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Well-posedness Results for Models of Elastomers

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Existence and uniqueness of weak solutions are shown for different models of the dynamic behavior of elastomers. The models are based on a nonlinear stress-strain relationship (satisfying a locally Lipschitz and affine domination property) and incorporate hysteretic effects as well. The results provide alternatives to previous theories that required monotonicity assumptions on the nonlinearities. Results with a nonlinear constitutive law and nonlinear internal dynamics are presented for the first time.

1. INTRODUCTION

In this paper we examine the theoretical foundations of a series of models for the dynamic behavior of elastomers (filled rubber-like materials). As outlined in previous papers [6, 7], the basic model describing the longitudinal motion of a viscoelastic bar with the upper end ($x = 0$) fixed and a tip mass on the lower end ($x = \ell$) is given by

$$\rho A_c u_{tt} - (A_c \sigma)_x = 0 \qquad \text{for } 0 < x < \ell \tag{1.1}$$

$$M u_{tt}(t, \ell) + A_c \sigma(t, \ell) = Mg + f(t) \tag{1.2}$$

$$u(t, 0) = 0 \tag{1.3}$$

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$$u(0, x) = u_0 \tag{1.4}$$

$$u_t(0, x) = u_1, \tag{1.5}$$

where ρ is the mass density, A_c is cross-sectional area, M is tip mass, g is gravitational acceleration, f is external force, and σ denotes the stress. If there is no tip mass and $A_c = \rho = 1$, then we can write the system in variational formulation as

$$u_{tt} - \sigma_x = F(t) \qquad \text{in} \quad V^* \tag{1.6}$$

$$u(0, x) = u_0 \in V \tag{1.7}$$

$$u_t(0, x) = u_1 \in H, \tag{1.8}$$

where we let $V = H^1_L(0, \ell)$, $H = L^2(0, \ell)$ and $F(t) = f(t)\delta_\ell(x)$. The crucial modeling question is what type of stress strain relationship best describes the material. In our earlier experimentally based investigations of elastomers we found that a nonlinear constitutive law is needed [4, 5], i.e.,

$$\sigma(u_x) = Eu_x + g_e(u_x) + C_D u_{tx}, \tag{1.9}$$

where the third term on the right side is a first approximation to a damping that these materials exhibit. (For simplicity of presentation here we will later make the assumption that $E = 1$.) Comparing the actual experimental data with this model, we demonstrated good agreement for unfilled and lightly filled rubber bars. However, the model is not adequate to describe medium or highly filled elastomers that exhibit significant hysteretic behavior. To account for this property of the material we included a Boltzmann integral term in the constitutive relationship

$$\sigma(u_x) = Eu_x + g_e(u_x) + C_D u_{tx} + \int_{-\infty}^{t} e^{-c_1(t-s)} \frac{d}{ds} g_v(u_x, u_{tx})\, ds. \tag{1.10}$$

As described in [5] under specific assumptions on the prehistory of the motion this relationship can be expressed in an equivalent internal variable formulation

$$\sigma(u_x) = Eu_x + g_e(u_x) + C_D u_{tx} + \varepsilon_1 \tag{1.11}$$

$$\varepsilon_{1t} + c_1 \varepsilon_1 = \frac{d}{dt} g_v(u_x, u_{tx}) \tag{1.12}$$

$$\varepsilon_1(0, x) = 0. \tag{1.13}$$

We can think of ε_1 as an internal strain variable whose dynamics is described by (1.12)–(1.13). We also note that although these constitutive relationships are expressed in terms of the infinitesimal strain u_x, the formulation is equivalent to considering finite strains $\varepsilon = u_x + \frac{1}{2}u_x^2$, with different nonlinear functions g_e and g_v. The internal variable system (1.11)–(1.13) can be further generalized by considering nonlinear internal dynamics, i.e., (1.11) with

$$\varepsilon_{1t} + c_1\varepsilon_1 = \frac{d}{dt}g_v(u_x, u_{tx}) + g_{in}(\varepsilon_1) \tag{1.14}$$

$$\varepsilon_1(0, x) = 0. \tag{1.15}$$

We note that the resulting constitutive law is no longer equivalent to a Boltzmann integral formulation. A further generalization that has proved important for highly filled elastomers involves multiple internal variables. That is, one replaces ε_1 in (1.11) by a finite sum $\sum \varepsilon_j$ of internal variables ε_j which satisfy systems of the form (1.12)–(1.13) or (1.14)–(1.15).

In this note we consider the well-posedness of the basic model with the three different constitutive laws. Thus we first show that the nonlinear problem with no hysteresis ((1.6)–(1.8) with (1.9)) is well posed under rather general assumptions on the nonlinear function g_e. Our first result guarantees the existence of a unique local weak solution under a local Lipschitz condition on the nonlinear function. If we impose an additional growth assumption on the nonlinearity, then the weak solution is global (i.e., exists for any time interval $[0, T]$). We then show that similar results can be obtained for the nonlinear problem (i.e., nonlinear g_e, g_v) with linear internal strain dynamics ((1.6)–(1.8) with (1.11)–(1.13)) and for the nonlinear problem with nonlinear internal strain dynamics ((1.6)–(1.8) with (1.11),(1.14)–(1.15)). The first two problems were previously studied under certain monotonicity and growth assumptions on the nonlinearities (g_e and g_v) in [2, 6], respectively. The results in this paper demonstrate that well-posedness can be achieved under relaxed assumptions and can be extended to the problem with nonlinear internal dynamics. The techniques we use were successfully employed to establish existence-uniqueness of weak solutions for linear evolution equations of second order in t in [8] and for semilinear second order evolution equations where the nonlinear forcing term satisfies a global Lipschitz condition in [9]. In [1] such techniques were extended to study a nonlinear beam equation where the nonlinearity satisfies only a local Lipschitz condition. Arguments for our results concerning the nonlinear problem with no hysteresis are very similar to the ones used in [1]. We give a fairly detailed exposition here in order to be able to easily refer to these as we extend the well-posedness result to the systems with internal variables.

