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Generalized set-valued variational inclusions in q-uniformly smooth Banach spaces ☆

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Abstract

In this paper, a class of generalized set-valued variational inclusions in Banach spaces are introduced and studied, which include many variational inclusions studied by others in recent years. By using some new and innovative techniques, several existence theorems for the generalized set-valued variational inclusions in q-uniformly smooth Banach spaces are established, and some perturbed iterative algorithms for solving this kind of set-valued variational inclusions are suggested and analyzed. Our results improve and generalize many known algorithms and results.

Keywords: Generalized set-valued variational inclusion; Iterative algorithm with error; q-uniformly smooth Banach space

1. Introduction

In recent years, variational inequalities have been extended and generalized in different directions, using novel and innovative techniques, both for their own sake and for the applications. Useful and important generalizations of variational inequalities are set-valued variational inclusions, which have been studied by [1–9].

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Recently, in [1], S.S. Chang introduced and studied the following class of set-valued variational inclusion problems in a Banach space E. For a given m-accretive mapping $A:D(A)\subset E\to 2^E$, a nonlinear mapping $N(\cdot,\cdot):E\times E\to E$, set-valued mappings $T,F:E\to CB(E)$, single-valued mapping $g:H\to H$, any given $f\in E$ and $\lambda>0$, find $q\in E$, $w\in T(q)$, $v\in F(q)$ such that

$$f\in N(w,v)+\lambda A\big(g(q)\big), \tag{1.1}$$

where $CB(E)$ denotes the family of all nonempty closed and bounded subsets of E. Under the setting of uniformly smooth Banach space, S.S. Chang [1] gave the existence and convergence theorem for the solution of variational inclusion (1.1).

For a suitable choice of the mappings T,F,N,g,A and $f\in E$, a number of known and new variational inequalities, variational inclusions, and related optimization problems introduced and studied by Noor et al. [2,3] can be obtained from (1.1).

Inspired and motivated by the results in S.S. Chang [1] and Noor et al. [2,3], the purpose of this paper is to introduce and study a class of more general set-valued variational inclusions. By using some new techniques, some existence theorems and approximate theorems for solving the set-valued variational inclusions in q-uniformly smooth Banach spaces are established and suggested. The results presented in this paper generalize, improve and unify the corresponding results of S.S. Chang [1], Noor et al. [2,3], Ding [4], Huang [5,6], Zeng [7], Kazmi [8], Jung and Morales [9], Agarwal et al. [10], Liu [11,12] and Osilike [13,14].

2. Preliminaries

Definition 2.1. Let E be a real Banach space, $A,B,C,G:E\to CB(E)$ be set-valued mappings, $W:D(W)\subset E\to 2^E$ be a set-valued mapping, $g:E\to E$ be a single-valued mapping, and $N(\cdot,\cdot),M(\cdot,\cdot):E\times E\to E$ be two nonlinear mappings, for any given $f\in E$ and $\lambda>0$, we consider the following problem of finding $u\in E$, $\bar{x}\in Au$, $\bar{y}\in Bu$, $z\in Cu$, $v\in Gu$ such that

$$f\in N(\bar{x},\bar{y})-M(z,v)+\lambda W\big(g(u)\big). \tag{2.1}$$

This problem is called the generalized set-valued variational inclusion problem in Banach space.

Next we consider some special cases of problem (2.1).

(1) If $M=0$, $W=A:D(A)\to 2^E$ is an m-accretive mapping, $A=T$, $B=F$ and $C=G=0$, then problem (2.1) is equivalent to finding $q\in E$, $w\in Tq$, $v\in Fq$ such that

$$f\in N(w,v)+\lambda A\big(g(q)\big). \tag{2.2}$$

This problem was introduced and studied by S.S. Chang [1].

(2) If $E=H$ is a Hilbert space, $M=0$ and $W=A:D(A)\to 2^E$ is an m-accretive mapping, then problem (2.1) is equivalent to finding $q\in H$, $w\in Tq$, $v\in Fq$ such that

$$f\in N(w,v)+\lambda A\big(g(q)\big). \tag{2.3}$$

This problem was introduced and studied by Noor et al. [2,3].

