Fractional Calculus Application in Problems of Non-Linear Vibrations of Thin Plates with Combinational Internal Resonances

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Abstract

In the present paper, the dynamic response of a nonlinear plate embedded into a fractional derivative viscoelastic medium is studied by the method of multiple time scales under the condition of the combinational internal resonances of difference type using a newly developed approach resulting in uncoupling the linear parts of equations of motion of the plate. The influence of viscosity on the energy exchange mechanism between interacting nonlinear modes has been analyzed.

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1. Introduction

It is well known that the nonlinear vibrations of plates are an important area of applied mechanics, since plates are used as structural elements in many fields of industry and technology [1]. Moreover, nonlinear vibrations could be accompanied by such a phenomenon as the internal resonance, resulting in multimode response with a strong interaction of the modes involved [2] accompanied by the energy exchange phenomenon.

Nonlinear free vibrations of a thin plate embedded into a fractional derivative viscoelastic medium have been considered recently in [3] for the case when the plate motion is described by three coupled nonlinear differential equations. It has been shown that the occurrence of the internal resonance results in the interaction of modes corresponding to the mutually orthogonal displacements. As this takes place, the displacement functions are

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determined in terms of eigen functions of linear vibrations. The procedure resulting in decoupling linear parts of equations has been proposed with the further utilization of the method of multiple scales for solving nonlinear governing equations of motion, in so doing the amplitude functions are expanded into power series in terms of the small parameter and depend on different time scales. It has been shown that the type of the resonance depends on the order of smallness of the fractional derivative entering in the equations of motion of the plate.

In authors’ recent paper [3] it has been shown that the following three combinational resonances could occur during vibrations of a free supported non-linear thin rectangular plate:

\[ \omega_1 + \omega_2 = 2\omega_3, \]  
\[ \omega_1 - \omega_2 = 2\omega_3, \]  
\[ \omega_2 - \omega_1 = 2\omega_3, \]

where \(\omega_1\) and \(\omega_2\) are some particular natural frequencies of in-plane vibrations, and \(\omega_3\) is one of the natural frequencies of the out-of-plane modes.

Reference to relationships (1)-(3) shows that the combinational resonance (1) is of the additive type (which was analyzed in detail in [4]), while combinational resonances (2) and (3) are of the difference type.

In the present paper, we will focus our attention on the qualitative analysis of the case of the difference \(\omega_1 - \omega_2 = 2\omega_3\) combinational internal resonance, when two different modes of in-plane vibrations are coupled with a certain mode of out-of-plane vibrations.

2. Problem formulation

Now let us consider the case of the difference combinational resonance (2). Using the approach suggested in [3], it could be shown that the set of equations describing the modulations of amplitudes \(a_i\) and phases \(\varphi_i\) \((i=1, 2, 3)\) has the following form:

\[
\begin{align*}
\left(a_i^2\right)' & + s_ia_i^2 = -2\omega_1^{-1}\zeta_1 k_i a_i a_2 a_3^2 \sin \delta , \\
\left(a_2^2\right)' & + s_2 a_2^2 = 2\omega_2^{-1}\zeta_2 k_i a_i a_2 a_3^2 \sin \delta , \\
\left(a_3^2\right)' & + s_3 a_3^2 = \omega_3^{-1} \left(\zeta_{11} k_{16} + \zeta_{23} k_{14}\right) a_i a_i a_3^2 \sin \delta , \\
\dot{\varphi}_1 &= \frac{1}{2} \sigma_i + \omega_1^{-1}\zeta_1 k_i a_i a_3^2 + \omega_2^{-1}\zeta_2 k_0 a_i a^2 a_3 \cos \delta , \\
\dot{\varphi}_2 &= \frac{1}{2} \sigma_i + \omega_1^{-1}\zeta_1 k_0 a_i a_3^2 + \omega_2^{-1}\zeta_2 k_0 a_i a^2 a_3 \cos \delta , \\
\dot{\varphi}_3 &= \frac{1}{2} \sigma_i + \frac{1}{2} \omega_1^{-1}\zeta_1 k_{16} a_i a_3^2 + \frac{1}{2} \omega_2^{-1}\zeta_2 k_{14} a_i a_3^2 + \frac{1}{2} \omega_3^{-1} \left(\zeta_{11} k_{16} + \zeta_{23} k_{14}\right) a_i a_3 \sin \delta ,
\end{align*}
\]