2. THE NONLINEAR PROBLEM WITH NO HYSTERESIS

In this section we investigate the system

$$u_{tt} - C_D u_{txx} - u_{xx} - (g_e(u_x))_x = F(t) \qquad \text{in} \quad V^* \tag{2.16}$$

$$u(0, x) = u_0 \in V \tag{2.17}$$

$$u_t(0, x) = u_1 \in H, \tag{2.18}$$

where $V = H_L^1(0, \ell) \hookrightarrow H = L^2(0, \ell) \hookrightarrow V^*$, $C_D > 0$, and E has been normalized only for the sake of convenience. We make the following assumptions:

(A_{g_e}) The nonlinear function g_e satisfies the following local Lipschitz condition: let $B_r(0)$ denote the ball of radius r centered at 0 in H and for some positive constant L_{B_r} we have

$$\|g_e(w) - g_e(v)\| \le L_{B_r} \|w - v\|$$

for all $w, v \in B_r(0)$.

(A_{b_e}) There exist constants C_1, C_2 such that

$$\|g_e(v)\| \le C_1 \|v\| + C_2,$$

for every $v \in H$.

(A_f) The forcing term F satisfies

$$F \in L^2(0, T; V^*).$$

We define the space of weak solutions to be

$$\mathcal{U}(0, T) = \{u \in L^2(0, T; V) \mid u_t \in L^2(0, T; V),\ u_{tt} \in L^2(0, T; V^*)\},$$

with norm given by

$$\|u\|_{\mathcal{U}(0,T)} = \left(\|u\|^2_{L^2(0,T;V)} + \|u_t\|^2_{L^2(0,T;V)} + \|u_{tt}\|^2_{L^2(0,T;V^*)} \right)^{\frac{1}{2}}.$$

DEFINITION 2.1. *We define $u \in \mathcal{U}(0, T)$ to be a weak solution of* (2.16)–(2.18) *if it satisfies*

$$\langle u_{tt}, \varphi \rangle_{V^*,V} + C_D \langle u_{tx}, \varphi_x \rangle + \langle u_x, \varphi_x \rangle + \langle g_e(u_x), \varphi_x \rangle = \langle F, \varphi \rangle_{V^*,V} \qquad \textit{for every } \varphi \in V \tag{2.19}$$

and

$$u(0, x) = u_0 \in V \tag{2.20}$$

$$u_t(0, x) = u_1 \in H. \tag{2.21}$$

Here $\langle \cdot, \cdot \rangle$ denotes the inner product in H while $\langle \cdot, \cdot \rangle_{V^*,V}$ represents the usual duality product. We first prove the following local existence theorem:

THEOREM 2.1. *Under assumptions* (A_{g_e}) *and* (A_f) *and for any* $u_0 \in V, u_1 \in H$, *the system* (2.19)–(2.21) *has a unique weak solution on some interval* $[0, t^*]$.

Proof. Let P denote the Hilbert space radial retraction onto the ball in H centered at u_{0x} with radius 1, and let

$$\hat{g}_e(v) = g_e(Pv).$$

Thus $\hat{g}_e$ satisfies a global Lipschitz condition

$$\|\hat{g}_e(w) - \hat{g}_e(v)\| \leq L_{B_{(1+\|u_{0x}\|)}} \|Pw - Pv\| \leq \tilde{L}\|w - v\| \tag{2.22}$$

for every $w, v \in H$ and also

$$\|\hat{g}_e(w)\| \leq \tilde{L}\|w\| + C, \tag{2.23}$$

where $C = \|g_e(0)\| \geq 0$.

Now we consider the problem

$$u_{tt} - C_D u_{txx} - u_{xx} - (\hat{g}_e(u_x))_x = F(t) \qquad \text{in} \quad V^* \tag{2.24}$$

$$u(0, x) = u_0 \in V \tag{2.25}$$

$$u_t(0, x) = u_1 \in H. \tag{2.26}$$

Let $\{\psi_i\}_{i=1}^{\infty}$ be any linearly independent total subset of V. For each m let

$$V^m = \text{span}\{\psi_1, \psi_2, \ldots, \psi_m\},$$

and choose $u_0^m, u_1^m \in V^m$ such that $u_0^m \to u_0$ in V and $u_1^m \to u_1$ in H as $m \to \infty$. We develop the standard Galerkin approximations for the problem (2.24)–(2.26). Let $u^m(t) = \sum_{i=1}^m C_i^m(t)\psi_i$ be the unique solution of the m-dimensional ordinary differential equation system

$$\langle u_{tt}^m, \psi_j \rangle_{V^*,V} + C_D \langle u_{tx}^m, \psi_{jx} \rangle + \langle u_x^m, \psi_{jx} \rangle + \langle \hat{g}_e(u_x^m), \psi_{jx} \rangle = \langle F, \psi_j \rangle_{V^*,V} \tag{2.27}$$

$$u^m(0, x) = u_0^m \tag{2.28}$$

$$u_t^m(0, x) = u_1^m. \tag{2.29}$$

To obtain an *a priori* estimate we multiply (2.27) by $\frac{d}{dt} C_j^m(t)$ and sum over j to arrive at

$$\langle u_{tt}^m, u_t^m \rangle_{V^*,V} + C_D \langle u_{tx}^m, u_{tx}^m \rangle + \langle u_x^m, u_{tx}^m \rangle + \langle \hat{g}_e(u_x^m), u_{tx}^m \rangle = \langle F, u_t^m \rangle_{V^*,V}. \tag{2.30}$$

