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Bekenstein's generalized second law of thermodynamics: The role of the hoop conjecture

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In memory of Prof. Jacob D. Bekenstein
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ABSTRACT

Bekenstein's generalized second law (GSL) of thermodynamics asserts that the sum of black-hole entropy, $S_{\text{BH}} = Ac^3/4\hbar G$ (here A is the black-hole surface area), and the ordinary entropy of matter and radiation fields in the black-hole exterior region never decreases. We here re-analyze an intriguing gedanken experiment which was designed by Bekenstein to challenge the GSL. In this historical gedanken experiment an entropy-bearing box is lowered into a charged Reissner–Nordström black hole. For the GSL to work, the resulting increase in the black-hole surface area (entropy) must compensate for the loss of the box's entropy. We show that if the box can be lowered adiabatically all the way down to the black-hole horizon, as previously assumed in the literature, then for *near-extremal* black holes the resulting increase in black-hole surface-area (due to the assimilation of the box by the black hole) may become too small to compensate for the loss of the box's entropy. In order to resolve this apparent violation of the GSL, we here suggest to use a generalized version of the hoop conjecture. In particular, assuming that a physical system of mass M and electric charge Q forms a black hole if its circumference radius r_c is equal to (or smaller than) the corresponding Reissner–Nordström black-hole radius $r_{\text{RN}} = M + \sqrt{M^2 - Q^2}$, we prove that a new (and larger) horizon is already formed before the entropy-bearing box reaches the horizon of the original near-extremal black hole. This result, which seems to have been overlooked in previous analyzes of the composed black-hole-box system, ensures the validity of Bekenstein's GSL in this famous gedanken experiment.

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1. Introduction

The legend says [1,2] that it all began with a cup of tea and two genius physicists, Professor John Archibald Wheeler and his young student Jacob David Bekenstein, who tried to figure out what happens to the second law of thermodynamics when the cup goes down a black hole.

In this gedanken experiment, the thermal entropy of the tea disappears behind the black-hole horizon. Hence, at first glance, it seems that the second law of thermodynamics, which states that entropy cannot decrease, is violated in this physical process. In particular, to external observers it seems that the entropy of the visible universe decreases as the (entropy-bearing) object disappears into the black hole.

It was while attempting to resolve this apparent paradox that Bekenstein came up with the bold idea to associate entropy with

black holes – entropy as the measure of (missing) information about the black-hole internal state which is inaccessible to external observers [3]. In particular, the formal analogy between the second law of thermodynamics and Hawking's area theorem [4], which states that black-hole surface area cannot decrease [5], motivated Bekenstein to conjecture that the required black-hole entropy [6] is proportional to its surface area A [3]:

$$S_{\text{BH}} = \frac{k_{\text{B}} A}{4l_{\text{P}}^2}. \quad (1)$$

The Planck length $l_{\text{P}} = \sqrt{\hbar G/c^3}$ was introduced into (1) by Wheeler on dimensional grounds [2,7], whereas the correct proportionality coefficient, $1/4$, was later found by Hawking [8].

Using the conjectured proportionality (1) between black-hole entropy and horizon area, Bekenstein proposed a generalized version of the second law of thermodynamics [3]: *The sum of black-hole entropy, S_{BH} , and the ordinary entropy of matter and radiation fields in the black-hole exterior region, S , cannot decrease.* This conjecture

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therefore asserts that physical processes involving black holes are characterized by the relation

$$\Delta(S_{\text{BH}} + S) \geq 0. \quad (2)$$

The generalized second law of thermodynamics (GSL) provides a unique relation between thermodynamics, gravitation, and quantum theory [9]. It therefore allows us a unique glimpse into the elusive theory of quantum gravity. It should be emphasized, however, that despite the general agreement that the GSL reflects a fundamental aspect of the quantum theory of gravity, there currently exists no general proof (that is, a proof which is based on the fundamental microscopic laws of quantum gravity) for the validity of this principle. It is therefore of physical interest to consider gedanken experiments in order to test the validity of the GSL in various physical situations.