where \(\delta = 2\varphi_3 + \varphi_2 - \varphi_1\) is the phase difference, a dot denotes differentiation with respect to \(T_2\), all coefficients \(k_i\) \((i=1, 2, \ldots, 8)\), \(\zeta_i\) and \(\zeta_{ij}\) \((j=1, 2)\) depend on geometry of the plate and characteristics of three modes coupled by the internal combinational resonance (2) are defined in [3], and

\[
s_i = \mu \tau_i \omega_i^{-1} \sin \psi , \quad \sigma_i = \mu \tau_i \omega_i^{-1} \cos \psi , \quad \psi = \frac{1}{2} \pi \gamma \quad (i = 1, 2, 3).
\]

Reference to (10) shows that damping coefficients \(s_i\) depend on natural frequencies of the coupled modes \(\omega_i\) and the fractional parameter \(\gamma\), i.e. the order of the fractional derivative. In other words, the application of the fractional calculus for describing the damping features of a viscoelastic medium allows one to obtain this result, what is in compliance with experimental data.
Introducing new functions $\xi_1(T_2)$, $\xi_2(T_2)$ and $\xi_3(T_2)$ such that
\[
a_1^2 = \frac{\zeta_1 k_6}{\omega_1} \xi_1 \exp(-s_1 T_2), \quad a_2^2 = \frac{\zeta_2 k_6}{\omega_2} \xi_2 \exp(-s_2 T_2), \quad a_3^2 = \frac{\zeta_3 k_6 + \zeta_3 k_2}{\omega_3} \xi_3 \exp(-s_3 T_2),
\]
and adding Eqs. (4)-(6) with due account for (11) yield
\[
2 \dot{\xi}_1 e^{-s_1 T_2} + \dot{\xi}_2 e^{-s_2 T_2} + 2 \dot{\xi}_3 e^{-s_3 T_2} = 0,
\]
while subtracting (7) from the sum of doubled (9) and (8) we obtain
\[
\dot{\delta} = 2 \phi_1 - \dot{\phi}_1 + \dot{\phi}_2 = \Sigma + \left[ \alpha_3^{-1} (\zeta_1 k_2 + \zeta_2 k_4) - \alpha_1^{-1} \zeta_1 k_5 + \alpha_2^{-1} \zeta_2 k_6 \right] a_1^2 + \left[ \alpha_3^{-1} (\zeta_1 k_6 + \zeta_2 k_7) a_1 a_2 - \alpha_1^{-1} \zeta_1 k_6 a_1^2 + \alpha_2^{-1} \zeta_2 k_7 \frac{a_1^2}{a_2} \right] \cos \delta,
\]
where $2 \Sigma = 2 \sigma_3 - \sigma_1 + \sigma_2$.

Considering (11), Eqs. (4), (6) and (13) could be rewritten in the following form:
\[
\begin{align*}
\dot{\xi}_1 &= -2b \sqrt{\frac{\rho_1}{\xi_1} \xi_1^p} e^{-\left(\tau_{123} - \left(\tau_2 + \tau_3\right)\right) T_2} \sin \delta, \\
\dot{\xi}_3 &= b \sqrt{\frac{\rho_1}{\xi_3} \xi_3^p} e^{-\left(\tau_2 + \tau_3\right) T_2} \sin \delta, \\
\dot{\delta} - \Sigma &= K_1 \xi_1 e^{-s_1 T_2} + K_2 \xi_2 e^{-s_2 T_2} + K_3 \xi_3 e^{-s_3 T_2} + \left( \frac{\dot{\xi}_1}{\xi_1} + \frac{\dot{\xi}_2}{\xi_2} + \frac{\dot{\xi}_3}{\xi_3} \right) \cot \delta = K_1 \xi_1 e^{-s_1 T_2} + K_2 \xi_2 e^{-s_2 T_2} + K_3 \xi_3 e^{-s_3 T_2} + b \left( \frac{\xi_1}{\xi_3} e^{-\left(\tau_2 + \tau_3\right) T_2} + \frac{\xi_2}{\xi_3} e^{-\left(\tau_2 + \tau_3\right) T_2} + \frac{\xi_3}{\xi_3} e^{-\left(\tau_2 + \tau_3\right) T_2} \right) \cos \delta,
\end{align*}
\]
where
\[
b = \frac{\zeta_1 k_6 + \zeta_2 k_2}{\omega_3} \sqrt{\frac{\xi_1 \xi_2 k_7}{\omega_1 \omega_2}}, \quad K_1 = \zeta_1 \xi_1 k_6 / \omega_1 \omega_2, \quad K_2 = \zeta_2 \xi_2 k_4 / \omega_1 \omega_3, \quad K_3 = \zeta_3 \xi_3 k_6 / \omega_3 \omega_2 + \zeta_3 k_7 / \omega_3 \omega_2.
\]