Thus

$$\frac{1}{2}\frac{d}{dt}\left\{\|u_t^m(t)\|^2 + \|u_x^m(t)\|^2\right\} + C_D\|u_{tx}^m\|^2 = \\ -\langle \hat{g}_e(u_x^m), u_{tx}^m\rangle + \langle F, u_t^m\rangle_{V^*,V}. \tag{2.31}$$

Integrating from 0 to t we obtain

$$\|u_t^m(t)\|^2 + \|u_x^m(t)\|^2 + 2C_D\int_0^t \|u_{\tau x}^m(\tau)\|^2 d\tau = \|u_1^m\|^2 \\ + \|u_{0x}^m\|^2 - 2\int_0^t \langle \hat{g}_e(u_x^m), u_{\tau x}^m\rangle d\tau + 2\int_0^t \langle F, u_\tau^m\rangle_{V^*,V} d\tau. \tag{2.32}$$

Using assumption (A_f), the boundedness property (2.23) and standard inequalities, we obtain

$$\|u_t^m(t)\|^2 + \|u_x^m(t)\|^2 + \nu\int_0^t \|u_{\tau x}^m(\tau)\|^2 d\tau \le \|u_1^m\|^2 + \|u_{0x}^m\|^2 \\ + \frac{\tilde{L}^2}{\delta}\int_0^t \|u_x^m(\tau)\|^2 d\tau + \frac{1}{\delta}\int_0^t \|F(\tau)\|_{V^*}^2 d\tau + \frac{C^2 T}{\delta}, \tag{2.33}$$

where δ is chosen such that $\nu = 2C_D - 3\delta > 0$. By applying the Gronwall inequality we can conclude that the sequence $\{\|u_x^m\|^2\}$ is bounded. Hence there exists a positive constant $\tilde{C} = \tilde{C}(u_1, u_0, g_e, F, T)$ independent of m such that

$$\|u_t^m(t)\|^2 + \|u_x^m(t)\|^2 + \nu\int_0^t \|u_{\tau x}^m(\tau)\|^2 d\tau \le \tilde{C}. \tag{2.34}$$

Now we can argue as in [1, 2, 3, 8] that there exists a subsequence, again denoted by $\{u^m\}$, and limit functions $u \in W^{1,2}(0,T;V)$ and $\tilde{g} \in L^2(0,T;H)$, such that

$$u^m \to u \qquad \text{weakly in } W^{1,2}(0,T;V) \tag{2.35}$$

$$\hat{g}_e(u_x^m) \to \tilde{g} \qquad \text{weakly in } L^2(0,T;H). \tag{2.36}$$

Using these convergences we obtain that u satisfies

$$\langle u_{tt}, \varphi\rangle_{V^*,V} + C_D\langle u_{tx}, \varphi_x\rangle + \langle u_x, \varphi_x\rangle \\ + \langle \tilde{g}, \varphi_x\rangle = \langle F, \varphi\rangle_{V^*,V} \qquad \text{for every } \varphi \in V \tag{2.37}$$

and

$$u(0,x) = u_0 \in V \tag{2.38}$$

$$u_t(0,x) = u_1 \in H. \tag{2.39}$$

We note that as in [1] we have that

$$\phi \to \int_0^T \langle u_{tt}, \phi \rangle_{V^*,V} d\tau$$

is continuous over $\mathscr{D}(0, T; V)$ equipped with the topology of $L^2(0, T; V)$ and thus by density over $L^2(0, T; V)$. So $u_{tt} \in (L^2(0, T; V))^* = L^2(0, T; V^*)$ and since we already established that $u \in W^{1,2}(0, T; V)$ we can conclude that $u \in \mathscr{U}(0, T)$. By [8, Remark 1, p. 555] this also provides the additional regularity:

$$u \in C([0, T]; V), \qquad u_t \in C([0, T]; H).$$

To conclude that u is indeed a weak solution of (2.24)–(2.26) we need to show that

$$\langle \tilde{g}, \varphi_x \rangle = \langle \hat{g}_e(u_x), \varphi_x \rangle \qquad \text{for every } \varphi \in V. \tag{2.40}$$

This is achieved by establishing the strong convergence $u_x^m(t) \to u_x(t)$ in H as $m \to \infty$. Let $z^m(t) = u^m(t) - u(t)$. Using u_t^m and u_t as test functions in (2.27) and (2.37), respectively, and integrating, we find

$$\|z_t^m(t)\|^2 + \|z_x^m(t)\|^2 + 2C_D \int_0^t \|z_{\tau x}^m(\tau)\|^2 d\tau = \|u_1^m - u_1\|^2 + \|u_{0x}^m - u_{0x}\|^2$$
$$-2\int_0^t \langle \hat{g}_e(u_x^m(\tau)) - \tilde{g}(\tau), z_{\tau x}^m(\tau)\rangle d\tau + 2\int_0^t \langle F(\tau), z_\tau^m(\tau)\rangle_{V^*,V} d\tau + X_m(t), \tag{2.41}$$

where

$$X_m(t) = 2\Big[-\langle u_t(t), u_t^m(t)\rangle - \langle u_x(t), u_x^m(t)\rangle - 2C_D \int_0^t \langle u_{\tau x}(\tau), u_{\tau x}^m(\tau)\rangle d\tau$$
$$+ \langle u_1, u_1^m\rangle + \langle u_{0x}, u_{0x}^m\rangle - \int_0^t \langle \hat{g}_e(u_x^m(\tau)), u_{\tau x}(\tau)\rangle d\tau$$
$$- \int_0^t \langle \tilde{g}(\tau), u_{\tau x}^m(\tau)\rangle d\tau + 2\int_0^t \langle F(\tau), u_\tau(\tau)\rangle d\tau \Big]. \tag{2.42}$$

