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Paths to marriage stability

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Abstract

We obtain a family of algorithms that determine stable matchings for the stable marriage problem by starting with an arbitrary matching and iteratively satisfying blocking pairs, that is, matching couples who both prefer to be together over the outcome of the current matching. The existence of such an algorithm is related to a question raised by Knuth (1976) and was recently resolved positively by Roth and Vande Vate (1992). The basic version of our method depends on a fixed ordering of all mutually acceptable man–woman pairs which is consistent with the preferences of either all men or of all women. Given such an ordering, we show that starting with an arbitrary matching and iteratively satisfying the highest blocking pair at each iteration will eventually yield a stable matching. We show that the single-proposal variant of the Gale–Shapley algorithm as well as the Roth–Vande Vate algorithm are instances of our approach. We also demonstrate that an arbitrary decentralized system does not guarantee convergence to a stable matching.

1. Introduction

We consider the *stable marriage problem* introduced by Gale and Shapley [2]. In this model there is a finite set of agents of two different types, say men and women. Each agent has a preference relation over the members of the opposite sex, where singlehood is permitted, i.e., an agent may prefer being unmatched to being with a particular member of the opposite sex. A (partial) matching is a set of disjoint man–woman pairs. A blocking pair for a matching is a pair consisting of a man and a woman who both prefer to be together over the outcome of the given matching. The goal is to find a matching that has no blocking pairs. Such a matching is called stable.

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Much research on the above matching problem and related modifications has followed the original work of Gale and Shapley. The books of Knuth [4], Gusfield and Irving [3] and Roth and Sotomayor [7] can be viewed as landmarks in the development; in particular, the latter two contain extended lists of references to contributions that have taken place in the last three decades. An interesting study of the labor market for medical interns and its relation to the stable matching problem is given in Roth [6].

Gale and Shapley [2] proved that a stable matching must exist. Their constructive proof identifies an algorithm which starts with the empty matching and generates a sequence of matchings where blocking pairs are matched at each iteration. Their original algorithm is based on block pivots, i.e., a number of blocking pairs are matched at each iteration. McVitie and Wilson [5] modified the Gale–Shapley algorithm by letting only a single blocking pair be matched at each iteration. Abeledo and Rothblum [1] showed that the resulting variants of the Gale–Shapley algorithm have a natural interpretation as execution of the dual simplex method for finding a feasible extreme point of a corresponding polyhedron.

Knuth [4] demonstrated that starting with an arbitrary matching and iteratively satisfying blocking pairs will not necessarily lead to a stable matching. The example raises the question of whether it is possible to obtain an algorithm which will start with an arbitrary matching, rather than the empty one, will match a blocking pair at each iteration, and will always terminate with a stable matching. In fact, Knuth suggested that at each iteration the coupling of matched members of a blocking pair be accompanied by the coupling of their abandoned mates. Knuth's original problem was answered negatively by Tamura [9] and Tan and Su [10]. However, Roth and Vande Vate [8] discovered an algorithm which starts with an arbitrary matching and reaches a stable matching by iteratively matching blocking pairs (without forcing the matching of the abandoned mates). Their algorithm introduces the players successively into the system and lets them iteratively satisfy blocking pairs at each stage by a decentralized system that does not rely on a central control, suggesting the presence of an "invisible hand" in the marriage market.