The non-linear set of Eqs. (12) and (14)-(16) with the initial conditions
\[
\begin{align*}
\xi_1 \big|_{T_2=0} &= \xi_{10}, & \xi_2 \big|_{T_2=0} &= \xi_{20}, & \xi_3 \big|_{T_2=0} &= \xi_{30}, & \delta \big|_{T_2=0} &= \delta_0
\end{align*}
\]
completely describes the vibrational process of the mechanical system being investigated under the condition of the difference combinational internal resonance (2), and could be solved numerically.

In the particular case at $\Sigma = 0$ and $s_1 = s_2 = s_3 = s$, Eq. (12) has the form
\[
2 \dot{\xi}_1 + \dot{\xi}_2 + 2 \dot{\xi}_3 = 0,
\]
whence it follows that
\[
2 \dot{\xi}_1 + \dot{\xi}_2 + 2 \dot{\xi}_3 = E_0,
\]
and
\[
\dot{\xi}_1 = 2E_0 (c_1 - \xi), \quad \dot{\xi}_2 = 2E_0 (c_2 + \xi), \quad \dot{\xi}_3 = E_0 (c_3 + \xi),
\]
where $c_1$, $c_2$ and $c_3$ are constants of integration, such that $4c_1 + 2c_2 + 2c_3 = 1$.

Considering (20), Eqs. (14) and (16) are reduced to
\[
\dot{\xi} = 2bE_0 \sqrt{(c_1 - \xi)(c_2 + \xi)(c_3 + \xi)} e^{-s_2} \sin \delta,
\]
\[
\begin{align*}
\dot{\delta} &= 2E_0 \left[ K_1(c_1 - \xi) + K_2(c_2 + \xi) + K_3(c_3 + \xi) \right] e^{-\tau_2} + \left( \frac{1}{2} \frac{\dot{\xi}}{c_1} + \frac{1}{2} \frac{\dot{\xi}}{c_2} + \frac{\dot{\xi}}{c_3} \right) \cot \delta \\
&= 2E_0 \left[ K_1(c_1 - \xi) + K_2(c_2 + \xi) + K_3(c_3 + \xi) \right] e^{-\tau_2} \\
&+ bE_0 \left[ 2 \sqrt{(c_1 - \xi)(c_2 + \xi)} - \frac{(c_2 - \xi)}{\sqrt{c_1 - \xi}} - \frac{(c_3 + \xi)}{\sqrt{c_2 + \xi}} \right] e^{-\tau_2} \cos \delta. \\
\end{align*}
\]

The set of Eqs. (21) and (22) could be integrated, resulting in its first integral

\[
G(\xi, \delta) = (c_3 + \xi) \sqrt{(c_1 - \xi)(c_2 + \xi)} \cos \delta - \frac{1}{2} K_1 b^{-1}(c_1 - \xi)^2 \\
+ \frac{1}{2} K_2 b^{-1}(c_2 + \xi)^2 + \frac{1}{2} K_3 b^{-1}(c_3 + \xi)^2 = G_0(\xi_0, \delta_0). \\
\tag{23}
\]

This first integral (23) defines the stream function \(G(\xi, \delta)\) such that

\[
\begin{align*}
\dot{\xi} &= -2bE_0 \frac{\partial G}{\partial \delta} e^{-\tau_2}, \\
\dot{\delta} &= 2bE_0 \frac{\partial G}{\partial \xi} e^{-\tau_2}, \\
\end{align*}
\tag{24}
\]

which describes steady-state vibrations of an elastic plate decaying with time.