Now

$$2\int_0^t \langle \hat{g}_e(u_x^m(\tau)) - \tilde{g}(\tau), z_{\tau x}^m(\tau)\rangle d\tau$$
$$= 2\int_0^t \langle \hat{g}_e(u_x^m(\tau)) - \hat{g}_e(u_x(\tau)), z_{\tau x}^m(\tau)\rangle d\tau$$
$$+2\int_0^t \langle \hat{g}_e(u_x(\tau)) - \tilde{g}(\tau), z_{\tau x}^m(\tau)\rangle d\tau, \tag{2.43}$$

and

$$
\begin{aligned}
&|2\int_0^t \langle \hat{g}_e(u_x^m(\tau)) - \hat{g}_e(u_x(\tau)), z_{\tau x}^m(\tau)\rangle d\tau| \\
&\quad \le 2\int_0^t \tilde{L}\|u_x^m(\tau) - u_x(\tau)\|\|z_{\tau x}^m(\tau)\| d\tau \\
&\quad \le \frac{\tilde{L}^2}{\delta}\int_0^t \|z_x^m(\tau)\|^2 d\tau + \delta\int_0^t \|z_{\tau x}^m(\tau)\|^2 d\tau. \qquad (2.44)
\end{aligned}
$$

Substituting (2.44) into (2.41), we obtain

$$
\begin{aligned}
&\|z_t^m(t)\|^2 + \|z_x^m(t)\|^2 + (2C_D - \delta)\int_0^t \|z_{\tau x}^m(\tau)\|^2 d\tau \le \|u_1^m - u_1\|^2 \\
&\qquad + \|u_{0x}^m - u_{0x}\|^2 + \frac{\tilde{L}^2}{\delta}\int_0^t \|z_x^m(\tau)\|^2 d\tau + |X_m(t)| + |Y_m(t)|, \qquad (2.45)
\end{aligned}
$$

where

$$
Y_m(t) = 2\int_0^t \langle \hat{g}_e(u_x(\tau)) - \tilde{g}(\tau), z_{\tau x}^m(\tau)\rangle d\tau + 2\int_0^t \langle F(\tau), z_\tau^m(\tau)\rangle_{V^*,V} d\tau.
$$

Choosing the same subsequence as before we have $\|u_1^m - u_1\| \to 0$, $\|u_{0x}^m - u_{0x}\| \to 0$. Since $z_t^m \to 0$ weakly in $L^2(0, T; V)$ we also have that $|Y_m(t)| \to 0$. We can argue that $|X_m(t)| \to 0$ as $m \to \infty$ for a.e. t in the following way: using the convergences (2.35) and (2.36) in (2.42) we obtain the integrated form of (2.37) with $\varphi = u_t$. The fact that u satisfies this equation gives the required result. Now by applying the generalized Gronwall inequality to (2.45) we can see that

$$
\|z_x^m(t)\|^2 \to 0 \qquad \text{for a.e. } t \in [0, T],
$$

and we can conclude that

$$
\hat{g}_e(u_x^m) \to \hat{g}_e(u_x) \qquad \text{strongly in } L^2(0, T; H).
$$

This guarantees (2.40) and thus u is a weak solution of (2.24)–(2.26).

Uniqueness of the weak solution of (2.24)–(2.26) can be shown in the standard way (e.g., see [2, 3, 6, 8]).

We now prove that (2.16)–(2.18) has a unique weak solution on some interval $[0, t^*]$. By the above remarks we know that the weak solution of (2.24)–(2.26) has the property that u_x is continuous in t. Thus, there exists t^* with $0 < t^* \le T$ such that

$$
\|u_x(t) - u_{0x}\| \le 1 \qquad \text{for all } t \in [0, t^*],
$$

and therefore

$$
\hat{g}_e(u_x(t)) = g_e(u_x(t)) \qquad \text{for all } t \in [0, t^*].
$$

Hence u is a weak solution of (2.16)–(2.18) on $[0, t^*]$. Uniqueness of the weak solution can again be shown in the standard way. This completes the proof of Theorem 2.1.

Now we use the additional assumption (A_{b_e}) to guarantee the existence of a *global* weak solution.

THEOREM 2.2. *Under assumptions* (A_{g_e}), (A_{b_e}), (A_f) *the system* (2.16)–(2.18) *admits a unique global weak solution*.

Proof. As before we can define Galerkin approximations $u^m(t) = \sum_{i=1}^m C_j(t)\psi_j$ to solve (2.27)–(2.29) with the nonlinear function g_e instead of $\hat{g}_e$. By assumption (A_{b_e}) we can develop a similar *a priori* bound:

$$\|u_t^m(t)\|^2 + \|u_x^m(t)\|^2 + \nu \int_0^t \|u_{\tau x}^m(\tau)\|^2 d\tau \le \tilde{C} = \tilde{C}(u_0, u_1, F, T, C_1, C_2). \quad (2.46)$$

Thus we can again obtain convergences (2.35)–(2.36). Additionally, as in [2, Lemma 5.1] we can show that

$$u_x^m(t) \to u_x(t) \qquad \text{weakly in } H.$$

(The arguments to obtain this convergence in [2] depend only on the *a priori* bound and the general Arzela–Ascoli Theorem and are independent of the specific assumptions on the nonlinear function.) Thus by the weak lower semicontinuity of the norm in H we obtain that

$$\|u_x(t)\|^2 \le \tilde{C}.$$

So the proof can be completed exactly as before using the local Lipschitz property of g_e in the ball $B_{\sqrt{\tilde{C}}}(0)$ in H.

