligned}$$

This shows that $A(S)$ is a uniformly bounded subset of X.

Next we note that the mapping $f(t,x)$ is uniformly continuous on $[0,T]\times[-r,r]$. Therefore for any $t,\tau\in[0,T]$ we have

$$|f(t,x)-f(\tau,x)|\to 0 \quad \text{as } t\to\tau$$

for all $x\in[-r,r]$. Similarly for any $x,y\in[-r,r]$

$$|f(t,x)-f(t,y)|\to 0 \quad \text{as } x\to y$$

for all $t\in[0,T]$. Hence for any $t,\tau\in[0,T]$ and for any $x\in S$ one has

$$\begin{aligned}|Ax(t)-Ax(\tau)| &= |f(t,x(t))-f(\tau,x(\tau))|\\ &\leq |f(t,x(t))-f(\tau,x(t))|+|f(\tau,x(t))-f(\tau,x(\tau))|\\ &\to 0 \quad \text{as } t\to\tau.\end{aligned}$$

Again if $t,\tau\in I_0$, then we have

$$|Ax(t)-Ax(\tau)|=0\leq|t-\tau|\to 0$$

for all $x\in A(S)$. Finally if $t\in I$ and $\tau\in I_0$, then

$$|Ax(t)-Ax(\tau)|\leq|Ax(\tau)-Ax(0)|+|Ax(t)-Ax(0)|\leq|f(t,x(t))-f(0,x(0))|$$

for all $x\in A(S)$. Now if $\tau\to t$, then $\tau\to 0$ and $t\to 0$. Therefore, from the uniform continuity of f, it follows that

$$\begin{aligned}|f(t,x(t))-f(0,x(0))| &\leq |f(t,x(t))-f(0,x(t))|+|f(0,x(t))-f(0,x(0))|\\ &\to 0 \quad \text{as } \tau\to t.\end{aligned}$$

Thus in all three cases, we have

$$|Ax(t)-Ax(\tau)|\to 0 \quad \text{as } \tau\to t.$$

This shows that $A(S)$ is an equi-continuous set in X. Now an application of Arzela–Ascoli theorem yields that A is a completely continuous operator on $[a,b]$.

Step III: *Step* I: Next, we show that B has compact values on $[a,b]$. Now the multi-valued operator B is equivalent to

$$Bx(t)=\begin{cases}(\mathcal{K}\circ S_G^1)(x)(t), & \text{if } t\in I\\ \phi(t), & \text{if } t\in I_0\end{cases} \tag{3.17}$$

where, $\mathcal{K}: L^1(I,\mathbb{R})\to C(I,\mathbb{R})$ is a continuous operator defined by

$$\mathcal{K}v(t)=\phi(0)+\int_0^t v(s)\mathrm{d}s,\ t\in I. \tag{3.18}$$

To show that B has compact values, it then suffices to prove that the composition operator $\mathcal{K}\circ S_G^1$ has compact values on $[a,b]$. Let $x\in[a,b]$ be arbitrary and let $\{v_n\}$ be a sequence in $S_G^1(x)$. Then, by the definition of S_G^1, $v_n(t)\in G(t,x_t)$ a.e. for $t\in J$. Since $G(t,x_t)$ is compact, there is a convergent subsequence of $v_n(t)$ (for simplicity call it $v_n(t)$ itself) that converges in measure to some $v(t)$, where $v(t)\in G(t,x_t)$ a.e. for $t\in J$. From the continuity of $\mathcal{L}$, it follows that $\mathcal{K}v_n(t)\to\mathcal{K}v(t)$ pointwise on I as $n\to\infty$. In order to show that the convergence is uniform, we first show that $\{\mathcal{K}v_n\}$ is an equi-continuous sequence. Let $t,\tau\in I$; then

$$\begin{aligned}|\mathcal{K}v_n(t)-\mathcal{K}v_n(\tau)| &\leq \left|\int_0^t v_n(s)\mathrm{d}s-\int_0^\tau v_n(s)\mathrm{d}s\right|\\ &\leq \left|\int_\tau^t |v_n(s)|\mathrm{d}s\right|.\end{aligned} \tag{3.19}$$