3. Phase portrait analysis

Now let us carry out the qualitative analysis of the case of the difference \(\omega_1 - \omega_2 = 2\omega_3\) combinational internal resonance, when two different modes of in-plane vibrations are coupled with a certain mode of out-of-plane vibrations.

For this case, the stream-function \(G(\xi, \delta)\) is defined by relationship (23), and the phase portrait to be constructed according to (23) depends essentially on the magnitudes of the coefficients \(K_1, K_2,\) and \(K_3\).

3.1. The case \(K_1 = K_2 = K_3 = 0\)

For this case, the stream-function \(G(\xi, \delta)\) (23) is reduced to

\[
G(\xi, \delta) = (c_3 + \xi) \sqrt{(c_1 - \xi)(c_2 + \xi)} \cos \delta, \\
\tag{25}
\]

whence it follows that the stream-function also depends on the constants of integration \(c_1, c_2\) and \(c_3\), coupled by the relationship \(4c_1 + 2c_2 + 2c_3 = 1\), that is why their different combinations should be considered.

Eliminating \(\delta\) from (21) and (22) and integrating over the time the resulting equation, we obtain

\[
\int_{\xi_0}^{\xi} \frac{d\xi}{\sqrt{(c_3 + \xi)^2(c_1 - \xi)(c_2 + \xi) - G_0}} = \frac{2bE_0}{s} \left( 1 - e^{-\tau_2} \right). \\
\tag{26}
\]

The solution of (26) allows one to find the value \(\xi(T)\), and thus, to solve the problem under consideration.

1) subcase: \(c_1 = \frac{1}{4}, c_2 = c_3 = 0\). Then the stream function (25) takes the form
while the stream-lines of the phase fluid in the phase plane $\xi - \delta$ are presented in Fig. 1. Magnitudes of $G$ are indicated by digits near the curves which correspond to the stream-lines; the flow direction of the phase fluid elements are shown by arrows on the stream-lines. In the case under consideration, the velocities of the phase fluid particles could be calculated as follows

$$
\nu_\xi = \dot{\xi} = 2bE_0\xi\sqrt{\left(\frac{1}{4} - \xi^2\right)}\sin \delta e^{-\nu_\xi} , \quad \nu_\delta = \dot{\delta} = 2bE_0\left(-\frac{1}{4}\sqrt{\frac{\xi}{1-4\xi}}\right)\cos \delta e^{-\nu_\delta} .
$$

Reference to Fig. 1 shows that the phase fluid flows within the circulation zones, which tend to be located around the perimeter of the rectangles bounded by the lines $\xi = 0$, $\xi = 1/4$ and $\delta = \pm \pi / 2 \pm 2\pi n$ $(n = 0, 1, 2, \ldots)$ . As this takes place, the flow in each such rectangle becomes isolated. On all sides of the rectangle, $G = 0$ and inside each rectangle

$$
G(\xi, \delta) = \xi\sqrt{\left(\frac{1}{4} - \xi^2\right)}\cos \delta = G_1(\xi, \delta) ,
$$
the value $G$ preserves its sign. Horizontal and vertical stream lines with $G = 0$ are separatrices connected with each other at saddle-like points with coordinates $\xi = 1/4$, $\delta = \pm \pi / 2 \pm 2\pi n$ and $\xi = \xi_0 = 0$, $\delta = \delta_0 = \pm \pi / 2 \pm 2\pi n$. All stream lines within each rectangle are closed, what corresponds to the periodic variation of amplitudes and phases of vibrations, while at the points with coordinates $\xi = \xi_0 = 3/16$, $\delta_0 = 0 \pm 2\pi n$ stable stationary vibrations are observed.