3. THE NONLINEAR PROBLEM WITH LINEAR INTERNAL DYNAMICS

In this section we consider the system

$$u_{tt} - C_D u_{txx} - u_{xx} - (g_e(u_x))_x - \varepsilon_{1x} = F(t) \qquad \text{in } V^* \quad (3.47)$$

$$\varepsilon_{1t} + c_1 \varepsilon_1 = \frac{d}{dt} g_v(u_x, u_{tx}) \quad (3.48)$$

$$\varepsilon_1(0) = 0 \quad (3.49)$$

$$u(0, x) = u_0 \in V \quad (3.50)$$

$$u_t(0, x) = u_1 \in H, \quad (3.51)$$

where

$$g_v(u_x, u_{tx}) = \begin{cases} g_{vi}(u_x) & \text{if } u_{tx} > 0 \\ g_{vd}(u_x) & \text{if } u_{tx} < 0, \end{cases}$$

i.e., the viscoelastic response function is different when the strain is increasing and when it is decreasing. The internal dynamics is interpreted in the sense that

$$\varepsilon_1(t) = \int_0^t e^{-c_1(t-s)} \frac{d}{ds} g_v(u_x, u_{sx}) ds. \tag{3.52}$$

Integrating (3.52) by parts we obtain

$$\varepsilon_1(t) = g_v(u_x, u_{tx}) - \int_0^t c_1 e^{-c_1(t-s)} g_v(u_x, u_{sx}) ds$$
$$+ \sum_{k=0}^{K} h(t - t_k) e^{-c_1(t-t_k)} (-1)^{k+1} \left[g_{vi}(u_x(t_k)) - g_{vd}(u_x(t_k)) \right], \tag{3.53}$$

where h is the Heaviside function and t_k, $k \geq 1$, are the points where, roughly speaking, $u_{tx}(t_k) = 0$ with $t_0 = 0$ (e.g., see [6]). More precisely, as explained in detail in [5, 6], the definition of g_v is based on the *a priori* given set of points $\{t_k\}$ where the value of g_v takes alternate values g_{vi} and g_{vd} on successive intervals $(t_k, t_{k+1}]$. That is, in the *system formulation* (based on experimental data) one is *given* functions g_{vi} and g_{vd} and a sequence of points $\{t_k\}$ so that g_v is *defined* by the alternating values g_{vi}, g_{vd} on intervals $(t_k, t_{k+1}]$. Thus in essence g_v depends on u_x and t and not on u_{tx}. In what follows we will therefore use the notation $g_v(u_x)$.

Definition 3.1. *We define* $(u, \varepsilon_1) \in \mathcal{U}(0, T) \times L^2(0, T; H)$ *to be a weak solution of* (3.47)–(3.51) *if it satisfies*

$$\langle u_{tt}, \varphi \rangle_{V^*,V} + C_D \langle u_{tx}, \varphi_x \rangle + \langle u_x, \varphi_x \rangle + \langle g_e(u_x), \varphi_x \rangle + \langle \varepsilon_1, \varphi_x \rangle$$
$$= \langle F, \varphi \rangle_{V^*,V} \text{ in } L^2(0, T) \qquad \text{for every } \varphi \in V \tag{3.54}$$

$$u(0, x) = u_0 \in V \tag{3.55}$$

$$u_t(0, x) = u_1 \in H \tag{3.56}$$

and

$$\varepsilon_1(t) = g_v(u_x(t)) - \int_0^t c_1 e^{-c_1(t-s)} g_v(u_x) ds$$
$$+ \sum_{k=0}^{K} h(t - t_k) e^{-c_1(t-t_k)} (-1)^{k+1} \left[g_{vi}(u_x(t_k)) - g_{vd}(u_x(t_k)) \right]. \tag{3.57}$$

We make similar assumptions on g_{vi}, g_{vd} as on g_e, namely,

(A_{g_v}) The nonlinear functions g_{vi}, g_{vd} satisfy the following local Lipschitz condition: let $B_r(0)$ denote the ball of radius r centered at 0 in H and for some positive constants $L^i_{B_r}$ and $L^d_{B_r}$ we have

$$\|g_{vi}(w) - g_{vi}(v)\| \le L^i_{B_r}\|w - v\|$$

$$\|g_{vd}(w) - g_{vd}(v)\| \le L^d_{B_r}\|w - v\|$$

for all $w, v \in B_r(0)$.

(A_{b_v}) There exist constants C_1^i, C_1^d, C_2^i, and C_2^d such that

$$\|g_{vi}(v)\| \le C_1^i\|v\| + C_2^i,$$

$$\|g_{vd}(v)\| \le C_1^d\|v\| + C_2^d,$$

for every $v \in H$.