For a suitable choice for the mappings $A, B, C, G, W, N, M, g, f$ and the space E, we can obtain a lot of known and new variational inequalities, variational inclusions and the related optimization problems. Furthermore, they can make us be able to study mathematics, physics and engineering science problems in a general and unified frame, see [1–9].

Definition 2.2 [10]. Let E be a real Banach space. The module of smoothness of E is defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}\big(\|x+y\| + \|x-y\|\big) - 1;\ \|x\| \leqslant 1,\ \|y\| \leqslant t\right\}.$$

The space E is said to be uniformly smooth if $\lim_{t\to 0}(\rho_E(t)/t) = 0$. Moreover, E is called q-uniformly smooth if there exists a constant $c > 0$ such that $\rho_E(t) \leqslant ct^q$.

Remark 2.1 [10]. All Hilbert spaces, L_p (or l_p) spaces ($p \geqslant 2$) and the Sobolev spaces, W_m^p ($p \geqslant 2$) are 2-uniformly smooth, while, for $1 < p \leqslant 2$, L_p (or l_p) and W_m^p spaces are p-uniformly smooth.

Definition 2.3. Let $S: E \to CB(E)$ be a set-valued mapping. S is said to be quasi-contractive, if there exists a constant $r \in (0,1)$ such that for any $p \in Sx$, $q \in Sy$,

$$\|p-q\| \leqslant r \max\big\{\|x-y\|, \|x-p\|, \|x-q\|, \|y-p\|, \|y-q\|\big\}.$$

To prove the main result, we need the following lemmas.

Lemma 2.1 [15]. *Let E be a q-uniformly smooth Banach space with $q > 1$. Then there exists a constant $c > 0$ such that*

$$\big\|tx + (1-t)y - z\big\|^q \leqslant \big[1 - t(q-1)\big]\|y-z\|^q + tc\|x-z\|^q - t\big(1 - t^{q-1}c\big)\|x-y\|^q$$

for all $x, y, z \in E$ and $t \in [0,1]$.

Lemma 2.2 [16]. *Suppose that $\{e_n\}$, $\{f_n\}$, $\{g_n\}$ and $\{\gamma_n\}$ are nonnegative real sequences such that*

$$e_{n+1} \leqslant (1-f_n)e_n + f_n g_n + \gamma_n, \quad n \geqslant 0,$$

with $\{f_n\} \subseteq [0,1]$, $\sum_{n=0}^{\infty} f_n = \infty$, $\lim_{n\to\infty} g_n = 0$ and $\sum_{n=0}^{\infty} \gamma_n < \infty$. Then

$$\lim_{n\to\infty} e_n = 0.$$

For the remainder of this paper, r and c denote the constants appearing in Definition 2.3 and Lemma 2.1. We assume that $H(\cdot,\cdot)$ is the Hausdorff metric on $CB(E)$ defined by

$$H(A,B) = \max\left\{\sup_{x\in A} d(x,B), \sup_{y\in B} d(y,A)\right\}, \quad A, B \in CB(E).$$

Lemma 2.3 [17]. *Let E be a complete metric space, $T : E \to CB(E)$ be a set-valued mapping. Then for any given $\varepsilon > 0$ and any given $x, y \in E$, $u \in Tx$, there exists $v \in Ty$ such that*

$$d(u, v) \leqslant (1 + \varepsilon)H(Tx, Ty).$$

Lemma 2.4 [18]. *Let X and Y be two Banach spaces, $T : X \to 2^Y$ be a lower semi-continuous mapping with nonempty closed and convex values. Then T admits a continuous selection, i.e., there exist a continuous mapping $h : X \to Y$ such that $h(x) \in Tx$ for each $x \in X$.*

Using Lemmas 2.3 and 2.4, we suggest the following algorithms for the generalized set-valued variational inclusion (2.1).