In the current paper we further explore mechanisms that convert arbitrary matchings into stable ones by iteratively matching blocking pairs. We first demonstrate that an arbitrary decentralized system does not guarantee convergence to a stable matching. Thus, the decentralized method of Roth and Vande Vate is very special as it does guarantee such convergence. We also describe a broad class of centralized mechanisms that assure convergence to a stable matching from an arbitrary one by matching blocking pairs. The basic version of our method depends on a fixed ordering of all mutually acceptable man–woman pairs. We say that such an ordering is *consistent with* the preferences of a particular individual if the sublist of pairs containing that individual induces a ranking of his/her potential mates that coincides with his/her preferences. We say that an ordering of all mutually acceptable man–woman pairs is *male* (*female*) *consistent* if it is consistent with the preferences of all men (women), and we say that it is *gender consistent* if it is either male consistent or female consistent. Given a gender consistent ordering, our algorithm starts with an arbitrary matching and at each iteration satisfies the highest blocking pair. We prove that this scheme always terminates and at termination identifies a stable matching. Thus, our results show that in a centralized system that prioritizes the pairs in an arbitrary way that is individually rational for either the men or the women, matching blocking pairs in accordance to the given priority-rule guarantees convergence to a stable matching. We note that the single-proposal variant of the Gale–Shapley algorithm and the Roth–Vande Vate algorithm are shown to be instances of our approach.

A formal formulation of the stable matching problem is given in Section 2. In Section 3 we discuss schemes that iteratively satisfy blocking pairs and do not necessarily converge to a stable matching. We present our new method in Section 4 and prove its convergence to a stable matching. Finally, in Section 5, we discuss some extensions of our basic algorithm, show that they capture earlier algorithms and obtain some complexity results.

2. Formulation of the stable marriage problem

Formally, a *stable marriage problem* is represented by a pair $(G; P)$ where $G = (V, E)$ is a bipartite directed graph with vertex set $V = M \cup W$, where $M \cap W = \emptyset$ and edge set E that is a subset of $M \times W$, and P is a function that maps each $v \in M \cup W$ into a preference relation $P(v)$ over the set of vertices that is adjacent to v. We refer to M and W as sets of *men* and *women*, respectively, and we refer to E as the set of acceptable pairs. Also, for $v \in V = M \cup W$, we denote the set of vertices that is adjacent to v in G by $N(v)$ and we say that members of $N(v)$ are *acceptable to v*. Of course, as G is biparite, if $v \in M$ then $N(v)$ is a subset of W and if $v \in W$ then $N(v)$ is a subset of M. Finally, for $v \in V = M \cup W$, the preference relation $P(v)$ is extended to $N(v) \cup \{v\}$ by ranking v below every element of $N(v)$, and we denote the resulting preference relation by $>_v$. The relations $<_v$, $\geqslant_v$ and $\leqslant_v$ are derived in the standard way. A stable matching problem is frequently represented by lists where for each vertex v, the members of $N(v)$ are listed in decreasing order of $P(v)$.

Suppose $(G; P)$ represents a stable matching problem. A *matching* is a subset of E such that no two edges have a common vertex. A matching μ defines a one-to-one correspondence $\mu(\cdot)$ from the set $V = M \cup W$ onto itself where $\mu(m) = w$ and $\mu(w) = m$ if $(m, w) \in \mu$ and $\mu(v) = v$ if no edge in μ contains v. Given a matching μ, we call $\mu(v)$ for $v \in V = M \cup W$ the *outcome* of μ. Also, we say that v is *single* or *unmatched under* μ if $\mu(v) = v$, otherwise we say that v *is matched under* μ and that $\mu(v)$ is the *mate of v under* μ.

A *blocking pair* of a matching μ is a pair $(m, w) \in E$ such that

$$w >_m \mu(m) \quad \text{and} \quad m >_w \mu(w). \tag{2.1}$$

A matching is called *stable* if it has no blocking pairs. Of course, a matching μ is stable if and only if for every $(m, w) \in E$

$$w \leqslant_m \mu(m) \quad \text{or} \quad m \leqslant_w \mu(w). \tag{2.2}$$

We refer to (2.2) as the stability condition for the pair (m, w).

Given a subset E' of E we let $V_{E'}$ denote the set of vertices occurring in any of the edges of E'. Also, the *restriction of* $(G; E)$ *to* E' is the stable marriage problem $(G_{E'}; P_{E'})$ where $G_{E'} = (V_{E'}, E')$ and $P_{E'}$ is the natural restriction of P to E', i.e., for $v \in V_{E'}$, $P_{E'}(v)$ is the induced order of $P(v)$ to $\{u \in V\colon \{u, v\} \in E'\}$.