Along the lines $\delta = \delta_0 = \pm \pi / 2 \pm 2\pi n$ ($n = 0, 1, 2, \ldots$) in the presence of conventional viscosity the solution could be written for the amplitude modulated regimes decaying with time

$$-\frac{8}{\xi} \sqrt{\xi \left(\frac{1}{4} - \xi\right)} \left(\frac{2bE_0}{s} \left(1 - e^{-\delta t}\right)\right) = \delta = \delta_0 = \pm \frac{\pi}{2},$$

while the low boundary regime is a stable one, since $\xi = \xi_0 = 0$ at any $\delta$.

2) subcase: $c_1 = c_2 = -\frac{1}{2}$, $c_3 = 2$. Then stream function takes the form

$$G(\xi, \delta) = (2 + \xi) \sqrt{\left(\frac{1}{4} - \xi^2\right)} \cos \delta = G_0(\xi_0, \delta_0),$$

phase fluid stream lines are shown in phase portrait in Fig. 2, while the velocities of phase fluid particles are defined by the following relationships:

$$\nu_x = \dot{\xi} = 2bE_0 (2 + \xi) \sqrt{\left(\frac{1}{4} - \xi^2\right)} \sin \delta e^{-\delta t}, \quad \nu_y = \dot{\delta} = 2bE_0 \left(\frac{\xi}{2} \sqrt{\left(\frac{1}{4} - \xi^2\right)} - (2 + \xi) \left(\frac{2\xi}{2} \sqrt{\left(\frac{1}{4} - \xi^2\right)}\right)\right) \cos \delta e^{-\delta t}.$$

From Fig. 2 it is evident that the phase fluid flows within the circulation zones, which tend to be located around the perimeter of the rectangles bounded by the lines $\xi = 0$, $\xi = 1/2$ and $\delta = \pm \pi / 2 \pm 2\pi n$ ($n = 0, 1, 2, \ldots$). As this takes place, the flow in each such rectangle becomes isolated. On all sides of the rectangle, $G = 0$ and inside each rectangle the value $G$ preserves its sign. Horizontal and vertical stream lines with $G = 0$ are separatrices connected with each other at saddle-like points with coordinates $\xi = 1/2$, $\delta = \pm \pi / 2 \pm 2\pi n$ and $\xi = \xi_0 = 0$, $\delta = \delta_0 = \pm \pi / 2 \pm 2\pi n$. Within each rectangle there are both unclosed and closed stream lines, what corresponds, respectively, to the aperiodic and periodic variations of amplitudes and phases of vibrations, while at the points with coordinates $\xi = \xi_0 = 0.112$, $\delta_0 = 0 \pm 2\pi n$ stable stationary vibrations are observed.

Along the lines $\delta = \delta_0 = \pm \pi / 2 \pm 2\pi n$ ($n = 0, 1, 2, \ldots$) in the presence of conventional viscosity the solution could be written for the amplitude modulated regimes decaying with time

$$\frac{4\xi + 1}{2} \left[\arcsin \frac{\sqrt{5}bE_0}{s} \left(1 - e^{-\delta t}\right)\right] = \sqrt{5}bE_0 \left(1 - e^{-\delta t}\right),\quad \delta = \delta_0 = \pm \frac{\pi}{2}.$$

3) subcase: $c_1 = 1$, $c_2 = 0$, $c_3 = -\frac{3}{2}$. Then stream function takes the form

$$G(\xi, \delta) = \left(\xi - \frac{3}{2}\right) \sqrt{\xi (1 - \xi)} \cos \delta = G_0(\xi_0, \delta_0),$$

phase fluid stream lines are shown in phase portrait in Fig. 3, while the velocities of phase fluid particles are defined.
by the following relationships:

\[ v_\xi = \dot{\xi} = 2bE_0 \left( \xi - \frac{3}{2} \right) \sqrt{\xi (1-\xi)} \sin \delta e^{-\alpha \xi}, \quad v_\delta = \dot{\delta} = 2bE_0 \left\{ -\frac{8\xi^2 - 12\xi + 3}{4\sqrt{1-\xi}} \right\} \cos \delta e^{-\alpha \xi}. \]