2. Bekenstein's universal entropy bound

In order to challenge the GSL, Bekenstein [3,10] analyzed a gedanken experiment in which a finite-sized object with negligible self-gravity is assimilated into a black hole [11]. In particular, Bekenstein showed that the capture of a spherical body of proper mass μ and radius R by a black hole produces an unavoidable increase ΔA in the black-hole surface area, whose *minimal* value is given by the relation [3,12]

$$(\Delta A)_{\min} = 8\pi\mu R. \quad (3)$$

Taking cognizance of Eqs. (1), (2), and (3), Bekenstein [3,10] conjectured the existence of a universal upper bound,

$$S \leq \frac{2\pi\mu R}{\hbar}, \quad (4)$$

on the entropy content of physical systems with negligible self-gravity [13–16]. In particular, as emphasized by Bekenstein [3,10], an entropy bound of the form (4) ensures that the generalized second law of thermodynamics (2) is respected in a physical process in which a spherical body with negligible self-gravity is captured by a black hole [17]. It is worth mentioning that Bekenstein and others [10,18–20] provided compelling evidence that the entropy bound (4) is respected in various physical systems in which gravity is negligible.

The main goal of the present paper is to highlight a non-trivial aspect of Bekenstein's famous gedanken experiment [3]. In particular, we shall challenge the GSL in a gedanken experiment in which an entropy-bearing spherical body is slowly lowered into a charged Reissner–Nordström black hole. We shall show below that *if* the body can be lowered adiabatically all the way down to the black-hole horizon, as previously assumed in the literature, then for *near-extremal* black holes the unavoidable increase in black-hole surface-area [see Eq. (21) below] may become too small to compensate for the loss of the body's entropy [21]. We shall further develop a possible resolution of this apparent paradox. In particular, we shall show that a generalized version of the hoop conjecture [22] may ensure the validity of Bekenstein's GSL in this type of gedanken experiments.

3. Challenging the generalized second law of thermodynamics

We consider an entropy-bearing box of proper radius R and rest mass μ which is lowered towards a Reissner–Nordström (RN) black hole of mass M and electric charge Q . The external gravitational field of the RN black-hole spacetime is described by the line element

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1}dr^2 + r^2d\Omega^2. \quad (5)$$

The black-hole (outer and inner) horizons are located at

$$r_{\pm} = M \pm \sqrt{M^2 - Q^2}. \quad (6)$$

The test-particle approximation imposes the constraints

$$\mu \ll R \ll M. \quad (7)$$

These strong inequalities imply that the lowered body (the entropy-bearing box) has negligible self-gravity and that it is much smaller than the geometric size of the black hole.

Our goal is to challenge the GSL in the most extreme situation. We shall therefore consider the case of an entropy-bearing body which is *slowly* lowered towards the black hole. As shown by Bekenstein [3], this strategy guarantees that the energy delivered to the black hole when it swallows the body is as small as possible [23]. The Bekenstein strategy of lowering the body adiabatically into the black hole also guarantees that, for given parameters of the body, the resulting increase in the surface area (entropy) of the black hole is minimized [3].

The red-shifted energy (energy-at-infinity) of a static body which is located at a radial coordinate r in the RN black-hole spacetime is given by [3]

$$\mathcal{E}(r) = \mu\sqrt{1 - \frac{2M}{r} + \frac{Q^2}{r^2}}. \quad (8)$$

This energy can be expressed in terms of the proper distance l of the body's center of mass above the black-hole horizon. Using the relation [3]

$$l(r) = \int_{r_+}^r \sqrt{g_{rr}}dr, \quad (9)$$

and taking cognizance of (5), one finds the exact relation

$$l(r) = \sqrt{(r - r_+)(r - r_-)} + 2M \ln \left(\frac{\sqrt{r - r_+} + \sqrt{r - r_-}}{\sqrt{r_+ - r_-}} \right). \quad (10)$$

From (10) one finds

$$r(l) = r_+ + (r_+ - r_-) \frac{l^2}{4r_+^2} [1 + O(l^2/r_+^2)] \quad (11)$$

in the near-horizon $l \ll r_+$ region. Substituting (11) into (8), one finds [3]

$$\mathcal{E}(l) = \frac{\mu l(r_+ - r_-)}{2r_+^2} \quad (12)$$

for the red-shifted energy of the box in the RN black-hole spacetime [24–27].