THEOREM 3.1. *Under assumptions* $(A_{g_e}), (A_{g_v}), (A_f)$ *the system* (3.47)–(3.51) *has a unique local weak solution.*

Proof. The proof is essentially the same as the proof of Theorem 2.1, so we just outline the crucial steps. First, as in [6] we consider the interval $[0, t_1]$. Let P denote the Hilbert space radial retraction onto the ball in H centered at u_{0x} with radius 1, and define

$$\hat{g}_e(v) = g_e(Pv), \tag{3.58}$$

$$\hat{g}_{vi}(v) = g_{vi}(Pv), \tag{3.59}$$

$$\hat{g}_{vd}(v) = g_{vd}(Pv). \tag{3.60}$$

Thus $\hat{g}_e, \hat{g}_{vi}, \hat{g}_{vd}$ satisfy the following global Lipschitz and boundedness properties:

$$\|\hat{g}_j(w) - \hat{g}_j(v)\| \le L_j\|w - v\|, \qquad j = e, vi, vd, \tag{3.61}$$

$$\|\hat{g}_j(w)\| \le C_1^j\|w\| + C_2^j. \tag{3.62}$$

Hence we can consider the problem

$$\begin{aligned} u_{tt} - C_D u_{txx} - u_{xx} - (\hat{g}_e(u_x))_x & \\ - \frac{\partial}{\partial x}\Big[\hat{g}_v(u_x) - e^{-c_1 t}\hat{g}_v(u_{0x}) & \\ - c_1\int_0^t e^{-c_1(t-s)}\hat{g}_v(u_x)ds\Big] &= F(t) \qquad \text{in } V^*, \end{aligned} \tag{3.63}$$

$$u(0, x) = u_0, \tag{3.64}$$

$$u_t(0, x) = u_1. \tag{3.65}$$

We develop Galerkin approximations $\{u^m\}$ to (3.63)–(3.65). The additional terms in this system as compared to (2.24)–(2.26) cause no difficulties in obtaining an *a priori* estimate similar to (2.34) due to the properties (3.62). Thus the convergences (2.35)–(2.36) can be obtained and the strong convergence

$$u_x^m(t) \to u_x(t) \qquad \text{in } H$$

can be established. One can also observe the additional regularity: $u \in C([0, t_1]; V)$, $u_t \in C([0, t_1]; H)$, and also that $\varepsilon_1 \in C([0, t_1]; H)$. By the continuity property of u_x in t there exists $t^* > 0$ such that for $t \in [0, t^*]$, $\|u_x(t) - u_{0x}\| \leq 1$. Hence $\hat{g}_j(u_x) = g_j(u_x)$, $j = e, vi, vd$ on $[0, t^*]$. So u is a weak solution of (3.47)–(3.51) on the interval $[0, t^*]$. If $t^* = t_1$ it makes sense to consider the next interval $[t_1, t_2]$, where local existence of a weak solution can be established similarly as before. Uniqueness of the weak solution is shown in the standard way.

The existence of a global weak solution can be guaranteed under additional boundedness assumptions.

THEOREM 3.2. *Under assumptions* $(A_{g_e}), (A_{g_v}), (A_{b_e}), (A_{b_v}), (A_f)$ *the system* (3.47)–(3.51) *admits a unique global weak solution.*

Proof. First we consider the interval $[0, t_1]$. We develop Galerkin approximations $\{u^m\}$ for (3.47)–(3.51) as before and by the boundedness properties $(A_{b_e}), (A_{b_v})$ we can obtain an *a priori* estimate like (2.46). The crucial step in the proof is that we can establish the convergence

$$u_x^m(t) \to u_x(t) \qquad \text{weakly in } H,$$

and thus guarantee that $\|u_x(t)\|^2 \leq \tilde{C}$. Now the local Lipschitz property of g_e and g_v can be used in the ball $B_{\sqrt{\tilde{C}}}(0)$ to yield the strong convergence $u_x^m(t) \to u_x(t)$ in H as before. Uniqueness can be shown in the standard way, and then the weak solution can be extended to the next intervals $[t_i, t_{i+1}]$, $i \geq 1$.

4. THE NONLINEAR PROBLEM WITH NONLINEAR INTERNAL DYNAMICS

We consider the system

$$u_{tt} - C_D u_{txx} - u_{xx} - (g_e(u_x))_x - \varepsilon_{1x} = F(t) \qquad \text{in } V^* \tag{4.66}$$

$$\varepsilon_{1t} + c_1 \varepsilon_1 = g_{in}(\varepsilon_1) + \frac{d}{dt} g_v(u_x, u_{tx}) \tag{4.67}$$

$$\varepsilon_1(0) = 0 \tag{4.68}$$

$$u(0,x) = u_0 \in V \tag{4.69}$$

$$u_t(0,x) = u_1 \in H, \tag{4.70}$$

where the nonlinear functions g_e, g_v are as in Section 3. We interpret the internal dynamics in the sense that the internal strain ε_1 solves

$$\varepsilon_1(t) = g_v(u_x(t)) + \int_0^t e^{-c_1(t-s)}\,[-c_1 g_v(u_x) + g_{in}(\varepsilon_1)]\,ds$$
$$+ \sum_{k=0}^{K} h(t-t_k)e^{-c_1(t-t_k)}(-1)^{k+1}\,[g_{vi}(u_x(t_k)) - g_{vd}(u_x(t_k))]\,, \tag{4.71}$$

where again h is the Heaviside function and $t_k, \quad k \geq 1$ are defined as in Section 3. We suppose that the nonlinear functions g_e, g_v, g_{in} satisfy global Lipschitz properties:

(A_L) For some positive constants L^e, L^{vi}, L^{vd} and L^{in} we have

$$\|g_e(w) - g_e(v)\| \leq L^e\|w - v\| \tag{4.72}$$

$$\|g_{vi}(w) - g_{vi}(v)\| \leq L^{vi}\|w - v\| \tag{4.73}$$

$$\|g_{vd}(w) - g_{vd}(v)\| \leq L^{vd}\|w - v\| \tag{4.74}$$

$$\|g_{in}(w) - g_{in}(v)\| \leq L^{in}\|w - v\| \tag{4.75}$$

for all $w, v \in H$.

We prove the following theorem:

THEOREM 4.1. *Under assumptions* $(A_L), (A_f)$ *the system* (4.66)–(4.70) *has a unique global weak solution* .