Algorithm 2.1. For any given $x_0 \in E$, $x'_0 \in Ax_0$, $y'_0 \in Bx_0$, $z'_0 \in Cx_0$, $v'_0 \in Gx_0$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative schemes such that

$$\begin{aligned}
&(1)\quad x_{n+1} \in (1-\alpha_n)x_n + \alpha_n\big(f + y_n - N(\bar{x}_n, \bar{y}_n) + M(z_n, v_n) - \lambda W\big(g(y_n)\big)\big),\\
&(2)\quad y_n \in (1-\beta_n)x_n + \beta_n\big(f + x_n - N(x'_n, y'_n) + M(z'_n, v'_n) - \lambda W\big(g(x_n)\big)\big),\\
&(3)\quad \bar{x}_n \in Ay_n, \quad \|\bar{x}_n - \bar{x}_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right)H(Ay_n, Ay_{n+1}),\\
&(4)\quad \bar{y}_n \in By_n, \quad \|\bar{y}_n - \bar{y}_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right)H(By_n, By_{n+1}),\\
&(5)\quad z_n \in Cy_n, \quad \|z_n - z_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right)H(Cy_n, Cy_{n+1}),\\
&(6)\quad v_n \in Gy_n, \quad \|v_n - v_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right)H(Gy_n, Gy_{n+1}),\\
&(7)\quad x'_n \in Ax_n, \quad \|x'_n - x'_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right)H(Ax_n, Ax_{n+1}),\\
&(8)\quad y'_n \in Bx_n, \quad \|y'_n - y'_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right)H(Bx_n, Bx_{n+1}),\\
&(9)\quad z'_n \in Cx_n, \quad \|z'_n - z'_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right)H(Cx_n, Cx_{n+1}),\\
&(10)\quad v'_n \in Gx_n, \quad \|v'_n - v'_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right)H(Gx_n, Gx_{n+1}), \quad n = 0, 1, 2, \ldots.
\end{aligned} \tag{2.4}$$

The sequence $\{x_n\}$ defined by (2.4), in the sequel, is called Ishikawa iterative sequence.

In Algorithm 2.1, if $\beta_n = 0$ for all $n \geqslant 0$, then $y_n = x_n$. Take $\bar{x}_n = x'_n$, $\bar{y}_n = y'_n$, $z_n = z'_n$ and $v_n = v'_n$ for all $n \geqslant 0$, and we obtain the following

Algorithm 2.2. For any given $x_0 \in E$, $\bar{x}_0 \in Ax_0$, $\bar{y}_0 \in Bx_0$, $z_0 \in Cx_0$, $v_0 \in Gx_0$, compute the sequences $\{x_n\}$, $\{\bar{x}_n\}$, $\{\bar{y}_n\}$, $\{z_n\}$ and $\{v_n\}$ by the iterative schemes such that

$$\begin{aligned}
&x_{n+1} \in (1-\alpha_n)x_n + \alpha_n\big(f + x_n - N(\bar{x}_n, \bar{y}_n) + M(z_n, v_n) - \lambda W\big(g(x_n)\big)\big),\\
&\bar{x}_n \in Ax_n, \quad \|\bar{x}_n - \bar{x}_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right) H(Ax_n, Ax_{n+1}),\\
&\bar{y}_n \in Bx_n, \quad \|\bar{y}_n - \bar{y}_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right) H(Bx_n, Bx_{n+1}),\\
&z_n \in Cx_n, \quad \|z_n - z_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right) H(Cx_n, Cx_{n+1}),\\
&v_n \in Gx_n, \quad \|v_n - v_{n+1}\| \leqslant \left(1 + \frac{1}{n+1}\right) H(Gx_n, Gx_{n+1}), \quad n = 0, 1, 2, \ldots. \qquad (2.5)
\end{aligned}$$

The sequence $\{x_n\}$ defined by (2.5), in the sequel, is called Mann iterative sequence.

3. An existence theorem for solutions of the generalized set-valued variational inclusions

In this section, we shall establish an existence theorem for solutions of the set-valued variational inclusion (2.1). We have the following results.

Theorem 3.1. *Let E be a q-uniformly smooth Banach space, $A, B, C, G, W : E \to CB(E)$ be five set-valued mappings, $N(\cdot,\cdot), M(\cdot,\cdot) : E \times E \to E$, $g : E \to E$ be three single-valued mappings. If $I - N(A(\cdot), B(\cdot)) + M(C(\cdot), G(\cdot)) - \lambda W(g(\cdot))$ is quasi-contractive, then there exist $u \in E$, $\bar{x} \in Au$, $\bar{y} \in Bu$, $v \in Cu$, $z \in Gu$ which is a solution of the generalized set-valued variational inclusion* (2.1)*.*

Proof. Let

$$\begin{aligned}
&S(u) = u - N\big(A(u), B(u)\big) + M\big(C(u), G(u)\big) - \lambda W\big(g(u)\big),\\
&S_1(u) = f + S(u).
\end{aligned}$$

Since S is set-valued quasi-contractive, S_1 is also set-valued quasi-contractive.