Now, $v_n \in L^1(I, \mathbb{R})$, so the right-hand side of (3.10) tends to 0 as $t \to \tau$. Hence, $\{\mathcal{K}v_n\}$ is equi-continuous, and an application of the Ascoli theorem implies that it has a uniformly convergent subsequence. We then have $\mathcal{K}v_{n_j} \to \mathcal{K}v \in (\mathcal{K} \circ S^1_G)(x)$ as $j \to \infty$, and so $(\mathcal{K} \circ S^1_G)(x)$ is compact. Therefore, A is a compact-valued multi-valued operator on $[a, b]$.

Step IV: Finally, we show that the operator C is a contraction on $[a, b]$. Let $x, y \in [a, b]$. Then by hypothesis (h_1),

$$\begin{aligned}\|Cx - Cy\| &= \sup_{t \in J} |Cx(t) - Cy(t)| \\ &\le \sup_{t \in J} |h(t, x(t)) - h(t, y(t))| \\ &\le \sup_{t \in J} \ell(t)|x(t) - y(t)| \\ &\le \|\ell\| \|x - y\|\end{aligned}$$

where $\|\ell\| < 1/2$. This shows that C is a contraction on $[a, b]$ with a contraction constant $\|\ell\| < 1/2$.

Now an application of Corollary 2.13 yields that the operator inclusion $x \in AxBx + Cx$ and consequently the FDI (3.11) has a minimal and a maximal solution on J. This completes the proof. □

Theorem 3.3. *Assume that the hypotheses* (f_2), (f_3), (h_1), (h_3) *and* (G_4)–(G_6) *hold. Further, if* $\|\ell_1\|(\|\phi\|_C + \|h_r\|_{L^1}) < 1/2$ *and* $\phi \ge 0$ *on* I_0, *then the FDI* (3.11) *has a minimal and a maximal solution on* J.

Proof. The proof is similar to Theorem 3.2 and now the conclusion follows by an application of Theorem 2.7. □

Note that we do not need any of the multi-valued functions involved in the functional differential inclusions (3.1) and (3.11) to have convex values on their domain of definitions. All the functions are also not required to be continuous, but are required to satisfy certain monotonicity conditions.

4. Remarks and conclusion

The method of upper and lower solutions has been in practice for a long time in the theory of nonlinear differential and integral equations. The monotonicity together with the upper and lower solutions yields the existence results for extremal solutions of nonlinear differential and integral equations. The exhaustive treatment of the subject appears in Heikkilä and Lakshmikantham [4]. The upper and lower solutions method for the differential and integral inclusions is relatively new and may be found in the works of Halidias and Papageorgiou [21]. Monotone theory for differential and integral inclusions is a most active area of research at present, but the study was initiated by Dhage [5] more than a decade ago. There are different types of monotonic conditions for the multi-valued functions. The strict upper and lower solutions together with the mild monotonic conditions of the multi-valued functions yield the existence results for differential and integral inclusions. The work along this direction appears in Dhage [1]. The hybrid fixed point theorems for the multi-valued mappings involving the mixed hypotheses from algebra, geometry and topology have been discussed in Dhage [19] under the mild monotonic conditions along with their applications to differential inclusions for the existence results. In that case the multi-valued operators need to have convex values on the domain of their definition and the solutions are obtained under mild monotonic conditions. In this paper, we used the strict upper and lower solutions together with the strict monotonic conditions of the multi-valued functions to yield the existence results for extremal solutions for the first order functional differential inclusions under the mixed Lipschitz, compactness and monotonic conditions, but without assuming the convexness of the multi-valued functions. Other existence results involving strict upper and lower solutions together with the strict monotonic conditions of multi-valued functions may be found in Dhage [7,19], Dhage and O'Regan [8], Agarwal et al. [9]. Again, our results of this paper generalize and extend the fixed point results given in Dhage [5,14,7] and some of the results are also new even to the single-valued case functional differential equations. Finally, the abstract fixed point results of this paper have some nice applications to a variety of perturbed differential and integral inclusions and some of the results in this direction will be reported elsewhere.