Proof. Let us first consider the interval $[0, t_1]$. On this interval $g_v = g_{vi}$ or $g_v = g_{vd}$ depending on the initial conditions and the forcing term. We define the following approximate sequence $\{u^N, \varepsilon_1^N\}$: let $\{u^N, \varepsilon_1^N\}$ be the unique weak solution of the system

$$\langle u_{tt}^N, \varphi\rangle_{V^*,V} + C_D\langle u_{tx}^N, \varphi_x\rangle + \langle u_x^N, \varphi_x\rangle + \langle g_e(u_x^N), \varphi_x\rangle$$
$$+ \langle \varepsilon_1^N, \varphi_x\rangle = \langle F, \varphi\rangle_{V^*,V} \tag{4.76}$$

$$u^N(0,x) = u_0 \in V \tag{4.77}$$

$$u_t^N(0,x) = u_1 \in H \tag{4.78}$$

and

$$\varepsilon_1^N = g_v(u_x^N) - e^{-c_1 t} g_v(u_{0x}) + \int_0^t e^{-c_1(t-s)}\left[-c_1 g_v(u_x^N) + g_{in}(\varepsilon_1^{N-1})\right] ds, \tag{4.79}$$

where $\varepsilon_1^0 = 0$. The sequence is well-defined since (4.76)–(4.79) is the same as (3.54)–(3.57) except for the known term $\int_0^t e^{-c_1(t-s)} g_{in}(\varepsilon_1^{N-1}(s))ds$. It is easy to see that Theorem 3.2 extends to this case as well. Also the assumption (A_L) for g_e and g_v guarantees that $(A_{g_e}), (A_{g_v}), (A_{b_e})$, and (A_{b_v}) are satisfied. Our goal is to show that $\{u^N(t)\}, \{u_x^N(t)\}, \{\varepsilon_1^N(t)\}$ are Cauchy sequences (uniformly in $t < t_1$).

Let $\hat{u}^N = u^N - u^{N-1}$ and $\hat{\varepsilon}_1^N = \varepsilon_1^N - \varepsilon_1^{N-1}$. Then we have

$$\langle \hat{u}_{tt}^N, \varphi \rangle + C_D \langle \hat{u}_{tx}^N, \varphi_x \rangle + \langle \hat{u}_x^N, \varphi_x \rangle + \langle g_e(u_x^N) - g_e(u_x^{N-1}), \varphi_x \rangle + \langle \hat{\epsilon}_1^N, \varphi_x \rangle = 0. \tag{4.80}$$

With $\varphi = \hat{u}_t^N$ this gives

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\{\|\hat{u}_t^N(t)\|^2 + \|\hat{u}_x^N(t)\|^2\} + C_D\|\hat{u}_{xt}^N(t)\|^2 &\le (L^e + L^v)\|\hat{u}_x^N(t)\|\|\hat{u}_{xt}^N(t)\| \\ &\quad + c_1 L^v \int_0^t \|\hat{u}_x^N(s)\| ds \|\hat{u}_{tx}^N(t)\| \\ &\quad + L^{in} \int_0^t \|\varepsilon_1^{N-1}(s) - \varepsilon_1^{N-2}(s)\| ds \|\hat{u}_{tx}^N(t)\|, \end{aligned} \tag{4.81}$$

where $L^v = \max\{L^{vi}, L^{vd}\}$. Thus,

$$\begin{aligned} \|\hat{u}_t^N(t)\|^2 + \|\hat{u}_x^N(t)\|^2 + 2C_D \int_0^t \|\hat{u}_{\tau x}^N(\tau)\|^2 d\tau &\le 2(L^e + L^v) \int_0^t \|\hat{u}_x^N(\tau)\|\|\hat{u}_{\tau x}^N(\tau)\| d\tau \\ &\quad + 2c_1 L^v \int_0^t \int_0^\tau \|\hat{u}_x^N(s)\| ds \|\hat{u}_{\tau x}^N(\tau)\| d\tau \\ &\quad + 2L^{in} \int_0^t \int_0^\tau \|\varepsilon_1^{N-1}(s) - \varepsilon_1^{N-2}(s)\| ds \|\hat{u}_{\tau x}^N(\tau)\| d\tau. \end{aligned} \tag{4.82}$$

We estimate the second term on the right side as

$$\begin{aligned} 2c_1 L^v \int_0^t \int_0^\tau \|\hat{u}_x^N(s)\| ds \|\hat{u}_{\tau x}^N(\tau)\| d\tau &\le 2c_1 L^v \int_0^t \|\hat{u}_x^N(\tau)\| d\tau \int_0^t \|\hat{u}_{\tau x}^N(\tau)\| d\tau \\ &\le \frac{c_1 L^v}{\delta_1^2} \left(\int_0^t \|\hat{u}_x^N(\tau)\| d\tau \right)^2 + c_1 L^v \delta_1^2 \left(\int_0^t \|\hat{u}_{\tau x}^N(\tau)\| d\tau \right)^2 \\ &\le \frac{c_1 L^v t_1}{\delta_1^2} \int_0^t \|\hat{u}_x^N(\tau)\|^2 d\tau + c_1 L^v \delta_1^2 t_1 \int_0^t \|\hat{u}_{\tau x}^N(\tau)\|^2 d\tau. \end{aligned} \tag{4.83}$$