From Fig. 3 it is seen that the phase fluid flows within the circulation zones, which tend to be located around the perimeter of the rectangles bounded by the lines \( \xi = 0, \quad \xi = 1 \) and \( \delta = \pm \pi / 2 \pm 2\pi n \quad (n = 0, 1, 2,...) \). As this takes place, the flow in each such rectangle becomes isolated. On all sides of the rectangle, \( G = 0 \) and inside each rectangle the value \( G \) preserves its sign. Within each rectangle all stream lines are closed and surround the center-like points with coordinates \( \xi = \xi_0 = 0.317, \quad \delta_0 = 0 \pm 2\pi n \) corresponding to stable stationary vibrations.

Along vertical separatrices the amplitude modulated motions take place

\[ \arcsin \left( \frac{-2 \xi - \frac{3}{2}}{\xi - \frac{3}{2}} \right) = \frac{\sqrt{3}bE_0}{s} \left( 1 - e^{-\alpha \xi} \right), \quad \delta = \delta_0 = \pm \pi / 2. \]
3.2. The influence of coefficients $K_1$, $K_2$, and $K_3$ on the character of the phase portraits

Now we will trace the influence of parameters $K_1$, $K_2$, and $K_3$ on the character of the phase portraits at the fixed magnitudes of the coefficients $c_1$, $c_2$, and $c_3$.

1) subcase: $K_1b^{-1} = K_2b^{-1} = K_3b^{-1} = 1$ and $c_1 = 1; \quad c_2 = 0; \quad c_3 = -\frac{3}{2}$. Thus, the stream function takes the form

$$G(\xi, \delta) = \left(\xi - \frac{3}{2}\right) \sqrt{(1-\xi)\xi} \cos \delta - \frac{1}{2}(1-\xi)^2 + \frac{1}{2} \xi^2 + \frac{1}{2} \left(\xi - \frac{3}{2}\right)^2 = G_0(\xi_0, \delta_0),$$

phase fluid stream lines are shown in phase portrait in Fig.4, while the velocities of phase fluid particles are defined by the following relationships:

$$v_\xi = 2bE_0 \left(\xi - \frac{3}{2}\right) \sqrt{(1-\xi)\xi} \sin \delta e^{-\gamma T}; \quad v_\delta = 2bE_0 \left(\frac{2\xi(1-\xi) - \left(\xi - \frac{3}{2}\right)(-1+2\xi)}{2\sqrt{(1-\xi)\xi}} \cos \delta + (1-\xi) + \xi + \left(\xi - \frac{3}{2}\right)\right) e^{-\gamma T}.$$

On the upper $\xi = \xi_0 = 1$ and lower $\xi = \xi_0 = 0$ boundaries $G= 0.625$. Comparison of Figs.3 and 4 shows that the vertical separatrices in Fig.3 have transformed into curvilinear branches of separatrices on which $G= 0.625$, in so doing the curvilinear and straight parts of separatrices are intersect at the saddle-like points with coordinates $\xi_0 = 1, \delta_0 = \pm \pi / 2 \pm \pi n$ and $\xi_0 = 0, \delta_0 = \pm \pi n$ (Fig.4). Inside each zone divided by separatrices, the phase fluid flows in one direction along closed trajectories around stationary center-like points with coordinates $\xi = \xi_0 = 0.358, \delta = \delta_0 = \pm 2\pi n$ and $\xi = \xi_0 = 0.26, \delta = \delta_0 = \pm \pi n$, wherein the function $G$ attains its extreme values of $-0.04$ and $1.07$, respectively. The distribution of phase fluid velocities along characteristic sections is also shown in Fig.4.