3. Satisfying blocking pairs

Suppose μ is an (unstable) matching with blocking pair (m, w). We say that a matching μ' *is obtained from* μ *by satisfying the blocking pair* (m, w), if m and w are matched to each other in μ', their mates (if any) in μ are unmatched in μ', and the status of all other individuals remains unchanged, i.e.,

$$\mu'(v) = \begin{cases} w & \text{if } v = m, \\ m & \text{if } v = w, \\ v & \text{if } v = \mu(m) \text{ and } \mu(m) \neq m, \\ v & \text{if } v = \mu(w) \text{ and } \mu(w) \neq w, \\ \mu(v) & \text{if } v \in (M \cup W) \setminus \{m, \mu(m), w, \mu(w)\}. \end{cases}$$

The natural method for searching for a stable matching is to start with a given matching and consecutively generate matchings by satisfying blocking pairs. One expects that such a procedure will eventually yield a matching having no blocking pairs, i.e., a stable matching. The following example, essentially due to Knuth [4], demonstrates that this intuitive approach can cycle and need not reach a stable matching.

Example 1. Consider the stable matching problem with $M = \{m_1, m_2, m_3\}$, $W = \{w_1, w_2, w_3\}$, $E = M \times W$ and preferences that are given by the following lists:

$$P(m_1) = w_2, w_1, w_3, \qquad P(w_1) = m_1, m_3, m_2,$$

$$P(m_2) = w_1, w_3, w_2, \qquad P(w_2) = m_3, m_1, m_2,$$

$$P(m_3) = w_1, w_2, w_3, \qquad P(w_3) = m_1, m_3, m_2.$$

Suppose we start with the initial matching $\mu_0 = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$. We next list a sequence of matchings where each is obtained from its predecessor by satisfying a blocking pair. In fact, for each matching we list all blocking pairs and the successor of each matching is obtained by satisfying the first corresponding blocking pair.

The matching	The blocking pairs
$\mu_0 = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$	$(m_1, w_2), (m_3, w_2)$
$\mu_1 = \{(m_1, w_2), (m_3, w_3)\}$	$(m_2, w_1), (m_3, w_2), (m_3, w_1)$
$\mu_2 = \{(m_1, w_2), (m_2, w_1), (m_3, w_3)\}$	$(m_3, w_2), (m_3, w_1)$
$\mu_3 = \{(m_2, w_1), (m_3, w_2)\}$	$(m_1, w_3), (m_1, w_1), (m_3, w_1)$
$\mu_4 = \{(m_1, w_3), (m_2, w_1), (m_3, w_2)\}$	$(m_3, w_1), (m_1, w_1)$
$\mu_5 = \{(m_1, w_3), (m_3, w_1)\}$	$(m_2, w_2), (m_1, w_1), (m_1, w_2)$
$\mu_6 = \{(m_1, w_3), (m_2, w_2), (m_3, w_1)\}$	$(m_1, w_1), (m_1, w_2)$
$\mu_7 = \{(m_1, w_1), (m_2, w_2)\}$	$(m_3, w_3), (m_1, w_2), (m_3, w_2)$

As μ_0 is obtained from μ_7 by satisfying the blocking pair (m_3, w_3), we conclude that this sequence cycles and will not produce a stable matching.

In Knuth's original example abandoned spouses of the members of each matched blocking pair are forced to be matched. The sequence of matchings he thereby obtains, are those of the above examples that have even indices.

Consider matchings μ_3 and μ_4 in Example 1. Matching μ_4 is obtained by satisfying the blocking pair (m_1, w_3) of μ_3. We observe that man m_1 can also form a blocking pair with w_1, whom he prefers to w_3. Thus, if the initiative of proposing is granted to the men and if man m_1 behaves greedily, he should actually propose to w_3 rather than w_1 and form the blocking pair (m_1, w_1) for μ_3. This observation suggests that the blocking pair that is satisfied in each iteration should consist of a man and his most preferred woman among those with whom he forms a blocking pair. This rule is next implemented on Example 1.