Suppose the entropy-bearing box is lowered slowly towards the black hole until its center of mass lies a proper distance l_0 (with $l_0 \geq R$) above the horizon. The box is then released to fall freely into the black hole. The energy (energy-at-infinity) delivered to the black hole when it captures the body is given by $\mathcal{E}(l = l_0)$. The increase

$$\Delta M = \mathcal{E}(l_0) = \frac{\mu l_0(r_+ - r_-)}{2r_+^2} \quad (13)$$

in the mass of the RN black hole results in a change [see Eq. (6)] [28]

$$\Delta A(l_0) = 4\pi \left[\left(M + \mathcal{E}(l_0) + \sqrt{[M + \mathcal{E}(l_0)]^2 - Q^2} \right)^2 - \left(M + \sqrt{M^2 - Q^2} \right)^2 \right] \quad (14)$$

in its surface area.

From (14) one immediately realizes that $\Delta A(l_0)$ is an increasing function of $\mathcal{E}(l_0)$, which also makes it an increasing function of the dropping point l_0 . Thus, in order to challenge the GSL in the most extreme situation, one should release the box to fall freely into the black hole from a point located as close as possible to the black-hole horizon. So, we are faced with the important question: How small can l_0 be made?

In his original analysis, Bekenstein [3] argued that the slow descent of the body towards the black hole must stop when its center of mass lies a proper distance R [29,30] above the horizon. At this point the bottom of the box almost touches the black-hole horizon [31] and the box should then be released to fall freely into the black hole [3]. Substituting $l_0 \rightarrow R$ into (14), one finds that the minimum increase in the black-hole surface area is given by

$$(\Delta A)_{\min} = 4\pi \left[\left(M + \mathcal{E}(R) + \sqrt{[M + \mathcal{E}(R)]^2 - Q^2} \right)^2 - \left(M + \sqrt{M^2 - Q^2} \right)^2 \right]. \quad (15)$$

In the regime [32]

$$\mu R \ll r_+(r_+ - r_-) \quad (16)$$

one finds [see Eq. (12)] $M\mathcal{E}(R) \ll M^2 - Q^2$, in which case the expression (15) can be expanded in the form [3]

$$(\Delta A)_{\min} = 16\pi \frac{r_+^2}{r_+ - r_-} \mathcal{E}(R) \cdot \{1 + O[\mathcal{E}(R)/(r_+ - r_-)]\}, \quad (17)$$

which yields [see Eqs. (1) and (12)] [3]

$$\Delta S_{\text{BH}} = \frac{2\pi\mu R}{\hbar} \quad (18)$$

for the minimal increase in the entropy of the black hole [33]. One therefore concludes [3] that an entropy bound of the form (4) ensures the validity of the GSL [that is, $\Delta S_{\text{total}} = \Delta S_{\text{BH}} - S_{\text{body}} \geq 0$] in the regime (16).

On the other hand, in the opposite regime [34]

$$r_+(r_+ - r_-) \ll \mu R \quad (19)$$

one finds [see Eq. (12)] $M\mathcal{E}(R) \gg M^2 - Q^2$, in which case the expression (15) can be expanded in the form

$$(\Delta A)_{\min} = 8\sqrt{2}\pi M^{3/2} \sqrt{\mathcal{E}(R)} \cdot \{1 + O[\sqrt{\mathcal{E}(R)}/M, (r_+ - r_-)^2/M\mathcal{E}(R)]\}, \quad (20)$$

which yields [see Eqs. (1) and (12)]

$$\Delta S_{\text{BH}} = \frac{M^{3/2} \sqrt{r_+ - r_-}}{r_+} \cdot \frac{2\pi \sqrt{\mu R}}{\hbar} \quad (21)$$

for the minimal increase in the entropy of the black hole. Perhaps somewhat surprisingly, the relation (21) tells us that the black-hole entropy increase (due to the assimilation of the body by the black hole) can be made arbitrarily small in the extremal $(r_+ - r_-)/r_+ \rightarrow 0$ limit.