It follows from Naddler [17] that S_1 has a unique fixed point $u \in E$, that is

$$u \in S_1(u) = f + u - N\big(A(u), B(u)\big) + M\big(C(u), G(u)\big) - \lambda W\big(g(u)\big).$$

Thus, there exist $\bar{x} \in Au$, $\bar{y} \in Bu$, $z \in Cu$, $v \in Gu$ such that

$$f \in N(\bar{x}, \bar{y}) - M(z, v) + \lambda W\big(g(u)\big),$$

that is, $u \in E$ is a solution of the generalized set-valued variational inclusion (2.1). □

4. Approximate problem of solutions for generalized set-valued variational inclusion

In Theorem 3.1, under some conditions, we have proved that there exist $u \in E$, $\bar{x} \in Au$, $\bar{y} \in Bu$, $z \in Cu$, $v \in Gu$ which is a solution of generalized set-valued variational inclusion (2.1). In this section, we shall study the approximate problem of solutions for generalized set-valued variational inclusion (2.1). We have the following result.

Theorem 4.1. *Let E be a q-uniformly smooth Banach space, $A, B, C, G, W : E \to CB(E)$ be five set-valued continuous mappings, $N(\cdot\,,\cdot), M(\cdot\,,\cdot) : E \times E \to E$, $g : E \to E$ be three single-valued continuous mappings, and $\{\alpha_n\}, \{\beta_n\}$ be two sequences in* $[0, 1]$ *satisfying the following conditions*:

(i) $I - N(A(\cdot), B(\cdot)) + M(C(\cdot), G(\cdot)) - \lambda W(g(\cdot))$ *is quasi-contractive,*
(ii) A, B, C, G *are M-Lipschitz continuous with the constants μ_1, μ_2, μ_3 and μ_4,*
(iii) $0 < h \leqslant \alpha_n$, $n \geqslant 0$, $0 < \mu_i < 1/2$, $i = 1, 2, 3, 4$,
(iv) $\alpha_n(q - 1 - cr^q) < 1$, $cr^q < q - 1$, $\beta_n^{q-1} < 1/c(1 - cr^q)$,

$$c\alpha_n^{q-1} + cr^q \beta_n (cr^q - q + 1) \leqslant 1 - cr^q.$$

Then for any given $x_0 \in E$, $x'_0 \in Ax_0$, $y'_0 \in Bx_0$, $z'_0 \in Cx_0$, $v'_0 \in Gx_0$, the sequences $\{x_n\}$, $\{\bar{x}_n\}$, $\{\bar{y}_n\}$, $\{z_n\}$ and $\{v_n\}$ defined by Algorithm 2.1 *strongly converge to the solution $u \in E$, $\bar{x} \in Au$, $\bar{y} \in Bu$, $z \in Cu$, $v \in Gu$ of the generalized set-valued variational inclusion* (2.1) *which is given in Theorem* 3.1*, respectively.*

Proof. In (1) and (2) of (2.4), choose $h_n \in W(g(x_n))$, $k_n \in W(g(y_n))$, such that

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n\big(f + y_n - N(\bar{x}_n, \bar{y}_n) + M(z_n, v_n) - \lambda k_n\big),$$
$$y_n = (1 - \beta_n)x_n + \beta_n\big(f + x_n - N(x'_n, y'_n) + M(z'_n, v'_n) - \lambda h_n\big).$$

Let

$$p_n = f + y_n - N(\bar{x}_n, \bar{y}_n) + M(z_n, v_n) - \lambda k_n,$$
$$r_n = f + x_n - N(x'_n, y'_n) + M(z'_n, v'_n) - \lambda h_n.$$

Then

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n p_n,$$
$$y_n = (1 - \beta_n)x_n + \beta_n r_n. \tag{4.1}$$

Since $I - N(A(\cdot), B(\cdot)) + M(C(\cdot), G(\cdot)) - \lambda W(g(\cdot))$ is quasi-contractive,

$$\|p_n - u\| \leqslant r \max\big\{\|y_n - u\|, \|p_n - y_n\|\big\},$$

which implies that

$$\|p_n - u\|^q \leqslant r^q\big(\|y_n - u\|^q + \|p_n - y_n\|^q\big). \tag{4.2}$$

Note that

$$\|p_n - r_n\| \leqslant r d(x_n, y_n), \quad n \geqslant 0.$$

We consider the following cases.