Example 1 (*continued*). Consider the modification of the sequence of matchings generated in Example 1 by using the blocking pair (m_1, w_1) for μ_3.

The matching	The blocking pairs
$\mu_3 = \{(m_2, w_1), (m_3, w_2)\}$	$(m_1, w_3), (m_1, w_1)$
$\mu_4 = \{(m_1, w_1), (m_3, w_2)\}$	(m_2, w_3)
$\mu_5 = \{(m_1, w_1), (m_2, w_3), (m_3, w_2)\}$	None

In particular, μ_5 is a stable matching.

The above example suggests that convergence to a stable matching can be assured when a sequence of matchings is generated by iteratively satisfying blocking pairs where members of one sex behave greedily. But, the following example demonstrates that this conjecture is false.

Example 2. Consider the following stable matching problem with $M = \{m_1, m_2, m_3, m_4\}$, $W = \{w_1, w_2, w_3, w_4\}$ and preferences that are given by the following lists:

$$P(m_1) = w_2, w_3, \qquad P(w_1) = m_3, m_4,$$

$$P(m_2) = w_3, w_4, \qquad P(w_2) = m_4, m_1,$$

$$P(m_3) = w_4, w_1, \qquad P(w_3) = m_1, m_2,$$

$$P(m_4) = w_1, w_2, \qquad P(w_4) = m_2, m_3.$$

Suppose we start with the initial matching $\mu_0 = \{(m_1, w_2), (m_2, w_4), (m_3, w_1)\}$. We next list a sequence of matchings where each is obtained from its predecessor by satisfying a blocking pair and this selection is greedy for the men. Again, for each matching we list all blocking pairs and the successor of each matching is obtained by satisfying the first corresponding blocking pair.

The matching	The blocking pairs
$\mu_0 = \{(m_1, w_2), (m_2, w_4), (m_3, w_1)\}$	$(m_4, w_2), (m_2, w_3)$
$\mu_1 = \{(m_2, w_4), (m_3, w_1), (m_4, w_2)\}$	$(m_2, w_3), (m_1, w_3)$
$\mu_2 = \{(m_2, w_3), (m_3, w_1), (m_4, w_2)\}$	$(m_1, w_3), (m_3, w_4)$
$\mu_3 = \{(m_1, w_3), (m_3, w_1), (m_4, w_2)\}$	$(m_3, w_4), (m_2, w_4)$
$\mu_4 = \{(m_1, w_3), (m_3, w_4), (m_4, w_2)\}$	$(m_2, w_4), (m_4, w_1)$
$\mu_5 = \{(m_1, w_3), (m_2, w_4), (m_4, w_2)\}$	$(m_4, w_1), (m_3, w_1)$
$\mu_6 = \{(m_1, w_3), (m_2, w_4), (m_4, w_1)\}$	$(m_3, w_1), (m_1, w_2)$
$\mu_7 = \{(m_1, w_3), (m_2, w_4), (m_3, w_1)\}$	$(m_1, w_2), (m_4, w_2)$

As μ_0 is obtained from μ_7 by satisfying the blocking pair (m_1, w_2), we conclude that this sequence cycles and will not produce a stable matching.

The above example shows that in our two-sided markets, leaving agents to pursue their best interests independently will not necessarily produce a sequence of matchings that converges to a stable matching. In the next section we show that a stable matching will be reached, if the greedy behavior is regulated by a centralized control.

4. The gender consistent algorithm

In this section we introduce a class of algorithms whose inputs are arbitrary matchings and whose outputs are stable matchings which are obtained from the inputs by iteratively satisfying blocking pairs.

Throughout this section we consider a given stable marriage problem $(G; P)$ where $G = (M \cup W, E)$. Let $\gg$ be a strict linear order over the set of acceptable couples E. We call $\gg$ *male consistent* if for every $m \in M$ and $w, w' \in N(m)$, $(m, w) \gg (m, w')$ if and only if $w >_m w'$, and we call $\gg$ *female consistent* if for every $w \in W$ and $m, m' \in N(w)$, $(m, w) \gg (m', w)$ if and only if $m >_w m'$. Finally, we call $\gg$ *gender consistent* if it is either male consistent or female consistent.