In particular, one finds from (21) that, in the near-extremal regime $(r_+ - r_-)/r_+ \ll 1$, the increase in black-hole entropy (due to the assimilation of the box by the black hole) may become too small to compensate for the loss of the entropy of the box [35]. Of

course, this situation is unacceptable from the point of view of the GSL. In the next section we shall discuss a possible resolution of this paradox which is based on (a generalized version of) the hoop conjecture [22].

4. Bekenstein's GSL and Thorne's hoop conjecture

We have seen that *if* the body can be lowered adiabatically all the way down to the black-hole horizon, as previously assumed in the literature, then for black holes in the *near-extremal* regime (19) the resulting increase (21) in the black-hole surface area (entropy) may become too small to compensate for the loss of the body's entropy. In order to resolve this apparent violation of the GSL, we shall henceforth concentrate on the dangerous regime (19) and examine the physical consequences of Thorne's hoop conjecture [22] in the context of our gedanken experiment.

Thorne [22] has conjectured that a physical system of mass M forms a black hole if its circumference radius r_c is equal to (or smaller than) the corresponding Schwarzschild black-hole radius $r_{\text{Sch}} = 2M$ [36]. Interestingly, there are several studies which support the validity of the hoop conjecture [37]. It should be emphasized, however, that there are also known counterexamples to this version of the hoop conjecture which involve *charged* matter configurations [38,39].

Hence, we would like to suggest here a natural generalization of the hoop conjecture to the charged case: *A physical system of mass M and electric charge Q forms a black hole if its circumference radius r_c is equal to (or smaller than) the corresponding Reissner–Nordström black-hole radius $r_{\text{RN}} = M + \sqrt{M^2 - Q^2}$.* Namely, we conjecture that

$$r_c \leq M + \sqrt{M^2 - Q^2} \implies \text{Black-hole horizon exists.} \quad (22)$$

This conjecture, if true, implies that a *new* horizon is formed if the body reaches $r = r_{\text{hoop}}$, where r_{hoop} is defined by the Reissner–Nordström relation [see Eq. (6)]

$$r_{\text{hoop}} = M + \mathcal{E}(r_{\text{hoop}}) + \sqrt{[M + \mathcal{E}(r_{\text{hoop}})]^2 - Q^2}. \quad (23)$$

Substituting (8) into (23), one finds

$$r_{\text{hoop}} = M + \sqrt{M^2 - Q^2 + 4\mu^2} \quad (24)$$

for the location of the new (and larger) horizon.

Taking cognizance of (7), one realizes that the dangerous regime (19) is characterized by the relations

$$r_+ - r_- \ll \mu \ll r_+. \quad (25)$$

Thus, the radius (24) of the new horizon can be written in the form

$$r_{\text{hoop}} = M + 2\mu \{1 + O[(r_+ - r_-)^2/\mu^2]\}. \quad (26)$$

Substituting (26) into (10), one finds

$$l(r_{\text{hoop}}) = M \cdot \left[\ln \left(\frac{\mu}{r_+ - r_-} \right) + O(1) \right] \gg M \gg R. \quad (27)$$

From the inequality $l(r_{\text{hoop}}) \gg R$ [see (27)] one learns that, in the dangerous regime (19), the new horizon [40] is already formed *before* the body reaches the horizon of the original near-extremal black hole.

The formation of the new (and larger) horizon (24) implies an increase [see Eqs. (6) and (26)]

$$\begin{aligned} \Delta A &= 4\pi (r_{\text{hoop}}^2 - r_+^2) \\ &= 16\pi M\mu \{1 + O[\mu/r_+, (r_+ - r_-)^2/\mu^2]\} \end{aligned} \quad (28)$$

in black-hole surface area. The corresponding increase in black-hole entropy is given by [41]

$$\Delta S_{\text{BH}} = \frac{4\pi M\mu}{\hbar} \gg \frac{2\pi\mu R}{\hbar}. \quad (29)$$

From (29) one learns that the (generalized) hoop conjecture (22) [22], together with Bekenstein's universal entropy bound (4), ensure the validity of the GSL [that is, $\Delta S_{\text{total}} = \Delta S_{\text{BH}} - S_{\text{body}} \geq 0$] in the dangerous regime (19).