Acknowledgment

The author is thankful to the referees for giving some useful suggestions for the improvement of the presentation of this paper.

References

[1] B.C. Dhage, A fixed point theorem for multivalued mappings on ordered Banach spaces with applications I, Nonlinear Anal. Forum 10 (2005) 105–126.
[2] A. Granas, J. Dugundji, Fixed Point Theory, Springer Verlag, 2003.
[3] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, New York, 1988.
[4] S. Heikkilä, V. Lakshmikantham, Monotone Iterative Technique for Nonlinear Discontinues Differential Equations, Marcel Dekker Inc., New York, 1994.
[5] B.C. Dhage, A lattice fixed point theorem for multi-valued mappings and applications, Chinese J. Math. 19 (1991) 11–22.
[6] B.C. Dhage, Multi-valued mappings and fixed points I, Nonlinear Funct. Anal. Appl. 10 (3) (2005) 359–378.
[7] B.C. Dhage, A fixed point theorem for multivalued mappings on ordered Banach spaces with applications II, Panamer. Math. J. 15 (2005) 15–34.
[8] B.C. Dhage, D. O'Regan, A lattice fixed point theorem and multi-valued differential equations, Funct. Differ. Equ. 9 (2002) 109–115.
[9] R.P. Agarwal, B.C. Dhage, D. O'Regan, The method of upper and lower solution for differential inclusions via a lattice fixed point theorem, Dynam. Systems Appl. 12 (2003) 1–7.
[10] K. Deimling, Multi-valued Differential Equations, De Gruyter, Berlin, 1998.
[11] E. Zeidler, Nonlinear Functional Analysis and its Applications: Part I, Springer Verlag, 1985.
[12] A. Petruşel, Multivalued operators and fixed points, PUMA 9 (1998) 165–170.
[13] L. Gorniewicz, Topological Fixed Point Theory for Multi-valued Mappings, Kluwer, Dordrecht, 1999.
[14] B.C. Dhage, Fixed point theorems for discontinuous multi-valued operators on ordered spaces with applications, Comput. Math. Appl. 51 (3–4) (2006) 589–604.
[15] M.A. Krasnoselskii, Topological Methods in the Theory of Nonlinear Integral Equations, Pergamon Press, 1964.
[16] AKP P.P. Akhmerov, M.I. Kamenskii, A.S. Potapov, B.N. Sadovskii, Measures of Noncompactness and Condensing Operators, Birkhauser, 1992.
[17] B.C. Dhage, Multi-valued operators and fixed point theorems in Banach algebras I, Taiwanese J. Math. 10 (4) (2006) 1025–1045.
[18] J. Banas, M. Lecko, Fixed points of the product of operators in Banach algebras, Panamer. Math. J. 12 (2002) 101–109.
[19] B.C. Dhage, A general multi-valued hybrid fixed point theorem and perturbed differential inclusions, Nonlinear Anal. 64 (2006) 2747–2772.
[20] B.C. Dhage, Existence results for extremal solutions of perturbed first and second order neutral functional differential equations, preprint.
[21] N. Halidias, N. Papageorgiou, Second order multi-valued boundary value problems, Arch. Math. (Brno) 34 (1998) 267–284.
[22] S. Hu, N.S. Papageorgiou, Handbook of Multivalued Analysis, vol. I: Theory, Kluwer Academic Publishers, Dordrecht, 1997.
[23] J.K. Hale, Theory of Functional Differential Equations, Springer, New York, 1977.
[24] A. Lasota, Z. Opial, An application of the Kakutani–Ky Fan theorem in the theory of ordinary differential equations, Bull. Acad. Pol. Sci. Ser. Sci. Math. Astronom. Phys. 13 (1965) 781–786.
[25] B.C. Dhage, S.N. Salunkhe, R. Agarwal, W. Zhang, A functional differential equation in Banach algebras, Math. Inequal Appl. 8 (1) (2005) 89–99.