2) subcase: $K_1 = 0, K_2b^{-1} = K_3b^{-1} = 1$, and $c_1 = 1; \quad c_2 = 0; \quad c_3 = -\frac{3}{2}$. Thus, the stream function takes the form:
Fig. 4. Phase portrait for the internal resonance (2) at $K b^{-1} = K b^{-1} = K b^{-1} = 1$ and $c_1 = 1, \ c_2 = 0, \ c_3 = -\frac{3}{2}$.

$$G(\xi, \delta) = \left(\frac{\xi - 3}{2}\right) \sqrt{(1 - \xi)} \xi \cos \delta + \frac{1}{2} \xi^2 + \frac{1}{2} \left(\frac{\xi - 3}{2}\right)^2 = G_0(\xi_0, \delta_0),$$

and its phase portrait is shown in Fig. 5, while the velocities of phase fluid particles could be calculated according to the following relationships:

$$v_\xi = 2bE_0 \left(\frac{\xi - 3}{2}\right) \sqrt{(1 - \xi)} \xi \sin \delta e^{-\alpha_1}, \ v_\delta = 2bE_0 \left[\sqrt{(1 - \xi)}(\xi) - \left(\frac{\xi - 3}{2}\right) \left(-1 + 2\xi \sqrt{(1 - \xi)} \xi \right)\right] \cos \delta + \xi + \left(\frac{\xi - 3}{2}\right) e^{-\alpha_1}.$$

On the upper $\xi = \xi_0 = 1$ and lower $\xi = \xi_0 = 0$ boundaries, the stream function takes on, respectively, the magnitudes $G = 0.625$ and $G = 1.125$, in so doing curvilinear branches of separatrices, on which $G = 0.625$ and $G = 1.125$, insect with the corresponding straight branches of separatrices at the saddle-like points with coordinates $\xi_0 = 1, \ \delta_0 = \pm\pi / 2 \pm \pi n$ and $\xi_0 = 0, \ \delta_0 = \pm\pi n$ (Fig. 5). Within each zone separated by separatrices, the phase fluid flows in one direction along closed trajectories around the center-like points with coordinates $\xi = \xi_0 = 0.5, \ \delta = \delta_0 = \pm 2\pi n$ and $\xi = \xi_0 = 0.12, \ \delta = \delta_0 = \pm \pi n$, at which the stream function $G$ attains the extreme magnitudes equal to 0.124 and 1.41, respectively. Despite to Fig. 4, in the given case between the upper and low zones of closed stream lines separated by curvilinear separatrices on which the stream function takes on magnitudes $G = 0.625$ and $G = 1.125$, respectively, there exists the region of nonclosed infinite stream lines corresponding to periodic change in amplitudes and aperiodic variation in phases of vibrations.

4. Conclusion

The proposed analytical approach for investigating the damped vibrations of a nonlinear plate in a fractional derivative viscoelastic medium subjected to the combinational internal resonances of additive-difference type has been possible owing to the new procedure suggested recently in [3], resulting in decoupling linear parts of equations with further utilization of the method of multiple scales for solving nonlinear governing equations of motion.

The phenomenological analysis carried out for the difference combinational internal resonance using the phase portraits constructed for different magnitudes of plate’s parameters reveals the great variety of vibrational motions:
stationary vibrations, two-sided energy exchange between two subsystems, and complete one-sided energy transfer. The analysis of the phase portraits for various oscillatory regimes shows that they contain closed and non-closed streamlines separated by the rectilinear and curvilinear separatrices, along which analytic solutions have been found, which define the irreversible energy transfer from one subsystem into another and are inherently soliton-like solutions in the theory of vibrations.

The location of horizontal rectilinear separatrices is defined by the constants of integration $c_i$ ($i=1, 2, 3$) governed by the initial conditions. These constants not only govern the distribution of the initial energy between the modes coupled by the difference internal resonance, but also influence the type of stream-lines locating between horizontal separatrices, in so doing the magnitudes of $c_i$ do not change the rectilinear character of the horizontal separatrices. The coefficients $K_i$ ($i=1, 2, 3$), which depend on the frequencies of interacting modes and on plate's parameters, define the character of separatrices connecting the boundaries $\xi = 0$ and $\xi = 1$. If all coefficients $K_i = 0$, then all separatrices are pure vertical, but if at least one of them is nonzero, then vertical separatrices transform into curvilinear separatrices, the initial and terminal points of which locate on horizontal separatrices and/or on boundary lines $\xi = 0$ and $\xi = 1$.

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