5. Summary

Historically, the idea to associate the black-hole surface area with entropy [see (1)] was suggested by Bekenstein [3] in order to save the validity of the second law of thermodynamics in a gedanken experiment in which an entropy-bearing object falls into a black hole [1,2].

Following this bold conjecture, Bekenstein [3] proposed a generalized version of the second law of thermodynamics, according to which the sum of black-hole entropy (given by $A/4\hbar$) and the ordinary entropy of matter and radiation fields in the black-hole exterior region never decreases [see (2)]. In particular, it was shown by Bekenstein [3] that, in the regime $r_+(r_+ - r_-) \gg \mu R$ of black holes which are far from extremality [12,32], an entropy bound of the form (4) ensures the validity of the GSL in the famous gedanken experiment in which an entropy-bearing spherical body of radius R and proper mass μ is slowly lowered into a black hole [42].

In the present paper we have highlighted a non-trivial aspect of Bekenstein's historical gedanken experiment: We have shown that *if* the body can be lowered slowly all the way down to the black-hole horizon, as previously assumed in the literature, then for *near-extremal* black holes the resulting increase in black-hole surface-area (due to the assimilation of the body by the black hole) may become too small to compensate for the loss of the body's entropy [see Eq. (21)].

In order to resolve this apparent violation of the GSL, we have suggested to use a (generalized) version of the hoop conjecture [22]. In particular, assuming that a physical system of mass M and electric charge Q forms a black hole if its circumference radius r_c is equal to (or smaller than) the corresponding Reissner–Nordström black-hole radius $r_{\text{RN}} = M + \sqrt{M^2 - Q^2}$ [see Eq. (22)], we have proved that a new (and *larger*) horizon is already formed *before* the body reaches the horizon of the original near-extremal black hole.

Finally, we have explicitly shown that the increase (29) in black-hole entropy, which is a direct consequence of the (generalized) hoop conjecture (22), when combined with the Bekenstein entropy bound (4), ensures the validity of Bekenstein's GSL in this historical gedanken experiment [43].