The third term can be estimated similarly to yield

$$2L^{in}\int_0^t\int_0^\tau \|\varepsilon_1^{N-1}(s)-\varepsilon_1^{N-2}(s)\|ds\|\hat{u}_{\tau x}^N(\tau)\|d\tau$$
$$\le \frac{L^{in}t_1}{\delta_1^2}\int_0^t \|\varepsilon_1^{N-1}(\tau)-\varepsilon_1^{N-2}(\tau)\|^2 d\tau$$
$$+L^{in}\delta_1^2 t_1\int_0^t \|\hat{u}_{\tau x}^N(\tau)\|^2 d\tau. \tag{4.84}$$

Hence (4.82) gives

$$\|\hat{u}_t^N(t)\|^2+\|\hat{u}_x^N(t)\|^2+\mu_1\int_0^t \|\hat{u}_{\tau x}^N(\tau)\|^2 d\tau$$
$$\le \mu_2\int_0^t \|\hat{u}_x^N(\tau)\|^2 d\tau$$
$$+\mu_3\int_0^t \|\varepsilon_1^{N-1}(\tau)-\varepsilon_1^{N-2}(\tau)\|^2 d\tau, \tag{4.85}$$

where $\mu_2=(L^e+L^v+c_1L^vt_1)/\delta_1^2$, $\mu_3=L^{in}t_1/\delta_1^2$, and δ_1 is chosen such that $\mu_1=2C_D-(L^e+L^v)\delta_1^2-c_1L^v\delta_1^2t_1-L^{in}\delta_1^2t_1>0$. By the Gronwall inequality we obtain

$$\|\hat{u}_x^N(t)\|^2\le\mu_4\int_0^t \|\varepsilon_1^{N-1}(\tau)-\varepsilon_1^{N-2}(\tau)\|^2 d\tau, \tag{4.86}$$

with $\mu_4=\mu_3e^{\mu_2t_1}$, so

$$\|\hat{u}_t^N(t)\|^2+\|\hat{u}_x^N(t)\|^2\le\mu_2\int_0^t\mu_4\int_0^\tau \|\varepsilon_1^{N-1}(s)-\varepsilon_1^{N-2}(s)\|^2 ds d\tau$$
$$+\mu_3\int_0^t \|\varepsilon_1^{N-1}(\tau)-\varepsilon_1^{N-2}(\tau)\|^2 d\tau$$
$$\le(\mu_2\mu_4t_1+\mu_3)\int_0^t \|\varepsilon_1^{N-1}(\tau)-\varepsilon_1^{N-2}(\tau)\|^2 d\tau. \tag{4.87}$$

Similarly,

$$\|\hat{\varepsilon}_1^N(t)\|^2=\langle g_v(u_x^N(t))-g_v(u_x^{N-1}(t)),\hat{\varepsilon}_1^N(t)\rangle$$
$$-c_1\langle\int_0^t e^{-c_1(t-s)}(g_v(u_x^N(s))-g_v(u_x^{N-1}(s)))ds,\hat{\varepsilon}_1^N(t)\rangle$$
$$+\langle\int_0^t e^{-c_1(t-s)}(g_{in}(\varepsilon_1^{N-1}(s))-g_{in}(\varepsilon_1^{N-2}(s)))ds,\hat{\varepsilon}_1^N(t)\rangle.$$

Using similar techniques to those above, we obtain that for some constants μ_5,μ_6,μ_7 independent of N

$$\|\hat{\varepsilon}_1^N(t)\|^2\le\mu_5\|\hat{u}_x^N(t)\|^2+\mu_6\int_0^t \|\hat{u}_x^N(\tau)\|^2 d\tau$$
$$+\mu_7\int_0^t \|\varepsilon_1^{N-1}(\tau)-\varepsilon_1^{N-2}(\tau)\|^2 d\tau. \tag{4.88}$$

Substituting (4.86) into (4.88) we find

$$\|\hat{\varepsilon}_1^N(t)\|^2 \le \mu_8 \int_0^t \|\varepsilon_1^{N-1}(\tau) - \varepsilon_1^{N-2}(\tau)\|^2 d\tau, \tag{4.89}$$

which together with (4.87) yields

$$\begin{aligned}\|\hat{u}_t^N(t)\|^2 + \|\hat{u}_x^N(t)\|^2 + \|\hat{\varepsilon}_1^N(t)\|^2 &\le \mu \int_0^t \|\varepsilon_1^{N-1}(s) - \varepsilon_1^{N-2}(s)\|^2 ds \\ &\le \mu \frac{(\mu_8 t_1)^{N-2}}{(N-2)!} \|\varepsilon_1^1\|^2_{L^2(0,t_1;H)},\end{aligned} \tag{4.90}$$

where $\mu = \mu_2\mu_4 t_1 + \mu_3 + \mu_8$. This guarantees that $\{u^N(t)\}, \{u_x^N(t)\}, \{\varepsilon_1^N(t)\}$ are Cauchy sequences. Using the strong convergence of these sequences we can take a limit in (4.76) and (4.79) to obtain the existence of a weak solution of (4.76)–(4.79) on the interval $[0, t_1]$. Uniqueness of the weak solution can be derived in the usual way. Now the weak solution can be extended to the intervals $[t_i, t_{i+1}]$, $i \ge 1$, as in [6]. Thus we proved that (4.76)–(4.79) has a unique global weak solution.

REMARK 4.1. *It is possible to establish the global existence of a weak solution under local Lipschitz properties and growth conditions on the nonlinear functions* $g_e, g_{vi}, g_{vd}, g_{in}$. *Taking* u_t^N *as a test function in* (4.76) *and using standard inequalities we can show that the iterates* $\{u^N(t)\}, \{\varepsilon_1^N(t)\}$ *are bounded by a constant, independent of N. Thus the computations* (4.80)–(4.90) *can be repeated in this ball using the local Lipschitz property of the nonlinear functions.*

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