Case 1. Suppose that $d(x_n, y_n) = \|x_n - y_n\|$ for some $n \geqslant 0$. It follows from (4.1) that

$$\|p_n - r_n\| \leqslant r\|x_n - y_n\| = r\big\|\beta_n(x_n - r_n)\big\| \leqslant r\beta_n\|x_n - r_n\|. \tag{4.3}$$

Case 2. Suppose that $d(x_n, y_n) = \|x_n - r_n\|$ for some $n \geqslant 0$. Then we have

$$\|p_n - r_n\| \leqslant r\|x_n - r_n\|. \tag{4.4}$$

Case 3. Suppose that $d(x_n, y_n) = \|y_n - r_n\|$ for some $n \geqslant 0$. Using (4.1), we have

$$\begin{aligned}\|p_n - r_n\| &\leqslant r\|y_n - r_n\| = r\big\|(1-\beta_n)(x_n - r_n)\big\| \\ &= r(1-\beta_n)\|x_n - r_n\| \leqslant r\|x_n - r_n\|.\end{aligned} \tag{4.5}$$

Case 4. Suppose that $d(x_n, y_n) = \|x_n - p_n\|$ for some $n \geqslant 0$. Then we have

$$\|p_n - r_n\| \leqslant r\|x_n - p_n\|. \tag{4.6}$$

Case 5. Suppose that $d(x_n, y_n) = \|y_n - p_n\|$ for some $n \geqslant 0$. It follows from (4.1) that

$$\begin{aligned}\|p_n - r_n\| &\leqslant r\|y_n - p_n\| = r\big\|(1-\beta_n)(x_n - p_n) + \beta_n(r_n - p_n)\big\| \\ &\leqslant r(1-\beta_n)\|x_n - p_n\| + r\beta_n\|p_n - r_n\|,\end{aligned}$$

which implies that

$$\|p_n - r_n\| \leqslant \frac{r(1-\beta_n)}{1 - r\beta_n}\|x_n - p_n\| \leqslant r\|x_n - p_n\|. \tag{4.7}$$

It follows from (4.3)–(4.7) that

$$\|p_n - r_n\|^q \leqslant r^q\|x_n - p_n\| + r^q\|x_n - r_n\|^q, \quad n \geqslant 0. \tag{4.8}$$

It follows from Lemma 2.1 that

$$\begin{aligned}\|y_n - u\|^q &= \big\|(1-\beta_n)x_n + \beta_n r_n - u\big\|^q = \big\|(1-\beta_n)(x_n - u) + \beta_n(r_n - u)\big\|^q \\ &\leqslant \big[1 - \beta_n(q-1)\big]\|x_n - u\|^q + \beta_n c\|r_n - u\|^q \\ &\quad - \beta_n\big(1 - \beta_n^{q-1}c\big)\|x_n - r_n\|^q, \quad n \geqslant 0.\end{aligned} \tag{4.9}$$

Similarly, we have

$$\begin{aligned}\|y_n - p_n\|^q &\leqslant \big[1 - \beta_n(q-1)\big]\|x_n - p_n\|^q + \beta_n c\|p_n - r_n\|^q \\ &\quad - \beta_n\big(1 - \beta_n^{q-1}c\big)\|x_n - r_n\|^q, \quad n \geqslant 0,\end{aligned} \tag{4.10}$$

and

$$\begin{aligned}\|x_{n+1} - u\|^q &\leqslant \big[1 - \alpha_n(q-1)\big]\|x_n - u\|^q + \alpha_n c\|p_n - u\|^q \\ &\quad - \alpha_n\big(1 - \alpha_n^{q-1}c\big)\|x_n - p_n\|^q, \quad n \geqslant 0.\end{aligned} \tag{4.11}$$