Henceforth, we assume that $\gg$ is a given strict linear order on E. We use this order to generate a sequence of matchings by iteratively satisfying blocking pairs according to the following procedure:

The $\gg$-Consistent Algorithm.

Initialization

0. Select a matching μ_0 and let $k = 1$.

Main iteration

1. If μ_{k-1} is stable, stop with output μ_{k-1}.
2. Otherwise, let (m_k, w_k) be the blocking pair of μ_{k-1} that is ranked highest with respect to $\gg$. Obtain the matching μ_k from μ_{k-1} by satisfying the blocking pair (m_k, w_k). Let $k := k + 1$. Go to step 1.

When the underlying order $\gg$ is gender consistent, we refer to the $\gg$-Consistent Algorithm as a *gender consistent algorithm*.

The $\gg$-Consistent Algorithm terminates only when it identifies a stable matching. We will show that this is always the case when $\gg$ is gender consistent. For that purpose we find the following lemma useful.

Lemma 4.1. *Suppose μ is a matching which is not stable and suppose there is a man m^* who is contained in all blocking pairs of μ and this man is single in μ. Let w^* be the first woman according to m^*'s preference among those women that form a blocking pair for μ with m^*, and let μ^* be the matching obtained from μ by satisfying the blocking pair (m^*, w^*). Then:*

(a) *If $\mu(w^*) = w^*$, then μ^* is stable.*

(b) *If $\mu(w^*) \neq w^*$ and $m' = \mu(w^*)$, then each blocking pair of μ^* has the form (m', w) where $w^* >_{m'} w$.*

Proof. We first observe that the selection of w^* assures that each woman w that m^* prefers to $\mu^*(m^*) = w^*$ is matched under μ^* to $\mu^*(w) = \mu(w)$ whom w prefers to m^*, for otherwise (m^*, w) would be a blocking pair for μ. Thus, no pair containing m^* is a blocking pair for μ^*.

To establish (a), assume that $\mu(w^*) = w^*$, i.e., w^* is single in μ. Note that $\mu^*(v) \geqslant_v \mu(v)$ for all $v \in M \cup W$, implying that all pairs that satisfy the stability condition (2.2) with respect to μ also satisfy the stability condition (2.2) with respect to μ^*. Thus, it remains to examine only the blocking pairs of μ. But, each blocking pair of μ contains man m^* and we have already seen that no pair containing m^* is a blocking pair of μ^*. We conclude that μ^* has no blocking pairs, i.e., it is stable.

Finally, to establish (b), assume that $\mu(w^*) \neq w^*$ and that $m' = \mu(w^*) \in M$. Note that $\mu^*(v) \geqslant_v \mu(v)$ for all $v \in (M \cup W) \setminus \{m'\}$. Hence, all pairs that satisfy the stability condition (2.2) with respect to μ and do not contain m' also satisfy the stability condition (2.2) with respect to μ^*. It remains to examine only the blocking pairs of μ or pairs that contain m'. Each blocking pair of μ contains man m^* and we have already seen that no such pair is a blocking pair of μ^*. Hence, each blocking pair of μ^* must contain m'. Further, if (m', w) is a blocking pair for μ' and $w >_{m'} \mu(m')$ then $\mu'(w) = \mu(w) >_w m'$, for otherwise (m', w) would be a blocking pair for μ in contradiction to the assumption that every blocking pair of μ contains m^*. So, indeed, each blocking pair of μ^* must have the form (m', w) where $w >_{m'} w^*$. □

We are now ready for our main result that asserts the convergence of gender consistent algorithms.