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- [5] It is important to emphasize that Hawking's area theorem is based on the *classical weak* (positive) energy condition [4]. Thus, quantum processes which violate the positive energy condition can reduce the surface area of a black hole.
- [6] As discussed above, this black-hole entropy is required in order to save the validity of the second law of thermodynamics in Bekenstein's gedanken experiment [1,2].
- [7] We shall henceforth use natural units in which $G = c = k_B = 1$.
- [8] S.W. Hawking, *Commun. Math. Phys.* 43 (1975) 199.
- [9] Note that the expression (1) for the black-hole entropy contains all known fundamental constants of nature: the Newton gravitational constant G , the speed of light c , and the quantum Planck constant \hbar (in addition, this relation also contains the thermodynamic Boltzmann constant k_B).
- [10] J.D. Bekenstein, *Phys. Rev. D* 23 (1981) 287.
- [11] Remember the cup of tea whose disappearance (along with its entropy) into the black hole initiated the research on black-hole thermodynamics [1,2].
- [12] It is worth emphasizing that in deriving the lower bound (3), it was implicitly assumed in [3] that the black hole is far from extremality in the sense that $r_+(r_+ - r_-) \gg \mu R$, where r_{\pm} are the black-hole horizon radii [see Eq. (16) below]. In the present paper we shall rigorously analyze this near-extremal case. In particular, we shall show below that near-extremal black holes in the regime $r_+(r_+ - r_-) \ll \mu R$ pose the most dangerous threat to the GSL.
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- [17] That is, the unavoidable increase (3) in the black-hole surface-area (entropy) guarantees that the GSL (2) is respected in this type of gedanken experiments provided the entropy S of the body is bounded from above by (4).
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- [21] To the best of our knowledge, this fact [see Eq. (21) below] has been overlooked in previous analyzes of this famous gedanken experiment.
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- [23] Note that the mass–energy of the box (energy as measured by asymptotic observers) is *red-shifted* during its slow descent towards the black hole [see Eq. (8) below].
- [24] It is worth noting that, due to quantum buoyancy effects [25], there is an additional quantum contribution to the energy of the (slowly lowered) body. However, as shown by Bekenstein [26] (see also [27]), for macroscopic and mesoscopic objects in the regime $\eta \equiv R/R_c \gg 1$ (here $R_c \equiv \hbar/\mu$ is the Compton length of the body), the quantum contribution to the energy of the body is very small. Specifically, for objects in the regime $R \gg R_c$ (note that this inequality is already valid for atomic-sized systems [26]), quantum buoyancy effects are suppressed by a large factor of order $O(\eta^{-1/3})$ relative to the mass–energy (12) of the body [26].
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- [27] S. Hod, *J. High Energy Phys.* 1012 (2010) 033, arXiv:1101.3151.
- [28] Here we have used the expression $A = 4\pi r_+^2$ for the surface area of the RN black hole.
- [29] Recall that R is the proper radius of the body. As emphasized by Bekenstein [3], the expression (12) for the energy of the body in the black-hole spacetime is only valid when every part of it is still *outside* the black-hole horizon.
- [30] It is worth noting that quantum buoyancy effects [25] shift the optimal dropping point of the body (that is, the radial location for which the energy of the body is minimized) to a point slightly above $l = R$. Specifically, as shown in [26] (see also [27]), for objects in the regime $\eta \equiv R/R_c \gg 1$ (as emphasized earlier, this inequality is already valid for atomic-sized systems [26]), the optimal dropping point of the body lies a proper distance $l_0 = R[1 + O(\eta^{-1/3})] \simeq R$ above the black-hole horizon.
- [31] That is, at this point the bottom of the box begins to be swallowed by the black hole.
- [32] It is worth emphasizing that the strong inequality (16) was implicitly assumed in the original analysis of Bekenstein [3]. In particular, the *first-order* relation

- $\Delta A = \frac{16\pi r_+^2}{r_+ - r_-} \Delta M$ used in [3] [see, in particular, Eqs. (8) and (A15) of J.D. Bekenstein [3]] is only valid in the $M\Delta M \ll (r_+ - r_-)^2$ regime [or equivalently, in the regime $\mu R \ll r_+(r_+ - r_-)$, see Eq. (13)].
- [33] As emphasized by Bekenstein [3], the black-hole entropy increase (18) in the regime (16) [32] is universal in the sense that it is independent of the black-hole parameters.
- [34] It is worth emphasizing again [12,32] that this regime was not analyzed in Bekenstein's seminal paper [3].
- [35] Namely, the relation (21) suggests $\Delta S_{\text{total}} = \Delta S_{\text{BH}} - S_{\text{body}} \rightarrow -S_{\text{body}} < 0$ in the near-extremal $(r_+ - r_-)/r_+ \rightarrow 0$ limit.
- [36] More precisely, for non-spherically symmetric quasi-static systems, like the one we consider here, the hoop conjecture [22] requires the circumference radius of the system to be equal to (or smaller than) the corresponding Schwarzschild black-hole radius in every direction.
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- [40] It is worth emphasizing again that this new horizon is expected to be formed according to the (generalized) hoop conjecture [22].
- [41] The last inequality in (29) follows from (7).
- [42] That is, the entropy of the box disappears from the visible universe, but the increase (18) in black-hole entropy guarantees that the GSL is respected: $\Delta S_{\text{total}} = \Delta S_{\text{BH}} - S_{\text{body}} \geq 0$.
- [43] To the best of our knowledge, the important role played by the (generalized) hoop conjecture in ensuring the validity of the GSL has not been discussed in the literature so far.