By virtue of the condition (iii) and (iv), (4.2) and (4.8)–(4.11), we have

$$\begin{aligned}\|x_{n+1}-u\|^q &\leqslant \big[1-\alpha_n(q-1)\big]\|x_n-u\|^q+\alpha_n c r^q\big[\|y_n-u\|^q+\|y_n-p_n\|^q\big]\\&\quad -\alpha_n\big(1-\alpha_n^{q-1}c\big)\|x_n-p_n\|^q\\&\leqslant \big[1-\alpha_n(q-1)+cr^q\alpha_n\big(1-\beta_n(q-1)\big)\big]\|x_n-u\|^q\\&\quad +c^2r^q\alpha_n\beta_n\|r_n-u\|^q+c^2r^q\alpha_n\beta_n\|p_n-r_n\|^q\\&\quad +\big[cr^q\alpha_n\big(1-\beta_n(q-1)\big)-\alpha_n\big(1-\alpha_n^{q-1}c\big)\big]\|x_n-p_n\|^q\\&\quad -2cr^q\alpha_n\beta_n\big(1-\beta_n^{q-1}c\big)\|x_n-r_n\|^q\\&\leqslant \big[1-\alpha_n(q-1)+cr^q\alpha_n\big(1-\beta_n(q-1)\big)+c^2r^{2q}\alpha_n\beta_n\big]\|x_n-u\|^q\\&\quad +2cr^q\alpha_n\beta_n\big(cr^q+\big(\beta_n^{q-1}c-1\big)\big)\|x_n-r_n\|^q\\&\quad +\alpha_n\big[c^2r^{2q}\beta_n+cr^q\big(1-\beta_n(q-1)\big)-\big(1-\alpha_n^{q-1}c\big)\big]\|x_n-p_n\|^q\\&\leqslant \big[1-\alpha_n(q-1-cr^q)(1+cr^q\beta_n)\big]\|x_n-u\|^q\\&\leqslant \big[1-\alpha_n(q-1-cr^q)\big]\|x_n-u\|^q.\end{aligned}$$

Set $e_n=\|x_n-u\|^q$, $f_n=\alpha_n(q-1-cr^q)$,

$$g_n=\gamma_n=0,\quad n\geqslant 0.$$

It follows from the conditions (iii), (iv) and Lemma 2.2 that

$$\lim_{n\to\infty} e_n=0,$$

that is,

$$\lim_{n\to\infty}\|x_n-u\|=0.$$

Note that

$$\|r_n-u\|\leqslant r\max\big\{\|x_n-u\|,\|r_n-x_n\|\big\},$$

which implies

$$r_n\to u\quad (n\to\infty).$$

Thus, from (4.1), we have

$$y_n=(1-\beta_n)x_n+\beta_n r_n\to u.$$

From (2.4) and conditions (ii) and (iii), we have

$$\|\bar{x}_n-\bar{x}_{n+1}\|\leqslant\left(1+\frac{1}{n+1}\right)H(Ay_n,Ay_{n+1})\leqslant\left(1+\frac{1}{n+1}\right)\mu_1\|y_n-y_{n+1}\|,$$

which implies that $\{\bar{x}_n\}$ is a Cauchy sequence in E.

So, there exists $\bar{x}\in E$ such that $\bar{x}_n\to\bar{x}$. Now we show that $\bar{x}\in Au$.
In fact,

$$\begin{aligned}d(\bar{x},Au)&\leqslant\|\bar{x}-\bar{x}_n\|+H(Ax_n,Au)\\&\leqslant\|\bar{x}-\bar{x}_n\|+\mu_1\left(1+\frac{1}{n+1}\right)\|x_n-u\|\to 0.\end{aligned}$$

Also, we have

$$\bar{y}_n \to \bar{y} \in Bu, \quad z_n \to z \in Cu, \quad v_n \to v \in Gu.$$

From (2.4), we have

$$u \in f + u - N(\bar{x}, \bar{y}) + M(z, v) - \lambda W\big(g(u)\big),$$

that is,

$$f \in N(\bar{x}, \bar{y}) - M(z, v) + \lambda W\big(g(u)\big).$$

We get the required results. □

Remark 4.1. Theorem 4.1 generalizes Theorem 4.1 in S.S. Chang [1], the corresponding results in Noor et al. [2,3], Agarwal et al. [10], Liu [11,12], Osilike [13,14] and others.

Remark 4.2. Since Algorithm 2.2 is a special case of Algorithm 2.1, from Theorem 4.1, we can obtain the convergence theorem for Algorithm 2.2, the details are omitted.

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