Theorem 4.2. *Suppose $\gg$ is a gender consistent strict linear order on E. Then the $\gg$-Consistent Algorithm will stop. At termination it will identify a stable matching.*

Proof. We consider the case where $\gg$ is male consistent as the case where $\gg$ is female consistent follows from symmetric arguments. Also, the definition of the algorithm assures that it will always generate a stable matching at termination. Thus, we have to show that the algorithm must stop. Since the number of possible matchings is finite, it suffices to show that a matching cannot recur. Suppose that this is not the case and a matching does recur. Since the algorithm is deterministic, i.e., given a matching μ the algorithm uniquely determines the next matching, the assertion that a matching recurs means that the algorithm will cycle. So, assume that the $\gg$-Consistent Algorithm does not stop and, possibly ignoring matchings that are generated before the cycle is entered, the generated matchings $\mu_0, \mu_1, \ldots$ satisfy $\mu_t = \mu_{t+q}$ for all $t = 0, 1, \ldots$, where q is a positive integer.

Let E' be the set of pairs in E which form the blocking pairs of $\mu_0, \mu_1, \ldots$ that are satisfied during the execution of the algorithm. Consider the restriction of $(G; P)$ to E', i.e., consider the stable marriage problem $(G_{E'}; P_{E'})$. Also, let $\gg'$ be the restriction of $\gg$ to E'. Obviously, $\gg'$ is male consistent with respect to $(G_{E'}; P_{E'})$. We also observe that if the $\gg'$-Consistent Algorithm is executed on $(G_{E'}; P_{E'})$ with initial matching $\mu_0 \cap E'$ it will consecutively generate the matchings $\mu_1 \cap E', \mu_2 \cap E', \ldots$ for $(G_{E'}; P_{E'})$, and the same blocking pairs will be satisfied, respectively, as in the execution of the original algorithm on $(G; P)$.

As $\gg'$ is a strict linear order and as E' is finite, E' has a smallest element with respect to $\gg'$, say (m^0, w^0). Then there is a matching μ_t for which (m^0, w^0) is the blocking pair that will be satisfied to obtain μ_{t+1}. Now, the definition of the algorithm and the minimality of (m^0, w^0) assure that (m^0, w^0) is the only blocking pair for $\mu_t \cap E'$ in E'. Also, as $\gg'$ is male consistent, as (m^0, w^0) is the minimal element in E' and as $\mu_t(m^0) <_{m^0} w^0$, we have that $\mu_t(m^0) = m^0$, i.e., m^0 is single in $\mu_t \cap E'$. Further, all blocking pairs (in fact, only one in this case) of $\mu_t \cap E'$ include m^0, and w^0 is the first (in fact, only) woman according to m^0's preferences among those with whom m^0 forms a blocking pair for $\mu_t \cap E'$. So, the assumptions of Lemma 4.1 with respect to $\mu_t \cap E'$ and $\mu_{t+1} \cap E'$ are satisfied; hence, as $\mu_{t+1} \cap E'$ is not stable, we conclude that each blocking pair of $\mu_{t+1} \cap E'$ contains $m^1 = \mu_t(w^0)$ and a woman that m^1 prefers less than w^0. So $\mu_{t+2} \cap E'$ will be generated from $\mu_{t+1} \cap E'$ by satisfying a blocking pair (m^1, w^1) where $w^0 >_{m^1} w^1$. The rules defining the $\gg'$-Consistent Algorithm ensure that w' is the first woman according to m^1's preferences among those with whom m^1 forms a blocking pair for $\mu_{t+1} \cap E'$. So, the assumptions of Lemma 4.1 with respect to $\mu_{t+1} \cap E'$ and $\mu_{t+2} \cap E'$ are satisfied; hence, as $\mu_{t+2} \cap E'$ is not stable, we conclude that each blocking pair of $\mu_{t+2} \cap E'$ contains $m^2 \equiv \mu_{t+1}(w^1)$ and a woman that m^2 prefers less than w^1. So, $\mu_{t+3} \cap E'$ is generated from $\mu_{t+2} \cap E'$ by satisfying a blocking pair (m^2, w^2) where $w^1 >_{m^2} w^2$.

Repeated applications of Lemma 4.1 show that for each positive integer s, m^s is single in μ_{t+s}, all blocking pairs for μ_{t+s} contains a man m^s, and μ_{t+s+1} is obtained from μ_{t+s} by satisfying a blocking pair (m^s, w^s) where $\mu_{t+s-1}(m^s) = w^{s-1} >_{m^s} w^s = \mu_{t+s+1}(m^s)$. So, following the tth iteration, each time a man is forced away from his partner he is single for one period and he gets a new mate in the following period which he prefers less than the one he had just before being single. In particular, following iteration t, no man can be rematched to any woman. This conclusion contradicts the assertion that $\gg'$-Consistent Algorithm cycles, and thereby completes the proof that the $\gg$-Consistent Algorithm terminates. □

Consider a variant of the algorithm, that does not depend on an ordering of the acceptable pairs, where at each iteration one of the men is matched with the woman he prefers most among those with whom he forms a blocking pair, provided one exists. Example 2 shows that the resulting algorithm need not terminate and might cycle. But, using standard arguments, we conclude from Theorem 4.2 that if the identity of the selected man in each iteration is determined by a random choice with positive probabilities for each relevant man, the probability that the algorithm will eventually terminate with a stable matching is one.

In the original Gale–Shapley algorithm men propose simultaneously at each iteration. As mentioned in the Introduction, McVitie and Wilson [5] observed that the Gale–Shapley algorithm can be modified by letting at each iteration only one man propose to the woman he prefers most among those who have not yet turned him down. Consider an execution of any modified version of the Gale–Shapley algorithm. This execution determines a sequence of proposals and the accepted proposals correspond to satisfying blocking pairs. Further, by possibly augmenting the above list, we obtain a male consistent order $\gg$ of the acceptable pairs such that the acceptance of proposals in the modified Gale–Shapley algorithm correspond to the iterations of the $\gg$-Consistent Algorithm with the initial matching being the empty one. So, the modified version of the Gale–Shapley algorithm is a special case of the $\gg$-Consistent Algorithm.

The modified (and original) versions of the Gale–Shapley algorithm have the remarkable property that the stable matchings that they output are independent of the order of the proposals; see [3] or [7]. We also saw in the above paragraph that such executions can be viewed as executions of a $\gg$-Consistent Algorithm with the empty matching being the starting point. These facts lead one to suspect that the same stable matchings will be generated when the $\gg$-Consistent Algorithm is executed with the same initial matching under different orderings of the acceptable pairs. But, the following example shows that this is not the case.

Example 3. Consider the stable marriage problem of Example 1 and let $\mu_0 = \{(m_1, w_1), (m_2, w_2), (m_3, w_3)\}$. If we execute the algorithm with a male consistent ordering $\gg$ for which $(m_1, w_2) \gg (m_3, w_1) \gg (m_3, w_2)$, we obtain the following sequence of matchings:

$$\mu_1 = \{(m_1, w_2), (m_3, w_3)\},$$

$$\mu_2 = \{(m_1, w_2), (m_3, w_1)\},$$

$$\mu_3 = \{(m_1, w_2), (m_3, w_1), (m_2, w_3)\},$$

and μ_3 is a stable matching. On the other hand, if we execute the algorithm with the same initial matching μ_0, but with a male consistent ordering $\gg$ for which $(m_3, w_1) \gg (m_3, w_2) \gg (m_1, w_2)$, we obtain the following sequence of matchings:

$$\mu'_1 = \{(m_1, w_1), (m_3, w_2)\},$$

$$\mu'_2 = \{(m_1, w_1), (m_3, w_2), (m_2, w_3)\},$$

and μ'_2 is a stable matching.

5. Discussion and extensions

The gender consistent algorithms described in the previous section depend on a fixed (gender consistent) order of the acceptable pairs and this order stays unchanged throughout the execution of the algorithm. We next introduce a simple variant of the method that relaxes this requirement.

Consider a strictly increasing sequence of subsets of the edge set E, say $T_0 = \mu_0$, $T_1, T_2, \ldots, T_q$ where μ_0 is a matching and $T_q = E$. For $i = 1, \ldots, q$, consider the stable matching problem $(G_{T_i}; P_{T_i})$ which is the restriction of $(G; P)$ to T_i; see Section 2 for a formal definition. In particular, each matching μ of G_{T_i} is a matching of G and a pair $(m, w) \in T_i$ is a blocking of μ with respect to $(G_{T_i}; P_{T_i})$ if and only if it is a blocking pair of μ with respect to $(G; P)$. Further, if μ and μ' are matchings on G_{T_i} and μ' is obtained from μ by satisfying a blocking pair $(m, w) \in T_i$ with respect to $(G_{T_i}; P_{T_i})$, then the same conclusion about μ, μ' and (m, w) holds with respect to $(G; P)$.

We can generate a sequence of matchings by sequentially satisfying a blocking pair in the following way. First generate a sequence for $(G_{T_1}; P_{T_1})$, starting with the matching μ_0 and eventually reaching a stable matching μ_1 of $(G_{T_1}; P_{T_1})$. The matching μ_1 is then used as an initial (not necessarily stable) matching for $(G_{T_2}; P_{T_2})$ and a corresponding sequence of matchings for $(G_{T_2}; P_{T_2})$ is generated, yielding a stable matching μ_2 of $(G_{T_2}; P_{T_2})$. This procedure is continued till a stable matching for $(G_{T_q}; P_{T_q}) = (G; P)$ is obtained. The important fact is that when blocking pairs are satisfied with respect to the restriction $(G_{T_i}; P_{T_i})$ of $(G; P)$, $i = 1, \ldots, q$, these blocking pairs are also satisfied with respect to $(G; P)$ itself. Gender consistent algorithms can be used in each step to move from a stable matching of $(G_{T_{i-1}}; P_{T_{i-1}})$, which is not necessarily stable for $(G_{T_i}; P_{T_i})$, to a matching which is stable for $(G_{T_i}; P_{T_i})$. We observe that the gender consistent orders of T_i for $i = 1, \ldots, q$ can be selected completely independently; in fact, we may switch from a male consistent order to a female consistent order. Viewing the combined algorithm as one in which blocking pairs are satisfied for $(G; P)$, we see that the order of the edges can be changed whenever all stability constraints for one of the T_i's are satisfied.

One way of constructing the increasing set T_i is to construct an increasing set of vertices V_i and letting T_i be the set of edges that are spanned by V_i, i.e., $T_i = E \cap (M_i \times W_i)$ where $M_i = V_i \cap M$ and $W_i = V_i \cap W$. This approach was taken by Roth and Vande Vate [8] and it is easy to verify that whenever they obtain a stable matching for the corresponding restriction of $(G; P)$ their steps follow those of a gender consistent algorithm.

We next determine the complexity of the algorithm composed of iterations that sequentially determine stable matchings for induced marriage problems $(G_{T_i}; P_{T_i})$, where $T_1, T_2, \ldots$ are increasing subsets of E and $|T_i| = i$ for each $i = 1, 2, \ldots$. We will also assume that for each i, the gender consistent ordering of the elements of T_i used to determine the sequence of satisfied blocking pairs in the ith iteration is male consistent and is induced by a common male consistent ordering of E. Evidently, at the beginning of the ith iteration we have a matching which satisfies all the stability constraints corresponding to the pairs of T_i, with the possible exception of a single pair. If there is no such pair we move to the next iteration. If such a pair exists, say (m^i, w^i), the arguments of the proof of Theorem 4.2 show that m^i is single at the beginning of iteration i, and that throughout the iteration the position of men deteriorates while the position of women improves. So no pair can be satisfied more than once and a total of no more than $|T_i| = i$ blocking pairs will be satisfied during the iteration. So a total of no more than $\sum_{i=1}^{|E|} i = O(|E|^2)$ will be satisfied throughout the execution of the algorithm.

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